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Address: Univerzitetski trg 2, 18000 Niš, YU, Tel: +381 18 547-095, Fax: +381 18 547-950
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FORMS OF HAMILTON'S PRINCIPLE FOR NONHOLONOMIC SYSTEMS

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Valentin Vitalievich Rumyantsev

Academician, Computing Center of RAS, Russia, Moscow, 117967, ul. Vavilova, 40

Abstract. *The conditions under which the three forms of Hamilon's principle were derived for nonholonomic systems with linear constraints by Hölder, Voronets and Suslov are analysed in the general case of nonlinear constraints. It is proved, that these three forms are equivalent and transformable to each other.*
The analogous questions are analysed for the case of nonlinear quasi-coordinates and quasi-velocities. In addition the forms of Hölder, Voronets and Suslov are excibited in the case of Legendre transformation reducing the motion's equations to canonical form in quasi-coordinates. Also the conditions under which Hamilton's principle for nonholonomic systems has the characterictics of the principle of stationary action are derived.

The conditions under which the three forms of Hamilon's principle were derived for nonholonomic systems with linear constraints by Hölder [1], Voronets [2] and Suslov [3] are analysed in the general case of nonlinear constraints. It is proved, that these three forms are equivalent and transformable to each other.

The analogous questions are analysed for the case of nonlinear quasi-coordinates and quasi-velocities. In addition the forms of Hölder, Voronets and Suslov are exhibited in the case of Legendre transformation reducing the motion's equations to canonical form in quasi-coordinates. Also the conditions under which Hamilton's principle for nonholonomic systems has the characterictics of the principle of stationary action are derived.

It was also shown, that the same conditions are the necessary and sufficient ones for applying generalized Hamilton – Jacobi method for integration of motion's equations for nonholonomic systems.

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1. THE TRANSITIVITY EQUATIONS

1.1. Lagrangian coordinates and velocities

Let us consider a nonholonomic system with k degrees of freedom, whose Lagrangian coordinates and velocities are q_i, $\dot{q}_i$ ($i = 1,...,n$). The system is subjected to forces, defined by the force function $U(q_i,t)$, and constrained by ideal nonintegrable relationships

$$f_l(q_i, \dot{q}_i, t) = 0, \quad l = 1,...,r < n, \quad rank\left\|\frac{\partial f_l}{\partial q_i}\right\| = r \tag{1.1}$$

which are generally nonlinear with respect to $\dot{q}_i \equiv \frac{dq_i}{dt}$, where t denotes time.

Equations (1.1) can be solved to some r dependent velocities and represented in the form

$$f_l(q_i, \dot{q}_i, t) = \dot{q}_{k+l}, \ -\varphi_l(q, \dot{q}_1,...,\dot{q}_k, t) = 0, \tag{1.2}$$

where the velocities $\dot{q}_s$ ($s = 1,...,k$, $k = n - r$) are assumed independent.

The basic principle of mechanics is the variational principle of d'Alembert-Lagrange

$$\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i = 0 \tag{1.3}$$

where $L(q, \dot{q}, t) = T + U$ is the Lagrange function, $T(q, \dot{q}, t)$ is kinetic energy, δq_i are virtual displacements that satisfy Chetaev's conditions

$$\frac{\partial f_l}{\partial \dot{q}_i}\delta q_i = 0, \quad l = 1,...,r, \quad i = 1,...,n \tag{1.4}$$

Throughout the paper we assume the summation condition over repeated indexes.

For constraints in the form (1.2) the conditions (1.4) are

$$\delta q_{k+l} = \frac{\partial \varphi_l}{\partial \dot{q}_s}\delta q_s, \quad l = 1,...,r \tag{1.5}$$

The Hamilton's principle can be obtained by integrating the equation (1.3) within some constant limits t_0 and t_1

$$\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i dt = 0$$

on the assumption that the function $\delta q_i \in C^2$ satisfy the conditions: $\delta q_i = 0$ for $t = t_0$, $t_1 (i = 1,...,n)$.

This equation is reduced to one

$$\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i\right)dt = 0 \tag{1.6}$$

in which the time derivatives $d\delta q_i/dt$ appear. Two equivalent points of view exist in analytic mechanics on the relation of these derivatives with variation of generalized velocities [4].

1) According to Hölder [1] the commutation relationships

$$\frac{d}{dt}\delta q_i = \delta \dot{q}_i, \quad i = 1,...,n, \tag{1.7}$$

are valid for all coordinates.

With this definition of $\delta\dot{q}_i$ the variation of function (1.1) over virtual displacements, with (1.4) taken into account, are of the form

$$\delta f_l = \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt}\frac{\partial f_l}{\partial \dot{q}_i}\right)\delta q_i, \quad l = 1,...,r \tag{1.8}$$

If Eqs. (1.1) are integrable, expressions (1.8) are identically zero, and if they are not integrable, then although not identically zero, they may become zero in the case of their nonlinearity on the strength of the motion equations [5]. Note, that the identities $\delta f_l \equiv 0$ ($l = 1,...,r$) and conditions (1.7) are compatible in the case of holonomic systems.

For relationships (1.2) formulas (1.8) assume the form

$$\delta f_l = \delta\dot{q}_{k+l} - \delta\varphi_l = A_s^{k+l}\delta q_s, \quad l = 1,...,r \tag{1.9}$$

where

$$A_s^{k+l} = \frac{d}{dt}\frac{\partial \varphi_l}{\partial \dot{q}_s} - \frac{\partial \varphi_l}{\partial q_s} - \frac{\partial \varphi_l}{\partial q_{k+\nu}}\frac{\partial \varphi_\nu}{\partial \dot{q}_s}; \quad l,\nu = 1,...,r \tag{1.10}$$

2) According to Appel and Suslov [3] the identities $\delta f_l \equiv 0$ ($l = 1,...,r$) are valid and this implies that formulas (1.7) are correct only for the independent velocities

$$\frac{d}{dt}\delta q_s = \delta\dot{q}_s, \quad s = 1,...,k \tag{1.11}$$

Expressions for the variations of dependent velocities $\dot{q}_{k+l}(l = 1,...,r)$, defined by Eqs. (1.2), are obtained from conditions $\delta f_l = 0$ in the form

$$\frac{d}{dt}\delta q_{k+l} - \bar{\delta}\dot{q}_{k+l} = A_s^{k+l}\delta q_s, \quad l = 1,...,r \tag{1.12}$$

where the symbol $\bar{\delta}$ denotes the variation in the Appel - Suslov sense.

Note that in the case of linear relationships (1.2) when

$$\varphi_l(q,\dot{q},t) = a_{ls}(q,t)\dot{q}_s + a_l(q,t), \quad l = 1,...,r, \quad s = 1,...,k \tag{1.13}$$

the coefficients in (1.10) are of the form [2]

$$A_s^{k+l} = \frac{da_{ls}}{dt} - \frac{\partial a_{li}}{\partial q_s}\dot{q}_i - \frac{\partial a_l}{\partial q_s} - a_{js}\left(\frac{\partial a_{li}}{\partial q_{k+j}}\dot{q}_i + \frac{\partial a_l}{\partial q_{k+j}}\right), \quad j,l = 1,...,r$$

and the right-hand of equality (1.12) can be represented in the form [3]

$$A_s^{k+l}\delta q_s = \dot{a}_{ls}\delta q_s - \dot{q}_s\delta a_{ls} - \delta a_l \,.$$

1.2. Quasi-coordinates and quasi-velocities

Hamel [6] has determined quasi-velocities for a holonomic system by equalities

$$\eta_i \equiv f_i(q,\dot{q},t), \ \det\left(\frac{\partial f_i}{\partial \dot{q}_j}\right) \neq 0, \ \ i,j = 1,...,n \tag{1.14}$$

where in general case $f_i(q,\dot{q},t)$ are nonlinear arbitrary functions. When the Eqs. (1.14) are solved

$$\dot{q}_i = F_i(q,\eta,t) \tag{1.15}$$

and (1.15) are inserted to (1.14), they satisfy them identically. Obviously

$$f_{si}F_{ir} = f_{ir}F_{si} = \delta_{sr} \ \text{- Kronecker symbol} \tag{1.16}$$

where

$$f_{si} \equiv \frac{\partial f_s}{\partial \dot{q}_i}, \ \ F_{ir} \equiv \frac{\partial F_i}{\partial \eta_r}, \ \ i,r,s = 1,...,n \ .$$

Quasi-coordinates π_i are determined by conditional notations $\dot{\pi}_i = \eta_i$ and moreover

$$\frac{\partial}{\partial \pi_s} \equiv F_{is}\frac{\partial}{\partial q_i}, \ \ \frac{\partial}{\partial q_i} \equiv f_{si}\frac{\partial}{\partial \pi_s} \ . \tag{1.17}$$

The virtual displacements in Lagrangian coordinates and quasi-coordinates satisfy the relationships

$$\delta q_i = F_{is}\delta\pi_s, \ \ \delta\pi_s = f_{si}\delta q_i \ . \tag{1.18}$$

Using the equalities (1.18) it is easy to transform the tratsitivity equations (1.7) to the forms

$$\frac{d\delta\pi_i}{dt} - \delta\eta_i = W_r^i\delta\pi_r \ \text{ or } \ \frac{d\delta\pi_i}{dt} - \delta\eta_i = -f_{ij}T_r^j\delta\pi_r, \ \ i,j,r = 1,...,n \tag{1.19}$$

Comparing the equations, we see that are valid equalities

$$W_r^i \equiv -f_{ij}T_r^j, \ \ i,j,r = 1,...,n \tag{1.20}$$

where we use the notations [5]

$$W_j^i \equiv F_{rj}\left(\frac{df_{ir}}{dt} - \frac{\partial f_i}{\partial q_r}\right), \ \ T_j^r \equiv \frac{dF_{rj}}{dt} - \frac{\partial F_r}{\partial \pi_j} \tag{1.21}$$

In the case of nonholonomic system with constraints (1.1) we pose last r of quasi-velocities (1.14) equal to left-hand side of (1.1): $n_\alpha = 0$ ($\alpha = k + 1,...,n$) at the same time the first k of (1.14) n_s ($s = 1,...,k$) are arbitrary.

The first k of both groups Eqs. (1.19) retain their form for nonholonomic system on condition that one has $\delta\pi_\alpha = 0$ ($\alpha = k + 1,...,n$) in their right-hand sides according to (1.4) while the remaining equations assume the form

$$\delta\eta_\alpha = -W_r^\alpha\delta\pi_r = f_{\alpha i}T_r^i\delta\pi_r \tag{1.22}$$

For the special form (1.2) we pose

$$\eta_\alpha \equiv \dot{q}_\alpha - \varphi_l(q, \dot{q}_1, ..., \dot{q}_k, t) = 0, \quad \eta_s = \dot{q}_s \tag{1.23}$$

In this case the Eqs. (1.22) turn into Eqs. (1.9), moreover $A_s^\alpha = W_s^\alpha$, and the another Eqs. (1.19) become identities.

2. THE FORMS OF HAMILTON'S PRINCIPLE IN GENERALIZED COORDINATES AND VELOCITIES

Let the relationships (1.7) be satisfied for all coordinates. Substituting (1.7) into (1.6) we obtain the Hölder form [1] of Hamilton's principle

$$\int_{t_0}^{t_1} \delta L dt = 0, \quad \delta q_i = 0 \text{ at } t = t_0, t_1 \tag{2.1}$$

The position of the system on real trajectory $q_i(t)$ is compared in (2.1) with simultaneous position obtained by moving from real motions position by virtual displacements δq_i which define a momentarily configuration. The sequence of displaced positions $q_i(t) + \delta q_i$ may be considered an roundabout path which generally does not satisfy the Eqs. (1.1). Indeed, if the roundabout path satisfies Eqs. (1.1), the equalities

$$f_l(q + \delta q, \dot{q} + \delta\dot{q}, t) = f_l(q, \dot{q}, t) + \frac{\partial f_l}{\partial q_i}\delta q_i + \frac{\partial f_l}{\partial \dot{q}_i}\delta\dot{q}_i + ... = 0$$

are correct; these equalities give $\delta f_l = 0$, that are accurate to smalls of the first order. But these conditions are not satisfied for nonholonomic system, hence Hamilton's principle (2.1) does not generally represent the principle of stationary action [7]

$$\delta\int_{t_0}^{t_1} L dt = 0\,, \quad \delta q_i = 0 \text{ at } t = t_0, t_1 \tag{2.2}$$

as in the case of holonomic systems.

The equations of motion for nonholonomic system are derived from (2.1), for example, in the form of Lagrange equations with factors μ_l

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \mu_l \frac{\partial f_l}{\partial \dot{q}_i}\,, \quad i = 1, ..., n, \tag{2.3}$$

which together with Eqs. (1.1) form a closed system of $n + r$ equations with the same number of unknowns. The generalized solution of these equations depends on $2n - r$ arbitrary constants.

If $\Theta(q, \dot{q}_1, ..., \dot{q}_k, t)$ denotes the kinetic energy $T(q, \dot{q}, t)$ from which the dependent velocities $\dot{q}$ are eliminated by means of formulas (1.2), there valid the relation [8]

$$\delta T = \delta\Theta + \frac{\partial T}{\partial \dot{q}_\alpha}(\delta\dot{q}_\alpha - \delta\varphi_l)\,, \quad \alpha = k + l, \quad l = 1, ..., r. \tag{2.4}$$

Substituting the right-hand side of (2.4) for δT into (2.1) we obtain the Voronets form of Hamilton's principle

$$\int_{t_0}^{t_1} [\delta(\Theta+U)+\frac{\partial T}{\partial \dot q_\alpha}(\delta\dot q_\alpha-\delta\varphi_l)]\,dt=0,\quad \delta q_i=0 \text{ at } t=t_0,t_1 \tag{2.5}$$

established by P. Voronets [2] in the case of linear constraints. The form (2.5) was neither substantiated nor named in [2].

The Voronets equations of motion for nonholonomic system are derived from (2.5)

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\dot q_s}-\frac{\partial(\Theta+U)}{\partial q_s}-\frac{\partial(\Theta+U)}{\partial q_\alpha}\frac{\partial\varphi_l}{\partial\dot q_s}-\frac{\partial T}{\partial\dot q_\alpha}A_s^\alpha=0,\ s=1,\dots,k,\ \alpha=k+1,\dots,n. \tag{2.6}$$

The general solution of Eqs. (2.6), (1.2), as well as of Eqs. (2.3), (1.1) depends on $2n-r$ arbitrary constants.

Now we substitute expressions (1.12) into (1.6) and obtain Hamilton's principle in Suslov's form [3]

$$\int_{t_0}^{t_1} [\bar\delta L+\frac{\partial T}{\partial\dot q_\alpha}A_s^\alpha\delta q_s]\,dt=0,\quad \delta q_i=0 \text{ at } t=t_0,t_1 \tag{2.7}$$

which was originally got by Suslov for the case of linear constraints (1.13) and called the modification of d'Alembert principle by him.

It is necessary to stress that the variations of Lagrange functions in (2.1) and (2.7) are calculated differently: allowance in (2.1) is made for equalities (1.7), but in (2.7) – for equalities (1.11) and (1.12). Note also that since in the last case the conditions $\delta f_l = 0$ are satisfied, the roundabout paths $q_i(t)+\delta q_i$ in (2.7) the conditions (1.2) satisfy in the first approximation. But (2.7), as well as (2.1), does not represent generally the principle of stationary action.

We point out that in conformity with Suslov's method of variation the formula (2.4) turns into $\bar\delta T=\delta\Theta$, the equality (2.7) assumes the form

$$\int_{t_0}^{t_1} [\delta(\Theta+U)+\frac{\partial T}{\partial\dot q_\alpha}A_s^\alpha\delta q_s]\,dt=0,\quad \delta q_i=0 \text{ at } t=t_0,t_1 \tag{2.8}$$

which with equalities (1.9) taken into account evidently represents the Voronets form (2.5).

Thus it has been shown that formulas (2.1), (2.5), (2.7) are equivalent and convert to each other by means of the considered transformations [8].

3. THE FORMS OF HAMILTON'S PRINCIPLE IN QUASI-COORDINATES AND QUASI-VELOCITIES.

Motion's equations in nonlinear quasi-coordinates were first deduced by Hamel [6] from the central Lagrange equation using the transitivity equations, which were also derived by Hamel. Novoselov [5] has deduced such equations from Hamilton's principle (2.1) also using transitivity equations. Without last equations the motion's equations were derived by Rumyantsev [10] from Maggi's equations

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}\right)F_{is}=0,\ \ i=1,...,n,\ \ s=1,...,k\ . \tag{3.1}$$

Indeed, replacing the velocities $\dot{q}_i$ in $L(q,\dot{q},t)$ by expressions (1.15) we obtain generalized Lagrange function $L^*(q,\eta,t)$.

Since

$$\frac{\partial L}{\partial q_i}=\frac{\partial L^*}{\partial q_i}+\frac{\partial L^*}{\partial \eta_r}\frac{\partial f_r}{\partial q_i},\ \ \frac{\partial L}{\partial \dot{q}_i}=\frac{\partial L^*}{\partial \eta_r}f_{ri},$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}=\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_r}\right)f_{ri}+\frac{\partial L^*}{\partial \eta_r}\frac{df_{ri}}{dt},\ \ i,r=1,...,n,$$

we receive from (3.1) the motion's equations for nonholonomic system in nonlinear quasi-coordinates and quasi-velocities in the notations (1.21)

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s}+\frac{\partial L^*}{\partial \eta_r}W_s^r-\frac{\partial L^*}{\partial \pi_s}=0,\ \ \eta_\alpha=0,\ \ s=1,...,k\ ,\ \alpha=k+1,...,n \tag{3.2}$$

or

$$\frac{d}{dt}\frac{\partial L^*}{\partial \eta_s}-\frac{\partial L^*}{\partial \eta_i}f_{ir}T_s^r-\frac{\partial L^*}{\partial \pi_s}=0,\ \ \eta_\alpha=0\ ,\ i=1,...,n,\ \ s=1,...,k. \tag{3.3}$$

The Eqs. (3.2) and (3.3) are identical with Hamel's equations (I) and (II) [6]. Novoselov [5] has named these equations of Voronets-Hamel and Chaplygin type, respectively (with the factor $\frac{\partial L^*}{\partial \eta_r}f_{ri}$ in (3.3) replaced by $\frac{\partial L}{\partial \dot{q}_i}$).

Note that one can set $\eta_\alpha = 0$ in Eqs. (3.2), (3.3) only after expressing them in explicit form, since they generally involve all derivatives $\frac{\partial L^*}{\partial \eta_r}$, $r=1,...,n$.

The Eqs. (3.2) or (3.3) enable one to deduce the Hölder form of Hamilton's principle in quasi-coordinates. Indeed, multiply the Eqs. (3.2) or (3.3) by $\delta\pi_s$, sum over all $s = 1,...,k$, integrate the result with respect to t, then using the Eqs. (1.19) and set $\delta\pi_s = 0$ at $t = t_0$, t_1 we obtain

$$\int_{t_0}^{t_1}\delta L^* dt=0,\ \ \delta\pi_s=0\ \text{ at }\ t=t_0,t_1\ . \tag{3.4}$$

Of course the (3.4) is equivalent to (2.1).

Using the generalized Legendre transformation [9]

$$y_s=\frac{\partial L^*}{\partial \eta_s},\ \ H^*(q,y,t)=y_s\eta_s-L^*(q,\eta,t) \tag{3.5}$$

we are able [11] to bring the Eqs. (3.2) or (3.3) to the canonical form of equations in quasi-coordinates

$$\frac{dy_s}{dt}+y_rW_s^r+\frac{\partial H^*}{\partial \pi_s}=0,\ \ \eta_s=\frac{\partial H^*}{\partial y_s},\ \ \eta_\alpha=\frac{\partial H^*}{\partial y_\alpha}=0, \tag{3.6}$$

or
$$\frac{dy_s}{dt} - y_r f_{ri} T_s^i + \frac{\partial H^*}{\partial \pi_s} = 0,\ \ \eta_s = \frac{\partial H^*}{\partial y_s},\ \ \eta_\alpha = \frac{\partial H^*}{\partial y_\alpha} = 0 \tag{3.7}$$

The coefficients W_s^r, T_s^r in these equations must be expressed in terms y_r.

The Eqs. (3.6) or (3.7) enable one to deduce the second Hölder form of Hamilton's principle
$$\int_{t_0}^{t_1} \delta(\eta_s y_s - H^*)dt = 0,\ \ \delta\pi_s = 0 \ \text{ at } \ t = t_0, t_1 \tag{3.8}$$

In turn the Eqs. (3.6) or (3.7) may be derived from the principle (3.8).

It should be noted that the principle (3.8) is significant in it's own right, considering the assumption that variations δy_s are arbitrary and independent of the $\delta\pi_s$ in the interior of the interval (t_0,t_1) [9].

We now proceed to derive Voronets equations in quasi-coordinates. To do this, we replace the kinetic energy $T^*(q,\eta,t)$ of a holonomic system, which figures in $L^*(q,\eta,t)$ in Eqs, (3.2), (3.3), by the kinetic energy $\Theta^*(q,\eta_1,\ldots,\eta_k,t)$ of nonholonomic system with constraints $\eta_\alpha = 0$. Since the relations
$$\frac{\partial L^*}{\partial \eta_s} = \frac{\partial \Theta^*}{\partial \eta_s},\ \ \frac{\partial L^*}{\partial \pi_s} = \frac{\partial(\Theta^* + U)}{\partial \pi_s},\ \ \frac{\partial L^*}{\partial \eta_\alpha} = \left(\frac{\partial T^*}{\partial \eta_\alpha}\right)_0 \tag{3.9}$$

hold when $\eta_\alpha = 0$, where $\left(\frac{\partial T^*}{\partial \eta_\alpha}\right)_0 = \left.\frac{\partial T^*}{\partial \eta_\alpha}\right|_{\eta_\beta = 0,}$ $\alpha,\beta = k+1,\ldots,n,$ it follows that the first k equations (3.2) or (3.3) may be transformed to the Voronets equations in quasi-coordinates [8]
$$\frac{d}{dt}\frac{\partial \Theta^*}{\partial \eta_s} - \frac{\partial(\Theta^* + U)}{\partial \pi_s} + \left(\frac{\partial T}{\partial \eta_i}\right)_0^* W_s^r = 0,\ \ s = 1,\ldots,k\ ,\ i = 1,\ldots,n, \tag{3.10}$$

or
$$\frac{d}{dt}\frac{\partial \Theta^*}{\partial \eta_s} - \frac{\partial(\Theta^* + U)}{\partial \pi_s} - \left(\frac{\partial T}{\partial \dot q_i}\right)^* T_s^i = 0,\ \ s = 1,\ldots,k\ ,\ i = 1,\ldots,n, \tag{3.11}$$

where $\left(\frac{\partial T}{\partial \dot q_i}\right)^*$ denotes the result of replacement $\dot q_i$ by (1.15) in the expression $\frac{\partial T}{\partial \dot q_i}$ $(i = 1,\ldots,n)$.

The Eqs. (3.10) or (3.11) imply the Voronets form of Hamilton's principle in quasi-coordinates
$$\int_{t_0}^{t_1} [\delta(\Theta^* + U) - \left(\frac{\partial T^*}{\partial \eta_\alpha}\right)_0 W_s^\alpha \delta\pi_s]\, dt = 0,\ \ \delta\pi_s = 0 \ \text{ at } \ t = t_0, t_1 \tag{3.12}$$

or
$$\int_{t_0}^{t_1} [\delta(\Theta^* + U) - \left(\frac{\partial T}{\partial \dot q_i}\right)_0^* T_s^i \delta\pi_s]\, dt = 0,\ \ \delta\pi_s = 0 \ \text{ at } \ t = t_0, t_1\,. \tag{3.13}$$

The Eqs. (3.10), (3.11), in turn, may be deduced from (3.12), (3.13) respectively [12].

Using the Legendre transformation

$$y_s = \frac{\partial \Theta^*}{\partial \eta_s}, \quad H^*(q, y, t) = y_s \eta_s - \Theta^*(q, \eta, t) - U(q, t) \tag{3.14}$$

by condition $\left\| \frac{\partial^2 \Theta^*}{\partial \eta_r \partial y_s} \right\| \neq 0$, one can reduce the Eqs. (3.10) and (3.11) to the canonical form in quasi-coordinates

$$\frac{dy_s}{dt} + \frac{\partial H^*}{\partial \pi_s} + \left(\frac{\partial T^*}{\partial \eta_\alpha} \right)_0 W_s^\alpha = 0, \quad \eta_s = \frac{\partial H^*}{\partial y_s}, \quad s = 1, \ldots, k, \quad \alpha = k+1, \ldots, n, \tag{3.15}$$

and

$$\frac{dy_s}{dt} + \frac{\partial H^*}{\partial \pi_s} - \left(\frac{\partial T^*}{\partial \dot{q}_i} \right) T_s^i = 0, \quad \eta_s = \frac{\partial H^*}{\partial y_s}, \quad s = 1, \ldots, k, \quad i = 1, \ldots, n. \tag{3.16}$$

The Eqs. (3.15) or (3.16) enable one to obtain the second Voronets form in quasi-coordinates

$$\int_{t_0}^{t_1} [\delta(y_s \eta_s - H^*) - \left(\frac{\partial T^*}{\partial \eta_\alpha} \right)_0 W_s^\alpha \delta \pi_s]\, dt = 0, \quad \delta \pi_s = 0 \text{ at } t = t_0, t_1 \tag{3.17}$$

or

$$\int_{t_0}^{t_1} [\delta(y_s \eta_s - H^*) + \left(\frac{\partial T}{\partial \dot{q}_i} \right)^* T_s^i \delta \pi_s]\, dt = 0, \quad \delta \pi_s = 0 \text{ at } t = t_0, t_1 . \tag{3.18}$$

Using the relations (3.9) it is not hard to verify the truth of the equalities

$$\delta L^* = \delta(\Theta^* + U) - \left(\frac{\partial T^*}{\partial \eta_\alpha} \right)_0 W_s^\alpha \delta \pi_s = \delta(\Theta^* + U) + \left(\frac{\partial T}{\partial \dot{q}_i} \right)^* T_s^i \delta \pi_s ,$$

which prove the equivalence of the Voronets forms to Hölder's form of Hamilton's principle in quasi-coordinates.

In conclusion we consider a special case (1.2) the nonholonomic constraints (1.1) and we pose

$$\eta_\alpha \equiv \dot{q}_\alpha - \varphi_l(q, \dot{q}_1, \ldots, \dot{q}_k, t) = 0, \quad \eta_s = \dot{q}_s, \quad s = 1, \ldots, k, \quad \alpha = k + l.$$

In this case the Voronets form of Hamilton's principle has the form (2.5), and the Voronets motion's equations – the form (2.6). [1]

Using the Legendre transformation

$$p_s = \frac{\partial \Theta}{\partial \dot{q}_s}, \quad H(q, p, t) = p_s \dot{q}_s - \Theta(q, \dot{q}_s, t) - U(q, t) \tag{3.19}$$

we reduce the Eqs. (2.6) to the canonical form

[1] Note that in [12] formula (5.1) which is equivalent to (2.5), and Section 6 were incorrectly referred to as Suslov's principle; the latter has the form (2.7).

$$\frac{dp_s}{dt}+\frac{\partial H}{\partial q_s}+\frac{\partial H}{\partial q_\alpha}\left(\frac{\partial \varphi_\alpha}{\partial \dot q_s}\right)^*-\left(\frac{\partial T}{\partial \dot q_\alpha}A_s^\alpha\right)^*=0,\ \ \frac{dq_s}{dt}=\frac{\partial H}{\partial p_s}, \tag{3.20}$$

where $(\psi)^*$ denotes an expression of ψ in terms of p_s.

The Eqs. (3.20) lead to second Voronets form of Hamilton's principle in quasi-coordinates and momenta

$$\int_{t_0}^{t_1}[\delta(p_s\dot q_s-H)+\left(\frac{\partial T}{\partial \dot q_\alpha}A_s^\alpha\right)^*]\,dt=0,\ \ \delta q_s=0\ \text{ at }\ t=t_0,t_1 \tag{3.21}$$

4. A COMPARISON WITH THE LAGRANGE PROBLEM

Let us compare the Hamilton's principle (2.1) with the Lagrange problem of stationary value of the action integral (2.2) in the class of curves that satisfy Eqs. (1.1). The introduction of indeterminate multiplies $\chi_l(t)$ reduces that problem of conditional extremum to the Lagrange problem of variations

$$\delta\int_{t_0}^{t_1}(L+\chi_l f_l)\,dt=0 \tag{4.1}$$

The Euler's equations for the problem (4.1) are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}-\frac{\partial L}{\partial q_i}=\chi_l\left(\frac{\partial f_l}{\partial q_i}-\frac{d}{dt}\frac{\partial f_l}{\partial \dot q_i}\right)-\dot\chi_l\frac{\partial f_l}{\partial \dot q_i}\quad i=1,\dots,n,\ l=1,\dots,r. \tag{4.2}$$

Obviously the motion's equations (2.3), (1.1) are not equivalent to Eqs. (4.1), (1.1). However the nonequivalence of these two systems of equations does not exclude a possibility some of their solutions being the same. Let the general or some particular solution $q_i(t)$ of Eqs. (2.3), (1.1) be also a solution of Eqs. (4.2), (1.1) for the same initial conditions.

Evidently the equalities

$$(\mu_l+\dot\chi_l)\frac{\partial f_l}{\partial \dot q_i}=\chi_l\left(\frac{\partial f_l}{\partial q_i}-\frac{d}{dt}\frac{\partial f_l}{\partial \dot q_i}\right) \tag{4.3}$$

are now valid. Taking into account (1.4) we multiply Eqs. (4.3) by δq_i and summing over all *i's* , we obtain the condition

$$\chi_l\left(\frac{\partial f_l}{\partial q_i}-\frac{d}{dt}\frac{\partial f_l}{\partial \dot q_i}\right)\delta q_i=0\,, \tag{4.4}$$

which is necessary if two systems have the same solution $q_i(t)$.

This condition is also sufficient. For proving this, let us assume that some solution of Eqs. (4.2), (1.1) satisfies (4.4) for any δq_i compatible with (1.4). Multiplying Eqs. (4.2) by δq_i and Eqs. (1.4) by μ_l and summing over all *i's* and *l's* with allowance for (4.4) and (1.4) we obtain the relationship

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}-\mu_l\frac{\partial f_l}{\partial \dot{q}_i}\right)\delta q_i=0,$$

which shows that the considered solution $q_i(t)$ also satisfies Eqs. (2.3), (1.1).

Thus condition (4.4) is necessary and sufficient for solution $q_i(t)$ of Eqs. (2.3), (1.1) to be among solutions of Eqs (4.2), (1.1) [8].

Thus when condition (4.4) is satisfied, the equations of motion (2.3) of nonholonomic system have the form of Euler's equations (4.2). Owing to this we say that Hamilton's principle (2.1) for the motion of a nonholonomic system defined by such solution has the characteristics of the principle of stationary action (2.2).

For relationship of the form (1.2) equality (4.4) reduces to conditions

$$\chi_l A_s^{k+l}=0,\quad s=1,\dots,k,\quad l=1,\dots,r. \tag{4.5}$$

Note that Suslov's form (2.7) has also the characteristics of the principle of stationary action then and only then when the condition

$$\int_{t_0}^{t_1}\frac{\partial T}{\partial \dot{q}_\alpha}A_s^\alpha \delta q_s dt=0\ ,\ \alpha=k+l$$

is satisfied. Since δq_s are arbitrary and independent, this condition is satisfied only when [13]

$$\frac{\partial T}{\partial \dot{q}_\alpha}A_s^\alpha=0,\quad s=1,\dots,k. \tag{4.6}$$

We stress that conditons (4.4)–(4.6) are seldom satisfied in the case of nonholonomic systems.

Two examples are given below. In the first one these conditions are satisfied for the general solution, in the second one only for some particular solutions of motion's equations of the nonholonomic system

Example 4.1. For Appel's example [4] from equations of the form (2.3) and (1.1)

$$m\ddot{q}_s=-\mu a\frac{\dot{q}_s}{\sqrt{\dot{q}_1^2+\dot{q}_2^2}},\quad m\ddot{q}_3=-mg+\mu$$

we have

$$\frac{d}{dt}\left(\frac{\dot{q}_s}{\sqrt{\dot{q}_1^2+\dot{q}_2^2}}\right)=0,\quad s=1,2$$

which show that the conditions (4.4)–(4.6) are satisfied for all motions of the material point.

Example 4.2. For a disk the Lagrange function is

$$L=\frac{m}{2}\{[\dot{x}-r(\cos\theta\sin\varphi\dot{\theta}+\sin\theta\cos\varphi\dot{\varphi})]^2+[\dot{y}+r(\cos\theta\cos\varphi\dot{\theta}+\sin\theta\sin\varphi\dot{\varphi})]^2\}+$$
$$+\frac{1}{2}[A(\dot{\theta}^2+\dot{\varphi}^2\cos^2\theta)+C(\dot{\varphi}+\dot{\varphi}\sin\theta)^2]-mgr\cos\theta$$

The conditions (4.6) assume the form

$$\frac{\partial T}{\partial \dot q_{3+l}} A_2^{3+l} = -mr^2\dot\theta\dot\varphi\cos\theta = 0,\ \ \frac{\partial T}{\partial \dot q_{3+l}} A_3^{3+l} = mr^2\dot\theta\dot\psi\cos\theta = 0$$

which are satisfied either when $\dot\theta = 0$, or $\dot\varphi = \dot\psi = 0$.

Hence in the case of motion of the disk whose plane form a constant angle θ to the vertical, as well as some highly special motions for which $\dot\varphi = \dot\psi = 0$, Hamilton's principle has the characteristic of the principle of stationary action, while for another motions this is not so.

5. CONDITIONS OF APPLICABILITY OF THE GENERALIZED HAMILTON – JACOBI METHOD OF INTEGRATION

Preceding results are closely related to the problem of extending to nonholonomic systems the generalized Hamilton – Jacobi method of integration for canonical equations of motion

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + \mu_l \frac{\partial f_l}{\partial \dot q_i},\ \ i = 1,\dots,n,\ l = k+1,\dots,n, \tag{5.1}$$

that are equivalent to Eqs. (2.3), (1.1). Here

$$p_i = \frac{\partial L}{\partial \dot q_i},\ \ H(q,p,t) = p_i\dot q_i - L(q,\dot q,t),\ \ i = 1,\dots n. \tag{5.2}$$

In essence the Hamilton – Jacobi method consists in the following [14,15,8].

The variables

$$\pi_i = p_i + \lambda_l \frac{\partial f_l}{\partial \dot q_i},\ \ i = 1,\dots,n \tag{5.3}$$

are introduced and used for reducing (5.2) to the form

$$L = \pi_i \dot q_i - H_1, \tag{5.4}$$

where function

$$H_1(q,\pi,t) = H(q,p,t) + \lambda_l \frac{\partial f_l}{\partial \dot q_i}\dot q_i \tag{5.5}$$

is obtained by substitution into its right-hand side of functions $p_i(q,\pi,t)$ and $\lambda_l(q,\pi,t)$ derived from Eqs. (1.1) and (5.3) and of the first group of Eqs. (5.1).

It is advisable to construct function (5.5) as follows. Using (1.2) we represent the Lagrange function in the form $L^*(q_1,\dots,q_n,\dot q_1,\dots,\dot q_k,t) = \Theta + U$ and introduce the generalized momenta and the Hamiltonian

$$P_s = \frac{\partial L^*}{\partial \dot q_s} = p_s + p_{k+l}\frac{\partial \varphi_l}{\partial \dot q_s},\ \ H^*(q,P,t) = P_s\dot q_s - L^*,\ \ s = 1,\dots,k,\ l = 1,\dots,r. \tag{5.6}$$

The function $H^*(q,P,t)$ is connected with $H(q,p,t)$ by the formula

$$H^*(q,P,t) = H(q,p,t) + p_{k+l}\left(\frac{\partial\varphi_l}{\partial\dot{q}_s}\dot{q}_s - \varphi_l\right), \; l = 1,...,r. \tag{5.7}$$

Since Eqs. (5.3) imply for relationships (1.2) the equalities

$$\lambda_l = \pi_{k+l} - p_{k+l}, \;\; P_s = \pi_s + \pi_{k+l}\frac{\partial\varphi_l}{\partial\dot{q}_s}$$

the function (5.5) with allowance for (5.7) assumes the form

$$H_l(q,\pi,t) = H^*(q,P,t) + \pi_{k+1}\left(\varphi_l - \frac{\partial\varphi_l}{\partial\dot{q}_s}\dot{q}_s\right)$$

The generalized Hamilton – Jacobi equation

$$\frac{\partial S}{\partial t} + H_1\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0 \tag{5.8}$$

has characteristic equations of the canonical form

$$\frac{dq_i}{dt} = \frac{\partial H_1}{\partial \pi_i}, \;\; \frac{d\pi_i}{dt} = -\frac{\partial H_1}{\partial q_i}, \;\; i = 1,...,n. \tag{5.9}$$

According to Jacobi's theorem the relationships

$$\frac{\partial S}{\partial q_i} = \pi_i\,, \;\; \frac{\partial S}{\partial \alpha_i} = \beta_i, \quad i = 1,...,n,$$

represent $2n$ integrals of Eqs. (5.9), if $S(q_i,\alpha_i,t)$ is a complete integral of Eq. (5.8) with arbitrary constants α_i and β_i.

It was shown in [16] that the solution of Eqs. (5.9) is also the solution of motion's equations (5.1) if and only if it satisfies the condition

$$\lambda_l\left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt}\frac{\partial f_l}{\partial \dot{q}_i}\right)\delta q_i = 0, \;\; i = 1,...,n, \quad l = 1,...,r. \tag{5.10}$$

Hence (5.10) is a necessary and sufficient condition for the considered generalized Hamilton – Jacobi method to be applicable to nonholonomic systems.

The condition (5.10), with (1.4) taken into account, follows from equations [16]

$$\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i} = \lambda_l\left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt}\frac{\partial f_l}{\partial \dot{q}_i}\right) - \dot{\lambda}_l\frac{\partial f_l}{\partial \dot{q}_i}, \;\; i = 1,...,n, \quad l = 1,...,r. \tag{5.11}$$

obtained by differentiating the expressions (5.3) with respect to t on the basis of (5.9). When $\lambda_l = \chi_l$ ($l = 1,...,r$) Eqs. (5.11) evidently match to Euler's equations (4.2) of the variational problem (4.1).

Hence the generalized Hamilton – Jacobi method of integrating Eqs. (5.1) of nonholonomic systems is applicable if and only if Hamilton's principle has the characteristics of the principle of stationary action

$$\delta\int_{t_0}^{t_1}(\pi_i\dot{q}_i - H_1)\,dt = 0,\quad \delta q_i = 0 \text{ at } t = t_0, t_1,$$

which with allowance for (5.4) is equivalent to the principle (2.2).

Example 5.1. The equations of motion in Appel's example and the equations of motion of the disk in case $\dot{\theta} = 0$ were integrated in [14] and [17] accordingly by the generalized Hamilton – Jacobi method.

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OBLICI HAMILTONOVOG PRINCIPA ZA NEHOLONOMNE SISTEME

Valentin Vitalievich Rumyantsev

Analiziraju se uslovi pod kojima su izvedena tri oblika Hamiltonovog principa za neholonomne sisteme sa linearnim ograničenjima po Hölder-u, Voronets-u i Suslov-u u opštem slučaju nelinearnih ograničenja. Dokazano je da su ova tri oblika međusobno ekvivalentna i da se mogu transformisati jedan u drugi.

Analizirana su i analogna pitanja za slučaj nelineranih kvazi-koordinata i kvazi brzina. Sem toga oblici Hölder-a, Voronets-a i Suslov-a su prikazani u slučaju Legendre-ove transformacije re-dukovanjem jednačina kretanja na kanonički oblik u kvazi-koordinatama. Takođe su izvedeni i uslovi pod kojima Hamiltonov princip za za neholonomne sisteme ima karakteristike principa stacionarne akcije.