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# A VARIATIONAL APPROACH TO THE PROBLEM OF TWO BODIES

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**Abstract**. *In this paper, we assume that the motion of celestial bodies is subject to the Maupertuis-Lagrange's principle of least kinetic potential action under some geometrical constraints. By the Lagrange multiplier, more general Newton's gravitational equations can be readily obtained. For the problem of two bodies, the results are same as those obtained by Veljko A. Vujicic.*

## 1. INTRODUCTION

The motion of a system of two bodies, observed as material points, is known in the celestial mechanics as " the problem of two bodies", which is subject to Kepler's law and Newton's law of universal gravitational force. According to Newton's law, the motion of the celestial bodies stems from the universal gravitational force attracting mutually each other.

Recently Vujicic [1] proposed a possible reconsideration of Newton's gravitation law, details can be found in [1,2]. To illustrate his basic idea, we first consider the concept of constraint force [2, 3]. If a motion is subject to a constraint, which reads

$$f(x, y, z) = 0\,, \tag{1}$$

then we have

$$f' = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = \nabla f \cdot \boldsymbol{v}\,, \tag{2}$$

where prime denotes the derivatives with respect to time, $\boldsymbol{v}$ is velocity vector.

From the relation (2), it follows that the velocity vector is perpendicular to the constraint's gradient, so we can introduce a "constraint force $\boldsymbol{R}$ [2,3]" , which is defined as

$$\boldsymbol{R} = \nabla f\ . \tag{3}$$

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Similar to Newton's 2nd law, Vujicic[2] proposes the following formula

$$m\frac{d\mathbf{v}}{dt} = \boldsymbol{F} + \lambda\boldsymbol{R} = \boldsymbol{F} + \lambda\nabla f \ , \tag{4}$$

where $\boldsymbol{F}$ is the sum acting vector forces, $\lambda$ is an indefinite multiplier called constraint multiplier [2].

The approach is very heustical indeed, but Vujicic's method will not be valid in case when there exist derivatives with respect to coordinates in the constraint's equation, for example

$$\tilde{f}(x, y, z, x', y', z') = 0 \, . \tag{5}$$

Hinted by the heustical ideas proposed by Vujicic [2], in this paper we will introduce a new axiom for celestial mechanics, that is all motion of the celestial bodies is subject to the Maupertuis-Lagrange's principle of least action under some geometrical constraints, as a result a generalized Newton's gravitation law can be obtained and the limitation of Vujicic's theory can be eliminated.

## 2. MAUPERTUIS-LAGRANGE'S PRINCIPLE

The Maupertuis-Lagrange's principle of least kinetic potential action for a particle with mass $m$ can be expressed as follows [3]

$$\int_{t_0}^{t_1} \frac{1}{2} m v^2 dt \to \min \, . \tag{6}$$

In this paper, we consider that all celestial bodies can be considered as material points with masses. So Maupertuis-Lagrange's principle can be applied to celestial mechanics. To verify the assumption, we consider the well-known problem of two bodies in celestial mechanics [4].

Assume that there are two bodies with masses $m_1$ and $m_2$ respectively, moving towards each other so that the distance between the two bodies is a time function, which reads

$$\rho(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{7}$$

Using the Lagrange multiplier, Maupertuis-Lagrange's principle can be written in the form

$$J(x_1, y_1, z_1, x_2, y_2, z_2) = \int_{t_0}^{t_1} \{\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \lambda f\} dt \tag{8}$$

where

$$v_1^2 = (x_1')^2 + (y_1')^2 + (z_1')^2 \, ,$$

$$v_2^2 = (x_2')^2 + (y_2')^2 + (z_2')^2 \, ,$$

$$f = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} - \rho \, .$$

Calculating the variation of the functional (6) results in the following stationary conditions

$$m_1 x_1'' = \frac{\lambda}{\rho}(x_1 - x_2)\,, \tag{9a}$$

$$m_1 y_1'' = \frac{\lambda}{\rho}(y_1 - y_2)\,, \tag{9b}$$

$$m_1 z_1'' = \frac{\lambda}{\rho}(z_1 - z_2)\,, \tag{9c}$$

$$m_2 x_2'' = -\frac{\lambda}{\rho}(x_1 - x_2)\,, \tag{10a}$$

$$m_2 y_2'' = -\frac{\lambda}{\rho}(y_1 - y_2)\,, \tag{10b}$$

$$m_2 z_2'' = -\frac{\lambda}{\rho}(z_1 - z_2)\,. \tag{10c}$$

From the above equations, by simple manipulation, we have the following relations

$$m_1 m_2 (x_1'' - x_2'') = \frac{\lambda}{\rho}(m_1 + m_2)(x_1 - x_2)\,, \tag{11a}$$

$$m_1 m_2 (y_1'' - y_2'') = \frac{\lambda}{\rho}(m_1 + m_2)(y_1 - y_2)\,, \tag{11b}$$

$$m_1 m_2 (z_1'' - z_2'') = \frac{\lambda}{\rho}(m_1 + m_2)(z_1 - z_2)\,, \tag{11c}$$

and

$$m_1 m_2 [(x_1'' - x_2'')(x_1 - x_2) + (y_1'' - y_2'')(y_1 - y_2) + (z_1'' - z_2'')(z_1 - z_2)] = \lambda\rho(m_1 + m_2)\,. \tag{12}$$

Differentiating equation (7) twice with respect to time results in

$$(x_1'' - x_2'')(x_1 - x_2) + (y_1'' - y_2'')(y_1 - y_2) + (z_1'' - z_2'')(z_1 - z_2) = \rho'^2 + \rho\rho'' - v_0^2\,, \tag{13}$$

where

$$v_0^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2$$

From equations (12) and (13), the multiplier can be determined as follows

$$\lambda = \frac{m_1 m_2}{m_1 + m_2} \frac{\rho'^2 + \rho\rho'' - v_0^2}{\rho} = \kappa \frac{m_1 m_2}{\rho}\,, \tag{14}$$

where

$$\kappa = \frac{\rho'^2 + \rho\rho'' - v_0^2}{m_1 + m_2}\,. \tag{15}$$

Substituting the identified multiplier into (9a–c) and (10a–c) , we obtain the following generalized Newton's gravitational equations

$$m_1 x_1'' = \kappa \frac{m_1 m_2}{\rho^2}(x_1 - x_2)\,, \tag{16a}$$

$$m_1 y_1'' = \kappa \frac{m_1 m_2}{\rho^2}(y_1 - y_2)\,, \tag{17b}$$

$$m_1 z_1'' = \kappa \frac{m_1 m_2}{\rho^2}(z_1 - z_2)\,, \tag{18c}$$

$$m_2 x_2'' = -\kappa \frac{m_1 m_2}{\rho^2}(x_1 - x_2)\,, \tag{19a}$$

$$m_2 y_2'' = -\kappa \frac{m_1 m_2}{\rho^2}(y_1 - y_2)\,, \tag{19b}$$

$$m_2 z_2'' = -\kappa \frac{m_1 m_2}{\rho^2}(z_1 - z_2)\,. \tag{19c}$$

The results are same as those in [2]. In [2], some special conditions are discussed, which agree with the Newton's gravitation law or Kepler's laws.

If we considering Kepler's 2nd law as a constraint of Maupertuis-Lagrange's principle, then the constraint's equation includes the derivatives with respect to coordinates as illustrated in the following equation

$$g(x_1', y_1', z_1', x_2', y_2', z_2') = \int_{t_0}^{t_1} \rho\sqrt{(x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2}\,dt - C = 0\,, \tag{20}$$

where $C$ is a constant

In such case we can not apply Vujicic's theory to obtain the generalized gravitational equations, however, our approach has no such restrict.

Applying the Lagrange multipliers $\lambda$ and $\beta$, we have the following modified functional:

$$\begin{aligned} J(x_1, y_1, z_1, x_2, y_2, z_2) = \int_{t_0}^{t_1} \{\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \lambda f\} dt + \\ + \beta[\int_{t_0}^{t_1} \rho\sqrt{(x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2}\,dt - C]\,, \end{aligned} \tag{21}$$

from which the generalized gravitational equations can be readily obtained, which read

$$m_1 x_1'' = \frac{\lambda}{\rho}\frac{\partial f}{\partial x_1} - \frac{d}{dt}(\beta \frac{\partial g}{\partial x_1'})\,, \tag{22a}$$

$$m_1 y_1'' = \frac{\lambda}{\rho}\frac{\partial f}{\partial y_1} - \frac{d}{dt}(\beta \frac{\partial g}{\partial y_1'})\,, \tag{22b}$$

$$m_1 z_1'' = \frac{\lambda}{\rho}\frac{\partial f}{\partial z_1} - \frac{d}{dt}(\beta \frac{\partial g}{\partial z_1'})\,, \tag{22c}$$

$$m_2 x_2'' = \frac{\lambda}{\rho}\frac{\partial f}{\partial x_2} - \frac{d}{dt}(\beta \frac{\partial g}{\partial x_2'})\,, \tag{23a}$$

$$m_2 y_2'' = \frac{\lambda}{\rho}\frac{\partial f}{\partial y_2} - \frac{d}{dt}(\beta \frac{\partial g}{\partial y_2'})\,, \tag{23b}$$

$$m_2 z_2'' = \frac{\lambda}{\rho}\frac{\partial f}{\partial z_2} - \frac{d}{dt}(\beta \frac{\partial g}{\partial z_2'}) . \tag{23c}$$

The multipliers can be identified in a similar way as above, but the multipliers might not be written in explicit expressions.

The present theory can be easily extended to the problem of three bodies or multi-bodies. For example, considering Earth's motion, there exist two constraints: the distances between Sun and Earth, Moon and Earth are time functions $\rho_1(t)$ and $\rho_2(t)$:

$$f_1 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} - \rho_1 , \tag{24}$$

$$f_2 = \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2} - \rho_2 . \tag{25}$$

So we have the following functional

$$J(x_i, y_i, z_i) = \int_{t_0}^{t_1} \{\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + \lambda_1 f_1 + \lambda_2 f_2\} dt \text{ , } (i = 1,2,3) \tag{26}$$

where $m_1$, $m_2$ and $m_3$ are the masses of Earth, Sun and Moon respectively.

From the functional (26) we can obtain 9 equations, and the multipliers $\lambda_1$ and $\lambda_2$ can be identified in a similar way as illustrated above. We will discuss the problem in detail in future.

## 3. CONCLUSION

In this paper, generalized gravitational equations for celestial bodies can be readily obtained from the Maupertuis-Lagrange's principle under some geometrical constraints.

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# VARIJACIONI PRISTUP PROBLEMU DVA TELA

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*U ovom radu pretpostavljamo da je kretanje nebeskih tela izloženo Maupertis-Lagrange-ovom principu minimalne kinetičke potencijalne akcije pri nekim geometrijskim ograničenjima. Pomoću Lagrange-ovog množioca lako se mogu dobiti opštije Newton-ove gravitacione jednačine. Za problem dva tela, rezultati su isti kao oni dobijeni kod Veljka Vujučića.*