# SOME SPECIAL FUNCTIONS RELATED TO FRACTIONAL CALCULUS AND FRACTIONAL (NON-INTEGER) ORDER CONTROL SYSTEMS AND EQUATIONS

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**Abstract**. *This survey aims to attract the attention of engineers and applied scientists, eager to discover new useful analytical techniques for treatment of real mathematical models, to some applications of the fractional calculus (FC) and of the Mittag-Leffler (M-L) type functions as a class of special functions of FC, to fractional-order control systems and other fractional-order mathematical models. We introduce multi-indices generalizations of the M-L functions and discuss their properties, relations to the generalized FC, corresponding Laplace type transforms. An unexpectedly long list of examples is given, for mathematical special functions (some of them - well known) with use in solving problems arising from applied science, including control theory.*

**Key words**: *Mittag-Leffler functions, special functions, fractional calculus, fractional order differential equations and systems, control theory, mechanics, engineering*

## 1. CONTROL SYSTEMS OF FRACTIONAL ORDER AND MODELS OF PHYSICAL PHENOMENA DESCRIBED BY MEANS OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

The classical operational calculus (of Heaviside-Mikusinski) and the method of Laplace transform are well known tools in treating problems in control theory, linear systems and circuits, signals, etc. They are used in natural way for solving problems modeling integer-order systems, basically in terms of the exponential and trigonometric functions. Indeed, most of the basic characteristics of the control systems (as the unit-impulse response, the unit-step response, the unit-ramp response, etc) can be found from the transfer functions by means of integral operators like the *Laplace transform and its inversion*:

$$F(s) = L\{f(t); s\} = \int_0^\infty \exp(-st) f(t) \, dt,$$

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_C \exp(st) F(s) \, ds, \tag{1}$$

by successive integer order integrations and differentiations, and by other (integer-order) integro-differential operators. The integer order control systems usually considered, are related to transfer functions: $G(s)$ – of the controlled system, or $G_c(s)$ – of the controller, of the form

$$G(s) = \frac{1}{P(s)} = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0} \ , \quad G_c(s) = K + T s^n, \ n \in \mathbb{N}. \tag{2}$$

However, *fractional (non-integer) order dynamic systems* have been already considered that allow more efficient control and more appropriate description of the real processes, taking account of the memory and hereditary properties of different substances. For examples, see: Caputo [6], Caputo and Mainardi [7], Rabotnov [48], Bagley and Calico [5], Podlubny [44], [46]; an extensive list of papers published the *FCAA* journal [25]; the number of such works nowadays is extremely increasing. These studies are based on model differential equations involving the *Fractional Calculus (FC)* operators (fractional derivatives, semi-integrals, integrals of fractional order), for a popular idea about this – see http://en.wikipedia.org/wiki/Fractional_calculus , and for detailed theory – the books [51], [46], [23], etc. In the classical FC, we deal with the *Riemann-Liouville (R-L) operator* of fractional (i.e. arbitrary, not necessarily integer) order $\delta \geq 0$ of integration (called also *R-L fractional integral*):

$$R^\delta f(z) = \frac{1}{\Gamma(\delta)} \int_0^z (z - \zeta)^{\delta - 1} f(\zeta) \, d\zeta \ , \quad R^0 f(z) = f(z). \tag{3}$$

This is an extension of the known formula for *n*-fold integration, when *n*! is replaced by a $\Gamma-$ function for $n \in \mathbb{N} \to \delta \in \mathbb{R}_+$. The *R-L fractional derivative* of order $\delta \geq 0$ is then defined as a composition of integer order derivative and fractional order integral (3), with a suitable $n \in \mathbb{N} : n - 1 < \delta < n,$

$$D^\delta f(z) := D^n \, R^{n-\delta} f(z) = (\frac{d}{dz})^n \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^z \frac{f(\zeta) \, d\zeta}{(z-\zeta)^{\delta-n+1}} \right\}. \tag{4}$$

Another, alternative definition of derivative of fractional order, that includes the initial conditions of integer order and happens to be more suitable in applications, has been introduced by Caputo [6], see also [7], [46], the so-called *Caputo derivative*:

$$D_*^\delta \, f(z) := R^{n-\delta} \, D^n \, f(z) = \frac{1}{\Gamma(n-\delta)} \int_0^z \frac{f^{(n)}(\zeta) \, d\zeta}{(z-\zeta)^{\delta-n+1}}. \tag{5}$$

In general, $D^n \, R^{n-\delta} \neq R^{n-\delta} \, D^n$ and therefore, the difference between the above two definitions is evident when the passage of the $n$-th derivative in (4) under the sign of integral is legitime. In this case,

$$D^\delta f(z) = D_*^\delta f(z) + \sum_{k=0}^{n-1} f^{(k)}(+0) \frac{z^{k\delta}}{\Gamma(k-\delta+1)} \, .$$

The R-L definition (4) is preferred in the mathematically oriented papers, but the Caputo one (5) allows consideration of easily interpreted conventional initial conditions, expressed in terms of integer order derivatives: $f(+0) = f_0, f'(+0) = f_1, \ldots,$ and its Laplace transform is given by (see [46]):

$$L\{D_*^\delta f(z);s\} = s^\delta \, L\{f(z);s\} - \sum_{k=0}^{n-1} f^{(k)}(+0) \, s^{n-k-1} \, , \quad n-1 < \delta < n \, , \, n \in \mathbb{N} \, .$$

Moreover, a Caputo derivative of a constant is 0 (*as in the classical calculus*!), while the R-L derivative of a constant does not vanish if $\delta$ is not integer, for example:

$$D^\delta \{1\} = \frac{z^{-\delta}}{\Gamma(1-\delta)}, \text{ but } D_*^\delta \{1\} = 0.$$

The fractional order transfer functions are powerful instrument for description of the memory and hereditary properties of substances and systems, since in the integer order models these effects are neglected. In fact, the fractional order systems happen to play the role of "reality" in the "fractal world", while the integer order "approximations" play the role of a simpler, but not so suitable model (see Podlubny, [44], [46]). The *fractional order transfer functions* of a controlled system can have the form

$$G_n(s) = \frac{1}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}} + \cdots + a_1 s^{\beta_1} + a_0 s^{\beta_0}} \, , \tag{6}$$

with arbitrary real exponents $\beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0 > 0$ and real coefficients. In the time-domain, they correspond to fractional order differential equations of the form

$$a_n D_*^{\beta_n} y(t) + a_{n-1} D_*^{\beta_{n-1}} y(t) + \cdots + a_1 D_*^{\beta_1} y(t) + a_0 D_*^{\beta_0} y(t) = u(t) \, , \tag{7}$$

where $y(t) = L^{-1}\{Y(s)\}$ is the output of the system and $u(t) = L^{-1}\{U(s)\}$ – the controller's output; $D_*^\delta$ is the Caputo derivative, as in (5).

Until the "*era of applicable fractional calculus*", fractional order systems have been used and studied only sporadically, mainly due to absence of suitable mathematical tools, or in fact – due to poor knowledge (both of pure mathematicians and applied scientists and engineers) on them. To deal successfully with fractional order systems and controllers, the problem to know the inverse Laplace transformation of functions like

$$\frac{s^{\beta-1}}{s^\beta + \lambda} \, , \, \frac{s^{\alpha-\beta}}{s^\beta + \lambda} \, , \, \cdots \, , \, G(s) = \frac{1}{a_n s^{\beta_n} + \cdots + a_0 s^{\beta_0}} \tag{8}$$

needed to have explicit analytical solution. However, in the handbooks containing tables of the Laplace transforms, such functions have not been presented as known images,

except for some very special cases when $\alpha, \beta = \frac{1}{2}, 1, 2,$ corresponding to the exponential and trigonometric functions, error functions, incomplete gamma functions, etc. Until recently, the M-L functions have been almost totally ignored therein, and even in the *AMS Subject Classification*, http://www.ams.org/msc/classification.pdf, were missing until the 2000-edition (now classified as 33E12).

Recently, a rapidly increasing number of authors have switched to considering systems and phenomena described by fractional-order state equations, involving the operators of fractional calculus (FC). The successful use of Laplace transform techniques and of the FC operators in such studies is based on the class of M-L type functions, as fractional-indices analogues of the exponential, trigonometric and error functions in the case of fractional order integro-differential equations modeling fractional order systems. In their terms, it is possible to find out explicit analytical expressions not only for the basic characteristics of some fractional-order control systems (like the corresponding systems' responses) and solutions for automatics problems, *but also for the solutions of mathematical models of problems in various other areas of applied sciences and engineering*, like: fractional viscoelastic materials, mechanics, diffusion and wave processes in porous media, transfer processes in fractals, quantum physics, free electrons motions, electrical circuits, electroanalytical chemistry, biology, mathematical economy, traffic regulation, temperature fields in oil strata, etc. Only for some of these applications, we mention the works of: Nigmatullin [36], [37]; Podlubny [46] (Chapters 4, 9, 10); Gorenflo and Mainardi [18]; Mainardi et al. [31], [32]; Hilfer [19]; Kilbas, Srivastava and Trujillo [23]; Nikolov's extended collections [38], [39]; Scherer, Kalla, Boyadjiev and Al-Saqabi [53], etc. That is why, the interest into M-L type functions from both analytical and numerical points of view, has become nowadays one of the powerful engines for development of the applied fractional calculus.

Let us consider only a few examples of fractional order differential equations concerned with some models from mechanics, as ultraslow and intermediate processes, and diffusion-wave phenomena. The fractional differential equations of order $\alpha > 0$:

$$\frac{d^{\alpha}u}{dt^{\alpha}} + u(t) = 0 \quad , \quad t > 0, \tag{9}$$

with initial value conditions of the form $u^{(k)}(+0) = c_k, \, k = 0, 1, 2, \ldots, m; \, m \in \mathbb{N}$:

$m - 1 < \alpha \leq m,$ are usually referred to as the *fractional relaxation* (if $0 < \alpha \leq 1$), or *fractional oscillation* (if $1 < \alpha \leq 2$) *equations*, see Mainardi [32]. For integer order $\alpha$, the IVPs for equation (9) can be solved by elementary functions ($\alpha$=1: eq. of relaxation, $u(t) = c_0 e^{-t}$; $\alpha$=2: eq. of oscillation, $u(t) = c_0 \cos t + c_1 \sin t$). In the fractional order cases, (9) has been investigated and solved, for example by Mainardi et al. (see details in [32]), showing the *key role of the Mittag-Leffler function*.

One of the *partial differential equations (PDE) of fractional order* that deserves special attention is obtained from the classical diffusion or wave equations by replacing the first–, or resp. the second–order time derivative by a fractional derivative of order $\alpha$ with $0 < \alpha < 2$. It has the form

$$D^{\alpha} u(z,t) = \lambda^2 \frac{\partial^2 u(z,t)}{\partial z^2}, \tag{10}$$

where $z$, $t$ denote the space-time variables (in one-dimensional space) and $u(z,t)$ is the response field variable. Equation (10) has been introduced in physics, with $0 < \alpha < 1$, by Nigmatullin [36],[37] to describe the diffusion process in media with fractal geometry, and for $1 < \alpha < 2$, by Mainardi (see [31]) to describe the propagation of mechanical diffusive waves in viscoelastic media which exhibit a power law creep. These types of equations, called as *fractional diffusion-wave equations*, have been treated by several authors by means of different approaches, but all of them finally leading to use of, what we call now, *special functions of fractional calculus*: either by Mellin transforms techniques and solutions in terms of *H*-functions of Fox, or by Laplace transform allowing to obtain for (10) the fundamental solution of the IVP as the so-called Green function. In this case it is represented by a special function called recently the *Mainardi function* and closely related to the *Wright function* $W$ from [56], [57] (see for example Podlubny: [46] and http://people.tuke.sk/igor.podlubny/USU/03_specfun.pdf):

$$M(z;\beta) = W(-z; -\beta, 1 - \beta), \quad \beta = \alpha/2.$$

As a fractional index generalization of the Bessel function, the Wright function $W(z; \mu, \nu + 1)$ is also often denoted by $J_{\nu}^{\mu}(-z)$ and called *Bessel-Maitland function* (misnamed after the second name of E. *Maitland* Wright). More details will be given in the last section.

The FC is as old as the classical calculus, and like many other mathematical ideas, has its origin in the striving for extension of meaning. In differential and integral calculus the question of extension of meaning was asked by l'Hospital: "What if $n$ be $\frac{1}{2}$ in $\frac{d^n y}{dx^n}$?" Leibnitz, in 1695 replied, "It will lead to a paradox, but one day from this apparent paradox useful consequences will be drawn..." Despite its old origin, FC had a rather controversial development. Although the honour of its first application belonged to Abel in 1823, for explicit solution of an integral equation (now called *Abel equation*), related to the practical isochrone (tautochrone) problem, for a long time FC was considered as an abstract play with symbols. It took 279 years from the l'Hospital – Leibniz correspondence, for a text to appear entirely devoted to the topic of FC and specialized conferences to start. Each of the Proceedings of the first three international conferences on FC (that took place in: Univ. of New Haven (USA) in 1974; Univ. of Strathclyde (Scotland, UK) in 1984; Nihon University (Tokyo, Japan) in 1989) was ended by a section of *open problems*. Therein, the following pessimistic *conjecture, originally stated by B. Ross* (initiator of the first conference) was continuously repeated:

*Conjecture.* Is there a geometrical representation of a fractional derivative? If not, can one prove that a graphical representation of a fractional derivative *does not* exist? ... The consensus of the experts ... is that there is, in general, *no* geometrical interpretation of a derivative of fractional order ... It can be asked, however at least for a geometrical meaning or a physical phenomena that can be represented by means of equations involving a derivative of a particular order such as ½ ... ?

The *Fractal Geometry* gave an answer for the geometrical meanings. Let us note that the first book on FC was written in 1974 by a chemist (K. Oldham) and applied mathematician (J. Spanier) and included applications of so-called *semi-derivatives* and *semi-integrals* (i.e. of order $\delta = 1/2$) to electrochemistry! Recently, along with other authors, Podlubny [47] contributed in some explicitly written form for *the failure of the above-said long-standing (more than 300-years old) conjecture*. Namely, he provided some interesting geometrical interpretations of fractional-order integrations, of Riesz and Feller potentials, as well as physical interpretations of the R-L integrals, Stijeltes integral, R-L and Caputo fractional derivatives, etc. The details of this paper, published in the *FCAA* journal and already cited hundreds of times by applied scientists, can be seen also online, at: http://people.tuke.sk/igor.podlubny/pspdf/pifcaa_r.pdf .

## 2. MITTAG-LEFFLER FUNCTIONS

The M-L functions seem to be unknown to the majority of applied scientists and mathematicians, even now. A description of their basic properties however appeared yet in vol.3 of the Bateman – Erdélyi Project [14], in a chapter devoted to "miscellaneous functions". They were introduced by Mittag-Leffler [35] and extended to two indices by Agarwal [4]. Now their definition and properties can be found in many recent books and surveys on fractional calculus, integral and differential equations, mechanics, control theory, etc. like: Podlubny [45], [46]; Gorenflo and Mainardi [18]; Kiryakova [24]; Kilbas, Srivastava and Trujillo [23]; etc., as seen for example, in the 10–years contents of the journal *"Fractional Calculus and Applied Analysis"* [25]. To repeat Gorenflo and Mainardi from [18], "the M-L function exited from its isolated life as *Cinderella of the SpecialFunctions*, to become the *Queen of Fractional Calculus"*!

The Mittag-Leffler functions $E_\alpha(z)$ ([35]) and $E_{\alpha,\beta}(z)$ ( [4]), defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \; \alpha > 0, \, \beta > 0, \quad (11)$$

are "fractional index" extension of the exponential and trigonometric (resp. hyperbolic) functions $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)}$, $\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(2k+1)}$, satisfying ODEs of the form $D^n y(\lambda z) = \lambda^n y(z), n = 1, 2, \ldots$ of 1st and 2nd (in general, integer) order. Indeed, in the case of (11) we have fractional order (FO) differential equations, for example like this:

$$D^\alpha y(z) = \lambda y(z) \quad \text{with} \quad y(z) = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha) \; , \; \alpha > 0 \, .$$

More complicated FO differential and integral equations solved by M-L functions can be seen as examples, in: Samko, Kilbas and Marichev [51]; Podlubny [46]; Kilbas, Srivastava and Trujillo [23]; Saqabi and Kiryakova [52]; and in series of other papers in the *FCAA* journal, [25].

Each Mittag-Leffler function $E_\alpha(z)$, $E_{\alpha,\beta}(z)$, $\alpha > 0$, is an entire function of order $\rho = \frac{1}{\alpha}$ and of type 1; in a sense, the simplest entire function of this order. In the limit case for $\alpha \to +0$, the analyticity in the whole complex plane is lost, since

$$E_0(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \ |z| < 1.$$ Many of the properties of the M-L functions follow from their integral representations, differential relations, including such in terms of fractional calculus operators, and their asymptotics, as

$$E_{\alpha,\beta}(-\lambda z^\alpha) \sim \begin{cases} \dfrac{1}{\Gamma(\beta)} - \lambda \dfrac{z^\alpha}{\Gamma(\beta - \alpha)}, & z \to +0, \\[2mm] \dfrac{1}{\lambda} \dfrac{z^{-\alpha+\beta-1}}{\Gamma(\beta - \alpha)}, & z \to \infty. \end{cases}$$

For $|z| \to \infty$ the asymptotic behavior of the M-L function in the complex plane is quite different, depending on the angle sectors and the values of the parameters; a fact analogous to the known Stokes phenomenon for the Airy functions. The fundamental role of the M-L function in FC and in solving problems by means of Laplace transform method, is due to the *general form of its Laplace transform image* (see [45], [46], [18]):

$$L\{z^{\beta-1}E_{\alpha,\beta}(-\lambda z^\alpha); s\} = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \quad \mathrm{Re}(s) > |\lambda|^{\frac{1}{\alpha}}. \tag{12}$$

As more general, the following *special functions of M-L type* are often used in FC, see Podlubny [44] – [46]:

$$\mathfrak{E}_k(z, \lambda; \alpha, \beta) := z^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(\lambda z^\alpha), \quad k = 0, 1, 2, \ldots \tag{13}$$

They satisfy more general fractional differential relations and have Laplace images, allowing to interpret, after suitable decomposition, almost all the rational functions of $s$ (for $\mathrm{Re}(s) > |\lambda|^{\frac{1}{\alpha}}$, $k = 0, 1, 2, \ldots$):

$$L\{\mathfrak{E}_k(z, \pm\lambda; \alpha, \beta); s\} = \int_0^\infty \exp(-st)\,\mathfrak{E}_k(z, \pm\lambda; \alpha, \beta)\,ds = \frac{k!\,s^{\alpha-\beta}}{(s^\alpha \mp \lambda)^{k+1}}. \tag{14}$$

Most of the basic properties of the M-L functions are contained, for example, in: Erdélyi et al. [14], vol. 3; Dzrbashjan [13]; Gorenflo and Mainardi [18]; Podlubny [44] – [46]; Kiryakova [24]; Kilbas, Srivastava and Trujillo [23]; etc. Earlier numerical results and plots of M-L functions for basic values of indices can be found, e.g. in Caputo and Mainardi [7], Gorenflo and Mainardi [18], and for some recent ones, see in the *FCAA* journal [25]. A generalization of the M-L function (11) with additional parameter $\rho$ has been considered by Kilbas et al. [21], [23] in the form $E_{\alpha,\beta}^\rho(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}$.

**Examples of M-L functions:**

- $\boxed{\alpha > 0, \beta = 1}$ : $E_{0,1}(z) = \dfrac{1}{1-z}$ ; $E_{1,1}(z) = \exp z$ ; $E_{2,1}(z^2) = \cosh z$ ,

$$E_{2,1}(-z^2) = \cos z \, , \, E_{1/2,1}(\sqrt{z}) = \exp z\,[1 + erf(\sqrt{z})] = \exp z\, erfc(-\sqrt{z})$$

$$= \exp z\,[1 + \frac{1}{\sqrt{\pi}}\gamma(\frac{1}{2}, z)] \quad \text{(the error functions and the incomplete gamma function);}$$

- $\boxed{\beta \neq 1}$ : $E_{1,2}(z) = \dfrac{\exp z - 1}{z}$ , $E_{1/2,2}(z) = \dfrac{sh\sqrt{z}}{z}$ , $E_{2,2}(z) = \dfrac{sh\sqrt{z}}{\sqrt{z}}$ , $etc$ .

It is useful to discuss the place of the M-L functions among the other special functions of mathematical physics. Most of them (all the classical orthogonal polynomials, the Bessel type and cylindrical functions, the Gauss and the generalized hypergeometric functions $_pF_q$ , etc), together with all their special cases and the basic elementary functions, can be represented in terms of the $_pF_q$-functions and of the so-called *Meijer's G-functions* (see e.g. Erdélyi et al. [14], vol. 1). However, the M-L functions of irrational index $\alpha \neq p/q$ are examples of special functions that do not fall even in this general class of special functions. Thus, they require further generalizations like the Wright's generalized hypergeometric functions $_q\Psi_q$ and the *H*-functions of Fox:

$$E_{\alpha,\beta}(z) = {}_1\Psi_1 \left[ \begin{array}{c} (1,1) \\ (\alpha,\beta) \end{array} \Big| z \right] = H_{1,2}^{1,1} \left[ -z \Big| \begin{array}{c} (0,1) \\ (0,1),(1-\beta,\alpha) \end{array} \right] \quad . \tag{15}$$

We remind the following definitions of special functions. By a *Fox's H-function* we mean a *generalized hypergeometric function*, defined by means of the Mellin-Barnes type contour integral

$$H_{p,q}^{m,n} \left[ \sigma \Big| \begin{array}{c} (a_j, Aj) \\ (b_k, B_k) \end{array} \right] = \frac{1}{2\pi i} \int_C \frac{\prod_{k=1}^{m}\Gamma(b_k - B_k s)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod_{k=m+1}^{q}\Gamma(1 - b_k + B_k s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)}\sigma^s \, ds \, , \tag{16}$$

where $C$ is a suitably chosen contour in $\mathbb{C}$ (one of three possible types); the orders $(m,n,p,q)$ are integers $0 \leq m \leq q$, $0 \leq n \leq p$ and the parameters $a_j \in \mathbb{R}, A_j > 0$, $j = 1,\ldots,p; b_k \in \mathbb{R}, B_k > 0, j = 1,\ldots,q$ are such that $A_j(b_k + l) \neq B_k(a_j - l' - 1)$, $l, l' = 0, 1, 2, \ldots$ For the various type of contours and conditions for existence and analyticity of *H*-function (16) in disks $\{|z| < \rho\} \subset \mathbb{C}$ whose radii are $\rho = \prod_{j=1}^{p} A_j^{-A_j} \prod_{k=1}^{q} B_k^{-B_k} > 0$, one can see the books [49], [22], [23], [24], etc. For $A_1 = \ldots = A_p = 1$, $B_1 = \ldots = B_q = 1$, the *H*-function (16) is reduced to the simpler *G-function of Meijer*, which can be seen also in Erdélyi et al. [14], vol. 1, Ch. 5. The *Wright generalized hypergeometric functions* ([56], [57]; see also [49], [22], [23], [24]):

$$
{}_p\Psi_q\left[\begin{matrix}(a_1,A_1),...,(a_p,A_p)\\(b_1,B_1),...,(b_q,B_q)\end{matrix}\Big|\sigma\right]=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)...\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)...\Gamma(b_q+kB_q)}\frac{z^k}{k!}
$$

$$
=H_{p,q+1}^{1,p}\left[-\sigma\,\Big|\begin{matrix}(1-a_1,A_1),...,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),...,(1-b_q,B_q)\end{matrix}\right],
$$

(17)

*together with the M-L function* and its *generalizations considered in the next sections,* will be termed as *special functions of fractional calculus*. These are examples of special functions, which in contrary to the known special functions of mathematical physics, are *not represented as ${}_pF_q$ - functions and Meijer G-functions*:

$$
{}_pF_q(a_1,...,a_p;b_1,...,b_q;\sigma)=\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k}\frac{z^k}{k!}
$$

$$
=\left[\frac{\Pi_{k=1}^{q}\Gamma(b_k)}{\Pi_{j=1}^{p}\Gamma(a_j)}\right]{}_p\Psi_q\left[\begin{matrix}(a_1,1),...,(a_p,1)\\(b_1,1),...,(b_q,1)\end{matrix}\Big|\sigma\right]
$$

(18)

$$
=\left[\frac{\Pi_{k=1}^{q}\Gamma(b_k)}{\Pi_{j=1}^{p}\Gamma(a_j)}\right]G_{p,q+1}^{1,p}\left[-\sigma\,\Big|\begin{matrix}1-a_1,...,1-a_p\\0,1-b_1,...,1-b_q\end{matrix}\right],
$$

and do not satisfy differential equations of mathematical physics of *integer* order.

In [27] we proposed a *general procedure for introducing classes of special functions by means of generalized fractional calculus operators of some basic elementary functions.* As simplest example, most of the classical orthogonal polynomials can be defined by means of *Rodrigues type formulas* including integer order differentiations. The idea to extend them by means of fractional order differentiations and so to obtain "fractional index" classes of special functions has already a long history. Recently, it has an extension in the papers of Boyadjiev et al. [17] and [20], where *fractional Jacobi polynomials, fractional Gauss functions and Caputo-type fractional Laguerre functions* have been considered. We like to mention also the *functions introduced by Virchenko* [55], as other typical cases of special functions of fractional calculus of the form (17), recently studied towards applications by many other authors.

### 3. MULTI-INDEX MITTAG-LEFFLER FUNCTIONS

We give here a brief survey on a new class of special functions, introduced in our recent papers [28],[29], closely related to the *generalized fractional calculus* (GFC, Kiryakova [24]). They are applicable for operational calculus techniques for some operators of the GFC, as well as for solving differential and integral equations of fractional multi-orders, as in Kiryakova et al. [3],[2]. These functions generalize simultaneously the M-L functions – with respect to number of indices (we replace the indices $\alpha := 1/\rho$, $\beta := \mu$, by 2 sets of indices $(1/\rho_1,...,1/\rho_m),(\mu_1,...,\mu_m)$ ), and the *hyper-Bessel functions of*

*Delerue* [8] (see also Kiryakova [24], Ch.3, Ch.4; [27]) – with respect to fractional indices (instead of the same number *m* of integer ones).

Namely, let $m > 1$ be an integer, and $\rho_1, ..., \rho_m > 0$ and $\mu_1, ..., \mu_m$ be arbitrary real parameters. By means of these "multi-indices", the *multi-index Mittag-Leffler functions (multi-M-L f.)* are defined as follows:

$$E_{(1/\rho_i),(\mu_i)}(z) = E_{(1/\rho_i),(\mu_i)}^{(m)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + {}^k\!/\!_{\rho_1})...\Gamma(\mu_m + {}^k\!/\!_{\rho_m})}, \qquad (19)$$

the upper index (*m*) usually omitted, unless we like to specify the multiplicity *m*.

In [29] it is proved that (19) are entire functions of order $\rho$ and type $\sigma$, with:

$$^1\!/\!_{\rho} := {}^1\!/\!_{\rho_1} + ... + {}^1\!/\!_{\rho_m}, \quad \sigma = ({}^{\rho_1}\!/\!_{\rho})^{\rho/\rho_1}...({}^{\rho_m}\!/\!_{\rho})^{\rho/\rho_m} > 1,$$

(note that only for *m=1*, the classical M-L functions have type $\sigma = 1$), from where also an asymptotic estimate follows for every positive $\varepsilon$ and $r_0(\varepsilon)$ sufficiently large:

$$| E_{(1/\rho_i),(\mu_i)}(z) |< \exp\left((\sigma + \varepsilon) | z |^{\rho}\right), \quad | z | \geq r_0(\varepsilon) > 0. \qquad (20)$$

A more exact asymptotics follows for large $| z |$ from the representation below of the multi-index M-L functions as $H$ - and ${}_q\Psi_q$ - functions, namely (see [3]):

$$| E_{(1/\rho_i),(\mu_i)}(z) |\leq M | z |^{\rho\left(\frac{1}{2} + \mu - \frac{m}{2}\right)} \exp\left(\sigma | z |^{\rho}\right), \quad | z | \to \infty.$$

Similarly to (15), the multi-index Mittag-Leffler functions (19) are *typical representatives of the fractional calculus' special functions:* the generalized Wright hypergeometric functions, and in general – the Fox *H*-functions, see Kiryakova [28],[29]:

$$E_{(1/\rho_i),(\mu_i)}(z) = {}_1\Psi_m\left(\begin{matrix} (1,1) \\ (\mu_i, {}^1\!/\!_{\rho_i})_1^m \end{matrix} \middle| z\right) = H_{1.m+1}^{1,1}\left[-z \middle| \begin{matrix} (0,1) \\ (0,1),(1-\mu_i, {}^1\!/\!_{\rho_i})_1^m \end{matrix}\right]. \quad (21)$$

This yields, from the known facts for the *H*-functions, a series of properties and rules how to operate and calculate with the multi-M-L functions, as well as the Mellin-Barnes type integral representation:

$$E_{(1/\rho_i),(\mu_i)}(z) = \frac{1}{2\pi i} \int_C \frac{\Gamma(s)\Gamma(1-s)}{\prod_{k=1}^{m}\Gamma(\mu_k - {}^s\!/\!_{\rho_k})} (-z)^{-s} \, ds. \qquad (22)$$

In the last section we shall provide a long list of special functions of fractional calculus that happen to be *particular cases of the multi-M-L functions* (19).

Now we continue the exposition on this class of functions with stressing to their *relations to the operators of the generalized fractional calculus (GFC)* and to the Gelfond-Leontiev operators for generalized differentiation and integration.

## 4. GENERALIZED FRACTIONAL CALCULUS OPERATORS RELATED TO THE MULTI-INDEX MITTAG-LEFFLER FUNCTIONS

In the generalized fractional calculus (GFC, Kiryakova [24]) we introduce generalized operators of integration and differentiation of fractional order, based on compositions of finite number ($m>1$) of classical fractional integration and differentiation operators, but written in terms of single *integrals involving special functions as kernels*. Let $m > 1$ be an integer and $(\delta_1 \geq 0,...,\delta_m \geq 0),(\gamma_1,...,\gamma_m)$ be two sets of real parameters. Instead of the repeated integral representation for a commutable product of *Erdélyi-Kober (E-K) fractional integrals*

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} := \prod_{k=1}^{m} I_{\beta_k}^{\gamma_k,\delta_k}, \text{ with denotation } I_{\beta}^{\gamma,\delta} f(z) = \int_0^1 (1-\sigma)^{\delta-1}\sigma^\gamma f(z\sigma^{\frac{1}{\beta}})d\sigma, \qquad (23)$$

(for $\gamma = 0, \beta = 1$, $I_{\beta}^{\gamma,\delta}$ reduces to the R-L fractional integral (3)), we consider the generalized fractional integral of multi-order $(\delta_1 \geq 0,...,\delta_m \geq 0)$ defined by

$$\tilde{I}f(z) = I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) = \int_0^1 H_{m,m}^{m,0}\left[\sigma \mid \begin{matrix}(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})\end{matrix}\right] f(z\sigma)d\sigma, \qquad (24)$$

if $\delta_1 + \delta_2 + ... + \delta_m > 0$, and by the identity operator $\tilde{I}f(z) = f(z)$, if $\delta_1 = \delta_2 = ... = \delta_m = 0$. The kernel-function is a specially chosen Fox's H-function (16). The corresponding generalized fractional derivatives in GFC, [24] are defined analogously to the idea of the R-L fractional derivative (4), as $D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} := D_\eta I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}$ with integers $\eta_k \geq \delta_k, k = 1,...,m$, and differential operator $D_\eta$ being a polynomial of $z\frac{d}{dz}$, for the details of the theory one can see Kiryakova [26],[24], [27], etc.

In [24], Ch.2, we considered the so-called *Gelfond-Leontiev (G-L) operators* of generalized integration and differentiation (see [16]), generated by the classical M-L function (11). Then it happened that these operators are special cases of the E-K fractional integrals (as defined in (23)) and the E-K fractional derivatives $D_\beta^{\gamma,\delta}$, and the corresponding analogue of the Laplace transform was called as *Borel-Dzrbashjan transform*. By means of suitable convolution operation and inversion formulas, elements of a respective operational calculus for the E-K operators have been demonstrated.

Now, we consider G-L operators for generalized integration and differentiation, related to the multi-index M-L functions (19). Let *f(z)* be an analytic function in a disk $\Delta_R = \{|z| < R\}, R > 0$, and $\rho_1,...,\rho_m > 0$ and $\mu_1,...,\mu_m$ be arbitrary parameters (the indices of function (19)). The correspondences:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \rightarrow \quad \tilde{D}f(z) := D_{(\rho_i),(\mu_i)} f(z), \tilde{L}f(z) := L_{(\rho_i),(\mu_i)} f(z) \qquad (25)$$

$$\tilde{D} f(z) = \sum_{k=1}^{\infty} a_k \frac{\Gamma(\mu_1 + \frac{k}{\rho_1})...\Gamma(\mu_m + \frac{k}{\rho_m})}{\Gamma(\mu_1 + \frac{k-1}{\rho_1})...\Gamma(\mu_m + \frac{k-1}{\rho_m})} z^{k-1},$$

$$\tilde{L} f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu_1 + \frac{k}{\rho_1})...\Gamma(\mu_m + \frac{k}{\rho_m})}{\Gamma(\mu_1 + \frac{k+1}{\rho_1})...\Gamma(\mu_m + \frac{k+1}{\rho_m})} z^{k+1},$$

(26)

we call *multiple Dzrbashjan-Gelfond-Leontiev (multi-D-G-L) differentiations*, resp. *integrations*, or: *G-L type operators* of generalized differentiation and integration *generated by the multi-M-L functions* (19).

Operators (26) can be analytically continued outside the disks of convergence $\Delta_R$, in spaces of holomorphic functions in starlike domains, *as elements of the GFC*, namely we have shown (see [28],[29]) them to be generalized fractional integrals and derivatives in the sense of (24) (see Kiryakova [24]):

$$\tilde{L} f(z) = z I_{(\rho_i),m}^{(\mu_i - 1),(\frac{1}{\rho_i})} f(z) \ , \ \ \tilde{D} f(z) = z^{-1} D_{(\rho_i),m}^{(\mu_i - 1),(\frac{1}{\rho_i})} f(z) - \frac{f(0)}{z} \left[ \prod_{i=1}^{m} \frac{\Gamma(\mu_i)}{\Gamma(\mu_i - \frac{1}{\rho_i})} \right].$$

(27)

It is very important to note that the multi-index M-L function (19) satisfies the following *fractional order* (exactly said, "*multi-order*" $(\frac{1}{\rho_1},...,\frac{1}{\rho_m})$) *differential equation (FODE)*:

$$D_{(\rho_i),(\mu_i)} E_{(1/\rho_i),(\mu_i)} (\lambda z) = \lambda E_{(1/\rho_i),(\mu_i)} (\lambda z), \quad \lambda \neq 0,$$

(28)

where the D-G-L generalized fractional derivative $D_{(\rho_i),(\mu_i)}$ can be written also symbolically in terms of composed R-L fractional derivatives, in the form

$$D_{(\rho_i),(\mu_i)} := z^{-1} \prod_{k=1}^{m} \left( z^{1+(1-\mu_i)\rho_i} D_{z^{\rho_i}}^{\frac{1}{\rho_i}} z^{(\mu_i-1)\rho_i} \right) = z^{\alpha_0} D_{z^{\rho_1}}^{\frac{1}{\rho_1}} z^{\alpha_1} D_{z^{\rho_2}}^{\frac{1}{\rho_2}} ... z^{\alpha_{m-1}} D_{z^{\rho_m}}^{\frac{1}{\rho_m}} z^{\alpha_m}.$$

(29)

Looking on the operator $\tilde{L}$ in (26),(27) as a linear (integral) operator mapping a linear space into itself, and following the general scheme of Dimovski for convolutional calculi (started by his paper [9], see also Dimovski and Kiryakova [10],[11], Kiryakova [24]) we were able to construct a family of *convolution operations for* $\tilde{L}$, the simplest one of them having the representation (in terms of the GFC operators (24)):

$$(f \tilde{*} g)(z) := I_{(\rho_i),m}^{(2\mu_i-1),(-\mu_i)} (f \circ g)(z), \quad \text{where}$$

$$(f \circ g)(z) = \int_0^1 ... \int_0^1 \prod_{i=1}^{m} [t_i (1-t_i)]^{\mu_i-1} f \left[ z \prod_{i=1}^{m} t_i^{\frac{1}{\rho_i}} \right] g \left[ z \prod_{i=1}^{m} (1-t_i)^{\frac{1}{\rho_i}} \right] dt_1 ... dt_m.$$

(30)

In terms of this convolution operation, which is a linear, bilinear and commutative operation for which $\tilde{L}$ satisfies $\tilde{L}(f \tilde{*} g) = (\tilde{L} f) \tilde{*} g$, we have the following "*product*" of *two multi-M-L functions* (evident analogue of the convolution properties of the exponential and classical M-L functions!):

$$E_{(\frac{1}{\rho_i}),(\mu_i)}(\alpha z) \tilde{*} E_{(\frac{1}{\rho_i}),(\mu_i)}(\beta z) = \frac{\alpha E_{(\frac{1}{\rho_i}),(\mu_i)}(\alpha z) - \beta E_{(\frac{1}{\rho_i}),(\mu_i)}(\beta z)}{(\alpha - \beta) \prod\limits_{i=1}^{m} \Gamma(\mu_i)}, \ \alpha \neq \beta. \quad (31)$$

For the details and proofs, see Kiryakova et al. [29], [2] and [3].

## 5. LAPLACE TYPE INTEGRAL TRANSFORM

In [11] we have shown that the role of the Laplace transform for the D-G-L operators $D_{\rho,\mu}, L_{\rho,\mu}$ ($m = 1$) related to classical M-L function $E_{\frac{1}{\rho},\mu}(z)$ can be played by the *Borel-Dzrbashjan integral transform*: $B_{\rho,\mu}\left\{\sum\limits_{k=0}^{\infty} a_k z^k\right\} = \sum\limits_{k=0}^{\infty} \frac{a_k}{s^{k+1}} \Gamma(\mu + \frac{k}{\rho})$, in general:

$$B_{\rho,\mu}\{f(z); s\} = \rho s^{\mu\rho-1} \int\limits_0^\infty \exp(-s^\rho z^\rho) \, z^{\mu\rho-1} f(z) \, dz, \quad (32)$$

$$f(z) = \frac{1}{2\pi i} \int\limits_C E_{\frac{1}{\rho},\mu}(sz) \, B_{\rho,\mu}\{f; s\} ds.$$

In the case of the multi-index M-L functions, the new Laplace type integral transform is the *H*-transform (for the *H*-transforms, in general, one can see [22]) introduced and studied by Kiryakova et al. [28],[29], [3], with a *H*-function (16) as a kernel-function:

$$\tilde{B}(s) = B_{(\rho_i),(\mu_i)}\{f(z); s\} = \int\limits_0^\infty H_{0,m}^{m,0}\left[ sz \, \Big|_{(\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i})}^{-} \right] f(z) \, dz, \quad (33)$$

called *multiple Borel-Dzrbasjan (multi-B-D) transform*, corresponding to the D-G-L operators (25),(26). In case of functions analytic in a disk, (33) has the more visible form

$$f(z) = \sum\limits_{k=0}^{\infty} a_k z^k \ \rightarrow \ \tilde{B}(s) = \sum\limits_{k=0}^{\infty} \frac{a_k}{s^{k+1}} \prod\limits_{i=1}^{m} \Gamma(\mu_i + \frac{k}{\rho_i}).$$

It is interesting to list the following properties relating (32) to multi-M-l functions and multi D-G-L operators, analogous to the relations of the Laplace transform to the exponential function and to classical integration and differentiation operators:

$$B_{(\rho_i),(\mu_i)}\left\{E_{(1/\rho_i),(\mu_i)}(z); s\right\} = \frac{1}{s-1},$$

$$B_{(\rho_i),(\mu_i)}\left\{L_{(\rho_i),(\mu_i)}f(z); s\right\} = \frac{1}{s} B_{(\rho_i),(\mu_i)}\left\{f(z); s\right\} \text{ (integration law)},$$

$$B_{(\rho_i),(\mu_i)}\left\{D_{(\rho_i),(\mu_i)}f(z); s\right\} = s \, B_{(\rho_i),(\mu_i)}\left\{f(z); s\right\} - f(0) \prod\limits_{i=1}^{m} \Gamma(\mu_i) \text{ (diff. law)},$$

$$f(z) = \int\limits_C \frac{z^{-q}}{\prod_{i=1}^{m} \Gamma(\mu_i - q/\rho_i)} dq \left\{ \int\limits_0^\infty s^{-q} \tilde{B}(s) \, ds \right\} \text{ (complex inversion formula)}.$$

For the details and proofs, the space of transformable functions, images of some basic originals, a Post-Widder type real inversion formula, special cases, see in [28], [29], [3].

Let us note only one nontrivial special case, widely known recently as the *Obrechkoff integral transform* (Dimovski and Kiryakova [10]; Kiryakova [24], Ch.3.; Kiryakova [30]), represented both as a *G*-transform (with a Meijer's G-function as kernel) and as a generalized Laplace type transform:

$$\mathfrak{O}\left\{f(z);s\right\} = \beta s^{\beta} \int_0^{\infty} G_{0,m}^{m,0}\left[(sz)^{\beta} \,\Big|\, {{-} \atop (\gamma_i + 1 - {1}/{\beta_i})}\right] f(z)\,dz$$

$$= \beta \int_0^{\infty} z^{\beta(\gamma_m + 1) - 1}\, \mathrm{K}\left[(sz)^{\beta}\right] f(z)\,dz\,, \quad \text{with kernel-function} \tag{34}$$

$$\mathrm{K}(z) = \int_0^{\infty} \cdots \int_0^{\infty} \exp\left(-u_1 - u_2 - \ldots - u_{m-1} - \frac{z}{u_1 \ldots u_m}\right) \prod_{i=1}^{m} u_i^{\gamma_m - \gamma_i - 1}\, du_1 \ldots du_m\,.$$

This integral transform plays the role of the Laplace transform in the operational calculus for the so-called *hyper-Bessel differential operators* of arbitrary order *m>1*, considered first by Dimovski [9] and studied further by Dimovski and Kiryakova (see [10], [24], [30]) and representable in the alternative forms:

$$B = z^{\alpha_0}\frac{d}{dz}z^{\alpha_1}\frac{d}{dz}\ldots z^{\alpha_{m-1}}\frac{d}{dz}z^{\alpha_m} = z^{-\beta}\prod_{k=1}^{m}\left(z\frac{d}{dz} + \beta\gamma_k\right)$$

$$= z^{-\beta}\left(z^m\frac{d^m}{dz^m} + a_1 z^{m-1}\frac{d^{m-1}}{dz^{m-1}} + \ldots + a_{m-1}z\frac{d}{dz} + a_m\right), \quad 0 < z < \infty\,, \tag{35}$$

as natural extensions of the *2nd order Bessel differential operators,* of the *m-th order differentiation* and of many *linear singular differential operators* with variable coefficients, appearing often in problems of mathematical physics and engineering.

## 6. EXAMPLES OF MULTI-INDEX MITTAG-LEFFLER FUNCTIONS

- $\boxed{m = 1}$ : This is the *classical Mittag-Leffler function* $E_{{1}/{\rho},\mu}(z)$ with all its special cases (see Sect. 2, and in the other References).

  - The *Rabotnov function,* used by him (see e.g. [50]) in viscoelasticity to describe the hereditary properties of materials (inexplicitly applying fractional calculus' models and techniques) is a special case of the M-L function:

$$\mathfrak{Э}_{\alpha}(\beta, z) = z^{\alpha}\sum_{k=0}^{\infty}\frac{\beta^k z^{k(\alpha+1)}}{\Gamma\big((k+1)(1+\alpha)\big)} = z^{\alpha}\,E_{\alpha+1,\alpha+1}(\beta z^{\alpha+1})\,, \tag{36}$$

see Podlubny [46], p.19. There, one can see also for similar functions called *fractional sine and cosine*, used by Plotnikov and Tseytlin, in solving BVPs of civil engineering.

- $\boxed{m = 2}$ : $E_{({1}/{\rho_1}, {1}/{\rho_2}),(\mu_1,\mu_2)}(z) := \Phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2) = \sum_{k=0}^{\infty}\dfrac{z^k}{\Gamma(\mu_1 + k/\rho_1)\Gamma(\mu_2 + k/\rho_2)}\,. \tag{37}$

This is the *Dzrbashjan function*, introduced and investigated in [12] in whole details, analogous to the known facts for the classical M-L function. Dzrbashjan used this study to develop a theory of integral transforms in the class $L_2$. There, he mentioned also as special cases of this M-L type function the following (incl. *M-L* and *Bessel functions*):

$$\Phi_{\infty,\infty}(z;1,1) = \frac{1}{1-z} ; \Phi_{\rho,\infty}(z;\mu,1) = E_{1/\rho,\mu}(z); \left(\frac{z}{2}\right)^{\nu} \Phi_{1,1}(-\frac{z^2}{4};1,\nu+1) = J_{\nu}(z). \quad (38)$$

It happens that we can enlarge substantially this list of known special functions, obtained from the case *m=2* of the multi-M-L functions:

● The functions usually called *Bessel-Maitland functions* (when denoted as $J_{\nu}^{r}(z)$) or *Wright functions* (when denoted as $W(z;\alpha,\beta)$), are typical examples of special functions of fractional calculus:

$$J_{\nu}^{r}(z) = W(-z; 1/\rho, \nu+1) = {}_{0}\Psi_{1}\left[\begin{array}{c} - \\ (\nu+1,r) \end{array} \bigg| -z \right] = H_{0,2}^{1,0}\left[ z \bigg| \begin{array}{c} - \\ (0,1),(-\nu,r) \end{array} \right] = E_{(r,1),(\nu+1,1)}^{(2)}(-z),$$

$$J_{\nu}^{r}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(\nu+rk+1)}, \quad W(z;\alpha,\beta) = \sum_{k=0}^{\infty} \frac{(z)^k}{k!\Gamma(\alpha k+\beta)}, \quad r = \alpha > -1, \nu+1 = \beta > 0. \quad (39)$$

Entire functions (39) were introduced by E.M. Wright in [56] and originally considered for $\alpha > 0$ in the framework of the asymptotic theory of partitions, and only later also for $-1 < \alpha < 0$, in [57]. Their definitions and properties could be seen, for example, in: Marichev [33], Prudnikov, Brychkov and Marichev [49], Kiryakova [24], Podlubny [46], Kilbas, Srivastava and Trujillo [23], etc. Wright's function has been used in the scheme of Laplace transform/operational calculus, by Mikusinski [34] and for integral transforms of Hankel type involving (39) instead of the Bessel functions, by Stankovic [54], Gajic and Stankovic [15].

● In a special case, Wright's function (39) has become recently popular as the *Mainardi function*, used in a series of his papers (as [31], [32], [18]) as an auxiliary function, to represent the Green function in solving IVPs for the fractional diffusion-wave equations (as already mentioned in Sect.1):

$$M(z;\alpha) := W(-z;-\alpha,1-\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\Gamma(-\alpha k+1-\alpha)}, \quad \text{with some examples as:}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (40)$$

$$M(z; 1/2) = \frac{1}{\sqrt{\pi}}\exp(-\frac{z^2}{4}) , \quad M(z; 1/3) = 3^{2/3} Ai(\frac{z}{3^{1/3}}) \quad \text{(the } \textit{Airy function)}, \text{ etc.}$$

Let us point out an *interesting relation between the Wright function* (multi-M-L f. with *m=2*) *and the M-L function* (multi-M-L f. with *m=1*): namely, the Laplace transform of the Wright function is expressed in terms of the M-L function (see [14], vol. 3, # 18.2; [46], p.39; [23], p.55):

$$L\{W(z;\alpha,\beta);s\} = \frac{1}{s} E_{\alpha,\beta}(\frac{1}{s}). \qquad\qquad (41)$$

Later, we shall comment an analogue of (41) in the general case *m>1*.

● The *generalized Bessel-Maitland function*, introduced by Pathak [43] is defined by an additional parameter $\lambda \in \mathbb{C}$ :

$$J^r_{\nu,\lambda}(z) = \left(z/2\right)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k \, (z/2)^{2k}}{\Gamma(\nu+rk+\lambda+1)\Gamma(\lambda+k+1)}, \tag{42}$$

for details see Marichev [33], Prudnikov, Brychkov and Marichev [49], Kiryakova [24]. This is a multi-M-L function with $m = 2$; $\rho_1 = 1/r, \rho_2 = 1; \mu_1 = \nu+\lambda+1, \mu_2 = \lambda+1$:

$$J^r_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda} \, {}_1\Psi_2 \left[ \begin{array}{c} (1,1) \\ (\nu+\lambda+1,r),(\lambda+1) \end{array} \middle| -\frac{z^2}{4} \right] = (z/2)^{\nu+2\lambda} \, E^{(2)}_{(r,1),(\nu+\lambda+1)}\left(-\frac{z^2}{4}\right). \tag{43}$$

- Let $\boxed{r=1}$ in (42), then (see [33], [24], Erdélyi et al. [14], vol.2):

$$J^1_{\nu,\lambda}(z) = \frac{2^{2-2\lambda-\nu}}{\Gamma(\lambda)\Gamma(\lambda+\nu)} \, s_{2\lambda+\nu-1,\nu}(z), \quad \text{the } \textit{Lommel function:}$$

$$s_{\alpha,\nu}(z) = \frac{z^{\alpha+1}}{(\alpha-\nu+1)(\alpha+\nu+1)} \, {}_1F_2\left(1; \frac{\alpha-\nu+3}{2}, \frac{\alpha+\nu+3}{2}; -\frac{z^2}{4}\right). \tag{44}$$

- If additionally, $\boxed{\lambda=1/2}$, (44) and therefore, (42)-(43), becomes the *Struve function* (see Erdélyi et al. [14], vol.2; [33], [28], [29]): $H_n(z) = \dfrac{2^{1-\nu}}{\sqrt{\pi} \, \Gamma(\nu+1/2)} \, s_{\nu,\nu}(z)$.

- The *generalized Lommel-Wright functions*, see de Oteiza, Kalla and Conde [40], Prieto, de Romero and Srivastava [48], Paneva-Konovska [42], are defined by

$$J^{r,n}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k \, (z/2)^k}{\left(\Gamma(\lambda+k+1)\right)^n \, \Gamma(\nu+rk+\lambda+1)}$$

$$= (z/2)^{\nu+2\lambda} \, {}_1\Psi_{n+1} \left[ \begin{array}{c} (1,1) \\ (\lambda+1,1),...,(\lambda+1,1), \, (\nu+\lambda+1,r) \end{array} \middle| -\frac{z^2}{4} \right] \tag{45}$$

$$= (z/2)^{\nu+2\lambda} \, E^{(2)}_{(1,1,...,1,r),(\lambda+1,\lambda+1,...,\lambda+1,\nu+\lambda+1)}\left(-\frac{z^2}{4}\right), \quad r>0, \; n\in\mathbb{N}, \; \nu,\lambda\in\mathbb{C},$$

and as seen, are also multi-index Mittag-Leffler functions with *m=2*.

Applications of functions (45) to fractional calculus and to problems of applied science were commented in [40],[48]. Recently, Paneva-Konovska [41],[42] extended her results concerning series in Bessel functions, to prove some theorems on the convergence of series in the special functions of fractional calculus of the form (39), (42) and (45), as analogues of the classical Cauchy-Hadamard, Abel and Tauber theorems for power series.

- For arbitrary $\boxed{m>2}$:

  - Let $\forall \rho_i = \infty$ (i.e.$1/\rho_i = 0$) and $\forall \mu_i = 1$, $i = 1,...,m$. Then from definition (19),

$$E^{(m)}_{(0,0,...,0),(1,1,...,1)}(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

  - Consider the case $m \geq 2$ with $\forall \rho_i = 1$, $i = 1,...,m$: then

$$E^{(m)}_{(1,1,\dots,1),(\mu_i+1)}(z) =_1 \Psi_m \left[ \begin{matrix} (1,1) \\ (\mu_i,1) \end{matrix} \,\Big|\, z \right] = \frac{1}{\prod_{k=1}^{m} \Gamma(\mu_i)} \,_1F_m(1;\mu_1,\dots,\mu_m;z)$$

reduces to $_1F_m$ – and to a *Meijer's* $G^{1,1}_{1,m+1}$ – *function.* Denote $\mu_i = \gamma_i + 1, i = 1,\dots,m$ and let additionally one of $\mu_i$ be equal to 1, for example, $\mu_m = 1$, i.e. $\gamma_m = 0$. Then the multi-index M-L functions become *hyper-Bessel functions of Delerue* [8]; see also Kiryakova [24], Ch. 3 and Ch. 4; [27]:

$$J^{(m-1)}_{\gamma_1,\gamma_2,\dots,\gamma_{m-1}}(z) = (z/m)^{\gamma_1+\dots+\gamma_{m-1}} \left[ \prod_{i=1}^{m=1} \Gamma(\gamma_k+1) \right]^{-1} \,_0F_{m-1}(\gamma_1+1,\dots,\gamma_{m-1}+1;-(z/m)^m)$$

$$= (z/m)^{\gamma_1+\dots+\gamma_{m-1}} \, E^{(m)}_{(1,1,\dots,1),(\gamma_1+1,\gamma_2+1,\dots,\gamma_{m-1}+1)} \left( -(z/m)^m \right). \tag{46}$$

Functions (46) are generalizing the classical Bessel functions $J_\nu$ with respect to number of indices, as $\nu \to (\gamma_1,\gamma_2,\dots,\gamma_{m-1})$, $m = 2 \to$ arbitrary $m > 2$. These functions, with a permutaion of indices, form a fundamental system of solutions of the hyper-Bessel ODEs of arbitrary order $m$, containg the hyper-Bessel differential operators (35). In view of the above relation, *the multi-index M-L functions can be seen as fractional-indices analogues of the hyper-Bessel functions* (46), related to the "fractional analogues" of the hyper-Bessel operators of the form (29).

In general, for rational values of $\forall \rho_i, i = 1,\dots,m$, the multi-M-L functions (19) are representable via $G$-functions of Meijer, otherwise these are typical $H$-functions related to generalized fractional calculus.

One more note is interesting, as a trial for extending relation (41). Let us take a *Laplace transform of an arbitrary multi-index M-L function.* According to the formula (2.5.16) from Kilbas and Saigo [21], p. 45, we get an *analogue of (41)*:

$$L\left\{ E^{(m)}_{(1/\rho_i),(\mu_i)}(z); s \right\} = L\left\{ H^{1,1}_{1,m+1}\left[ -z \,\Big|\, \begin{matrix} (0,1) \\ (0,1),(1-\mu_i,1/\rho_i)^m_1 \end{matrix} \right]; s \right\} = \dots$$

$$= \frac{1}{s} H^{1,1}_{1,m}\left[ \frac{1}{s} \,\Big|\, \begin{matrix} (0,1) \\ (1-\mu_i,1/\rho_i)^m_1 \end{matrix} \right] = \frac{1}{s} E^{(m-1)}_{(1/\rho_i),(\mu_i)}\left( \frac{1}{s} \right), \text{ if we take } \mu_m = \rho_m = 1. \tag{47}$$

Continuing in this way, by taking $\forall \mu_i = 1$, $\forall \rho_i = 1$, we shall finally reach to (41) and to special cases of (12),(13). On the other hand, the analogue (33) of the Laplace transform corresponding to multi-M-L functions (19) and respective operators (26),(27), is nothing but the simplest expression $B_{(\rho_i),(\mu_i)}\left\{ E_{(1/\rho_i),(\mu_i)}(z); s \right\} = \frac{1}{s-1}$, given in Sect. 5.

● And what about the multi-M-L functions, if we suppose $\forall \mu_i = 1$, $\forall \rho_i = 1$, as above? We have the special function

$$E^{(m)}_{(1,\dots,1),(1,\dots,1)}(z) =_1 F_m(1;1,\dots,1;z) =_0 F_{m-1}(-;1,\dots,1;z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^m}, \tag{48}$$

*analogous to a Bessel function of index* $\nu = 0$ *when* $m=2$ *in (46).* It is seen from Erdélyi et al. [14], vol.1, Ch.4, after skilled manipulations, that function (48) satisfies the singular

ODE of the simple form: $\left( z\dfrac{d}{dz} \right)^m y(z) = 0$, related to the "hyper-Bessel" differential

operator (35) of arbitrary order $m>1$, but of the simple form $B_m = \dfrac{1}{z}\left( z\dfrac{d}{dz} \right)^m$, $m>1$. For it, operational calculi (as particular to one done by Dimovski [9] in the general case) and Laplace type integral transforms (as later shown to be special cases of the earlier introduced Obrechkoff transform (34)) have been developed in the years 1960-1965 by several authors as Meller, Ditkin, Prudnikov and Botashev. Historical details can be seen in Dimovski and Kiryakova [10], Kiryakova [24], [30]. A fundamental system of solutions for the ODE with the operator $(z\dfrac{d}{dz})^m + z$, has been proposed in terms of the functions (48) by Adamchik and Marichev [1].

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# NEKE SPECIJANE FUNKCIJE KOJE SE ODNOSE NA RAČUN RAZLOMAKA I UPRAVLJAČKE SISTEME I JEDNAČINE RAZLOMLJENOG REDA

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Ovaj rad ima za cilj da privuče pažnju inženjera i naučnika koji se bave primenjenom matematikom, a koji su voljni da otkriju nove i korisne analitičke metode za rad sa realnim matematičkim modelima, od aplikacija u kojima se sreće račun razlomaka kao i M-L tip funkcija kao poseban tip FC funkcija, do urpavljačkog sistema razlomljenog reda i drugih matematičkih modela razlomljenog reda.) Predstavljamo generalizaciju multi indeksa M-L funkcija i diskutuje se o njihovim osobinama, odnosu prema generalizovanim FC kao i odgovorajaućem tipu Laplasove transformacije. Neočekivano dugačka lista primera je data, za matematičke specijalne funkcije (neke od njih – dobro poznate ) sa upotrebom na rešavanje problema koji prositiču iz primenjene matematike, uključujući i teoriju upravljanja.

Kljucne reci: *Mittlag-Leffler funkcije, posebne funkcije, račun razlomaka,diferencijalne jednačine i sistemi razlomljenog reda, teorija upravljanja, mašinstvo, upravljanje.*