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ON A JERK DYNAMICAL SYSTEM II

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This paper is dedicated to Professor Bratislav M. Danković

Abstract. *Chaotic systems of J.C. Sprott [11-16] based on electric circuits turn to be attractive examples of week chaos, a form of chaos that eventually might occur in such sensible applications like automatic control or robotics. This note contains some further considerations concerning some modifications of a 3D dynamic flow, known as jerk dynamical system of J.C. Sprott [15]. More specifically, the type of equilibrium point is analyzed and the Poincaré maps and bifurcation diagrams are constructed.*

Key words: *jerk dynamics, chaos, spiral saddle, Poincaré map, bifurcation*

1. INTRODUCTION

Trying to simplify famous Rössler's dynamical model [10] given in the form of 3D system of ODE's, Sprott [11] came up with fourteen algebraically distinct cases with six terms and a single nonlinearity and five cases with five terms and two nonlinearities. One of the consequences of simplicity is that majority of these systems can be written in the form of an explicit 3rd-order ordinary differential equation of the "jerky" form

$$\dddot{x} = J(\ddot{x},\ \dot{x},\ x), \tag{1}$$

after the mechanical term *jerk* - the third order time derivative of the displacement. Thus, in order to study different aspects of chaos, differential equation (1) can be considered instead of 3D system. Sprott's work inspired Gottlieb [2] to pose the question of finding the simplest jerk function that still generates chaos. Answering to this, 1997 Sprott [12] proposed chaotic jerk circuit containing just 5 terms and one quadratic nonlinearity

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$$J(\ddot{x},\ \dot{x},\ x) = -A\,\ddot{x} + \dot{x}^2 - x\,, \tag{2}$$

in which chaos occurs for damping parameter values $2.0168 < A < 2.0577$. This seems to be the simplest quadratic jerk function that is able to produce chaos for. The system has the greatest Lyapunov exponent $\lambda \cong 0.0551$, for $A = 2.017$, which implies maximal chaos in this "minimal chaotic system" ([15], [16]).

Trying to simplify (2), Sprott did many experiments (see [12], [15], [14], [13]), but equation (2) remains the simplest jerk equation with quadratic nonlinearity. In [16] Sprott and Linz have experimented with reducing the quadratic term $\dot{x}^2$ in (2) by $|\dot{x}|$, ending up with a modified jerk function

$$J(\ddot{x},\ \dot{x},\ x) = -A\,\ddot{x} + |\dot{x}| - x. \tag{3}$$

But, as it is shown in [16] this replacement leads to a non-chaotic system. For further results with Sprott type chaotic systems see [1], [4], [6]-[9].

Finally in the first part of this note [5], the authors investigate existence of chaotic trajectories in a jerk dynamical system that is placed somehow in between jerk systems given by (2) and (3) by setting

$$\dddot{x} = -A\,\ddot{x} + g_k(\dot{x}) - x, \tag{4}$$

and

$$\dddot{x} = -A\,\ddot{x} + h_k(\dot{x}) - x, \tag{5}$$

where the *left semi-quadratic* function g_k, and *right semi-quadratic* function h_k are defined as a compromise between nonlinear terms $\dot{x}^2$ and $|\dot{x}|$, namely

$$g_k(\xi) = \begin{cases} \xi^2, & \xi \le 0, \\ k\xi, & \xi > 0, \end{cases} \quad \text{or} \quad h_k(\xi) = \begin{cases} -k\xi, & \xi \le 0, \\ \xi^2, & \xi > 0, \end{cases} \tag{6}$$

where $k \ge 0$ is a real parameter.

The method of verifying existence of chaos was fourfold: 1^o Construction of trajectories of a given system in phase 3D space $(\dot{x},\ \ddot{x},\ \dddot{x})$; 2^o Visualization at least one of the variable's solution, say $x(t)$, for $0 \le t \le t_{max}$; 3^o Finding the discrete Fourier power spectra of $x(t)$; 4^o Evaluation of the first Lyapunov exponent λ_1, which in the case of chaos should be strictly greater than zero.

Table 1. Lyapunov λ_1 exponent for dynamical system (4)

A	k	λ_1	
0.90	1.50	0.0554254	**weak chaos**
1.10	1.25	0.0386954	
1.30	1.00	0.0338091	
	0.85	0.0421858	
	0.75	0.0286423	
1.32		0.0260195	
1.34		0.0173497	
1.36	0.75	0.0074672	**regular flow**
1.38		–0.0033521	
1.40		–0.0030971	
1.30	0.60	0.0011777	
	0.35	0.0018932	
	0.30	0.0015730	
	0.20	–0.0003643	
	0.10	0.0002048	
	0.00	0.0018932	

Experiment reveal existence of weak chaos in the system (4) with left semi-quadratic function g_k, while in (5) no chaotic regime was found. Related diagrams and data are given in in [5]. Table 1 displays the first Lyapunov exponent λ_1 for various values of damping parameter A and slope k of the linear part of the function g_k. It is noticeable that chaotic regime is preserved only for values of A, much bellow the "chaotic" range [2.0168, 2.0577] needed for original system (2). Also, smaller values A are compensated by higher values of slope. More precise, chaos exists for the family of pairs $(A, k) = (0.9 + 0.4\tau, 1.5 - 0.75\tau)$, $\tau \in [0, 1]$, which makes A and k run through the intervals $A \in [0.9, 1.34]$ and $k \in [0.75, 1.5]$, but with inverse monotonicity. This makes the largest Lyapunov exponent to run in the interval $0.0286423 < \lambda_1 < 0.0554254$ which is certain indicator of chaotic dynamics in (4). The values of $k < 0.6$ can produce only regular flow, no matter the value of A.

In addition to phase trajectory, time diagram, Fourier spectra and leading Lyapunov exponent, two more indicators are customary in identifying chaotic dynamics are Poincaré sections and bifurcation diagrams. Both will be discussed for system (4) in the sequel.

2. Jacobian Analysis

Putting equation (4) in the form given by (1) the jerk equation is obtained

$$\dddot{x} = J_{(A,k)}(\ddot{x},\ \dot{x},\ x) = -A\ddot{x} + g_k(\dot{x}) - x, \quad A > 0, \tag{7}$$

where g_k is left semi-quadratic function, defined by (6). Introducing usual substitutions $y(t) = \dot{x}(t)$ and $z(t) = \ddot{x}(t)$, (7) is recasted to the system of three coupled ODJ's

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & g_k(\cdot) & -A \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{8}$$

where $A > 0$. Solution of the system $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$, which is $y = 0$, $z = 0$, $-x + g_k(y) - Az = 0$, yields the equilibrium point $y^* = 0$, $z^* = 0$ and $x^* = g_k(0) = 0$. The nature of the equilibrium point is characterized by the eigenvalues of the Jacobian matrix $\mathbf{J} = (x, y, z)$ associated with (8)

$$\mathbf{J} = (x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \dfrac{dg_k(y)}{dy} & -A \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2y & -A \end{pmatrix}, & y < 0, \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & k & -A \end{pmatrix}, & y > 0. \end{cases}$$

An obvious consequence is that eigenvalues of Jacobian gravely depends on the sign of $y(t) = \dot{x}(t)$ in the neighborhood of equilibrium point. More precisely, the characteristic equation approaches to

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & 0 & -A-\lambda \end{vmatrix} = \lambda^3 + A\lambda^2 + 1 = 0 \quad \text{when} \quad y \to 0_- \tag{9}$$

and

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & k & -A-\lambda \end{vmatrix} = \lambda^3 + A\lambda^2 - k\lambda + 1 = 0 \quad \text{when} \quad y \to 0_+ . \tag{10}$$

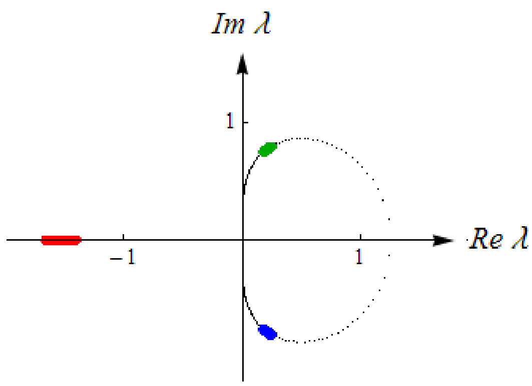

Fig. 1 The loci of eigenvalues of the Jacobian in equilibrium in complex plane for $0.9 \le A \le 1.3$ and $0.75 \le k \le 1.5$ for $y \to 0_-$

The solutions $\lambda_1(A, k)$, $\lambda_2(A, k)$ and $\lambda_3(A, k)$ of (10) in the neighborhood of the equilibrium point contain the solutions of (9) as its subset (for $k = 0$). Since pretty cumbersome, these exact expressions will not be presented here.

What is of interest is graphical insight into positions of the clusters of points for values $A = 0.9 + 0.4\tau$, $k = 1.5 - 0.75\tau$, $\tau \in [0, 1]$. Fig. 1 shows positions of these clusters as thick spots in complex plane while dotted line represents the extension of (A, k) trajectory when τ exceeds the unit interval.

The same conclusion refers to Fig. 2, where the clusters, although configured a little bit differently (following different dotted trajectories) repeat the same pattern.

According to this graphical analysis, it can be recognized that, as in the case of the Sprott original system (2), the point (0, 0, 0) of phase space is spiral saddle with index 2 (see [5]). Since, index as usually, indicates the dimension of unstable manifold (outset) we face here with a radial flow toward the equilibrium point along the inset line and then, after passing equilibrium point, an outward spiral in the outset plane.

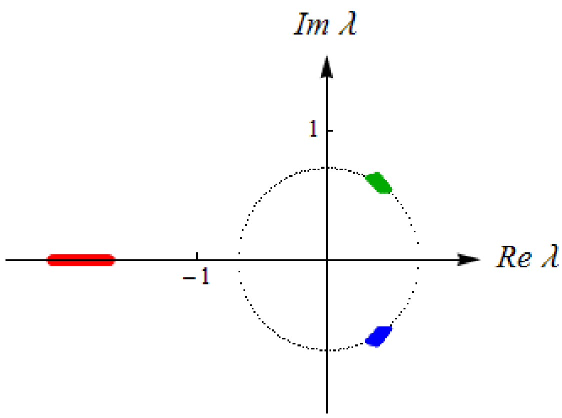

Fig. 2 The loci of eigenvalues of the Jacobian in equilibrium in complex plane for $0.9 \le A \le 1.3$ and $0.75 \le k \le 1.5$ for $y \to 0_+$

3. Poincaré Maps

In the case of autonomous system of dimension n, the *Poincaré map* is the intersection of its phase trajectory and a hyperplane Σ of dimension $n - 1$ (called *Poincaré plane or section*) that is transversal to this trajectory. The system (4) is autonomous dissipative flow with $n = 3$. Its phase trajectory $P(t)$ is the mapping

$$P : t \mapsto (x(t),\ y(t),\ z(t)),\ 0 \le t \le t_{\max}\ ,$$

where one of its components, say z(t) change its sign, as the graph in Fig. 3 (upper, left) shows. So, it is reasonable to take the plane Σ: $z = 0$ as Poincaré plane. The Poincaré map π is the set of intersection points $\pi = P(t) \cap \Sigma$, distributed in time in strictly increasing, infinite sequence of sample "moments" t_1, t_2, t_3,…. Thereby, in the case of periodic orbit, it consists of isolated points $\pi = \{(x(t_i), y(t_i)), i = 1, 2, 3,\ldots\}$ while when dynamical regime slides to chaos, π are represented by a continuous subset of Σ. Since the analytic methods for finding π are very complicated, a numerical calculation is an acceptable alternative. π– sets for the system (4) with A = 1.3 and k = 0.7 are calculated in 400, 1200 and 5000 points $\{t_i\}$. They are shown in Fig. 3. Increasing the number of sampling points makes π to approach to a continuous set, which indicates chaos.

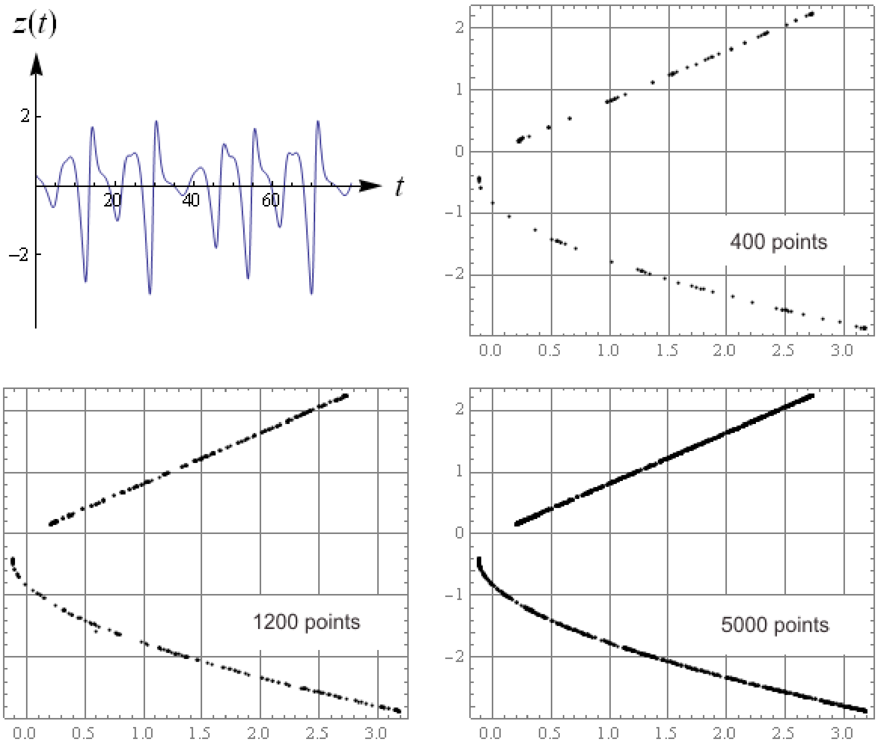

Fig. 3 Poincaré map for system (4) for A = 1.3 and k = 0.7

4. BIFURCATIONS

We conclude this investigation of modified Sprott jerk dynamical system (4) by constructing bifurcation diagrams using the value of the slope k of the left part of the semi-parabolic function (6) as bifurcation parameter. As expected, the diagrams confirm once again that we have characteristic chaotic behavior of period doubling. For fixed parameter A (1.25, 1.3 and 1.35) increasing the slope k from 0 to 2, leads to cascade of bifurcations with characteristic non-chaotic windows, and many smaller bifurcations all around. All three diagrams are presented in Figures 4, 5 and 6.

Numerical method for ODE solving used for creating diagrams in Fig. 3, Fig. 4, Fig. 5 and Fig. 6 are implicit Runge-Kutta with max. step 0.001. The initial conditions are set to $x_0 = -0.1$, $y_0 = 0.1$ and $z_0 = 0.3$. The bifurcation diagrams are calculated in 2000 points for $k \in [0, 2]$. All programs are implemented in Mathematica 7.0.

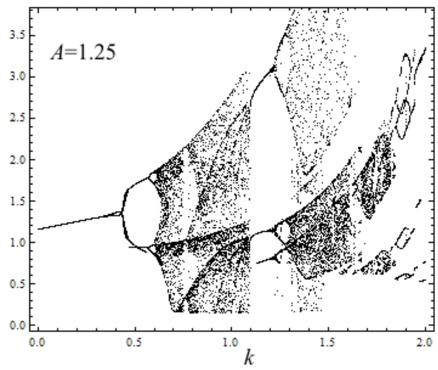

Fig. 4 The bifurcation diagram for dynamical system (4) with $A = 1.25$

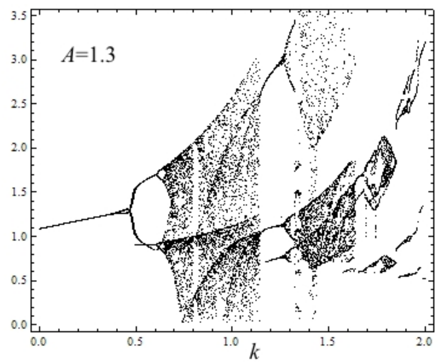

Fig. 5 The bifurcation diagram for dynamical system (4) with $A = 1.3$

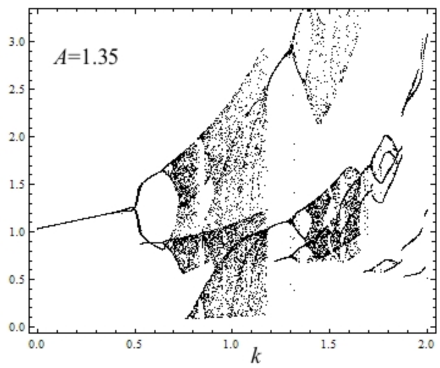

Fig. 6 The bifurcation diagram for dynamical system (4) with $A = 1.35$

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O JEDNOM "JERK" DINAMIČKOM SISTEMU II

Sonja Gegovska-Zajkova, Ljubiša M. Kocić

Haotični sistemi koje je najpre u obliku elektronskih kola uveo J.C. Sprott [11-16] pokazali su se kao zanimljivi primeri izvora slabog haosa – oblika haosa koji je jedino dozvoljen u osetljivim sistemima kao što su sistemi automatske kontrole i robotike. Ovaj rad je nastavak istraživanja jedne modifikacije 3D dinamičkog toka, koji je poznat kao jerk-dinamički sistem J.C. Sprott-a [15]. Konkretno, analiziran je tip tačke ekvilibrijuma, Poincaré-ovi preseci kao i bifurkacioni dijagrami.

Ključne reči: *"jerk"dinamoka, haos, spiralno sedlo, Poincaré-ov presek, bifurkacija*