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ON STABILITY CRITERIA FOR GYROSCOPIC SYSTEMS WITH NEGATIVE DEFINITE STIFFNESS

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Abstract*. A critical survey of published criteria - expressed by the properties of the system matrices - for the stability of linear conservative gyroscopic systems with negative definite stiffness matrix is presented. In addition a sufficient stability condition of the same type is derived. A simple two degree of freedom example is used to illustrate the usefulness of each.*

INTRODUCTION

One of the most interesting phenomena for linear gyroscopic dynamic systems is that gyroscopic forces may stabilize a conservative system which would have been unstable in their absence. Applications vary from the classical problem of the spinning top to more complicated rotating bodies (such as satellites and cd players), and to the motion of fluids in flexible pipes. It is well known that the stability properties of such systems can be checked by means of eigenvalue analysis for the case that all of the system's physical parameters are specified. However, if the physical parameters are not specified and the design problem of interest is to choose the physical parameters such that the system will be stable, then spectral (eigenvalue) analysis is not *directly* possible and of limited utility. Therefore, stability conditions which are stated in terms of the coefficients of system matrices without solving the spectrum of the entire system (nonspectral conditions) are of practical interest and importance. Nonspectral criteria may yield design constraints in terms of the physical parameters of the system.

This note presents a survey of selected published stability criteria for conservative gyroscopic systems in a design setting. Attention is restricted to systems with negative definite stiffness matrix. In addition, using a Lyapunov-type approach, sufficient stability conditions are derived. A standard example is used to illustrate the advantages and disadvantages of each condition.

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BACKGROUND

Systems of interest here are linear conservative gyroscopic systems described by the equation

$$M\ddot{q} + \hat{G}\dot{q} + \hat{K}q = 0 \tag{1}$$

where M , $\hat{G}$ and $\hat{K}$ are real $n \times n$ matrices, q is the n-vector, $. = d/dt$, and

- M is symmetric and positive definite $(M^T = M > 0)$;
- $\hat{G}$ is skew-symmetric $(\hat{G}^T = -\hat{G})$;
- $\hat{K}$ is symmetric and negative definite $(\hat{K}^T = K < 0)$.

The vector q represents the generalized coordinates, M is the mass matrix, $\hat{G}$ describes the gyroscopic forces, and $\hat{K}$ the potential forces.

It is convenient, although not necessary, to rewrite Eq. (1) in the form

$$\ddot{x} + G\dot{x} + Kx = 0\,, \tag{2}$$

using the transformation $x = M^{1/2}q$, $G = M^{-1/2}\hat{G}M^{-1/2}$, $K = M^{-1/2}\hat{K}M^{-1/2}$. Here the exponent ½ indicates the unique positive definite square root of the matrix M. Clearly, $G^T = -G$ and $K^T = K$.

The system is said to be stable if all solutions $x(t)$ of (2) are bounded for all non-negative t. All solutions of (2) can be characterized algebraically using properties of the quadratic matrix polynomial

$$L(\lambda) := \lambda^2 I + \lambda G + K\,, \tag{3}$$

where I is the identity matrix. The eigenvalues of the system (2) are zeros of the characteristic polynomial

$$\Delta(\lambda) := \det(L(\lambda)) \tag{4}$$

and the multiplicity of an eigenvalue is the order of the corresponding zero in $\Delta(\lambda)$. If λ is an eigenvalue, the nonzero vectors in the nullspace of $L(\lambda)$ are the eigenvectors associated with λ. In general, eigenvalues and eigenvectors may be real or complex. Since $G^T = -G$ and $K^T = K$, then $L(\lambda)^T = L(-\lambda)$and, consequently, $\Delta(\lambda) = \Delta(-\lambda)$. Thus, the system (2) is stable only when every eigenvalue is on the imaginary axis and semi-simple, i.e., if the eigenvalue has multiplicity k, there are k linearly independent associated eigenvectors.

Since the computation of eigenvalue problem (3) requires numerical values of the system's parameter it is not suitable when system's matrices are not fully specified. On the other hand, eigenvalue analysis becomes extremely difficult as the degree of freedom increases. As a result, alternative criteria such as those which provide simpler conditions directly in terms of the matrices G and K prove to be more attractive.

STABILITY CRITERIA

For the potential system, which can be formally obtained for (2) by setting $G = 0$, it is well known that the system is "completely unstable". If gyroscopic forces are introduced, then the system (2) may be stable or unstable. According to classical result, given by

Thomson (Lord Kelvin) and Tait [1] and Chetayev [2], the system (2) may be stable if and only if the degree of freedom is even. Then it is known that if G is nonsingular and sufficiently large, the system (2) is stable [3]. In this case, many attempts have been made to establish related stability criteria (necessary and/or sufficient) - expressed by the properties of the system matrices G and K - for system (2) with reasonably small gyroscopic forces. In the following, the main results, in the form of theorems, will be presented. Some of these results were found from several different authors and so they will be named for their founder.

Theorem 1 (Pozharitckii [4], see also [5])

The system (2) is unstable if

$$G^T G + 4K < 0 \,.$$

Moreover, under this condition the system (2) has no eigenvalues on the imaginary axis [6]. The reverse inequality does not generally ensure stability (see, for instance, the example in [7]).

Theorem 2 (Huseyin-Hagedorn-Teshner [8])

If $GK = KG$ *, then*

$$G^T G + 4K > 0$$

is necessary and sufficient for the stability of system (2).

Note that $GK = KG$ is a very restrictive condition, for example, if $n = 2$ then K is proportional to identity matrix.

Theorem 3 (Bulatović [9])

If G is nonsingular and matrices KG^2 *and* $(KG)^2$ *are symmetric, then*

$$G^T G - ((-K)^{1/2} + (G^T K G^{-1})^{1/2})^2 > 0$$

is necessary and sufficient for the stability of system (2).

When $GK = KG$, this theorem coincides with Theorem 2. The conditions for matrices KG^2 and $(KG)^2$ to be symmetric, contrary to the matrix commutation assumption in Theorem 2, do not confine system (2) with two degrees of freedom.

Theorem 4 (Barkwell-Lancaster [7])

The system is stable if the maximum eigenvalue of the negative matrix K (i.e. $k_M = \lambda_{\max}(-K)$*) times 4 is smaller than the smallest eigenvalue of the negative matrix* G^2 *(i.e.* $g_m = \lambda_{\max}(-G^2)$*), i. e.,*

$$g_m > 4k_M \,.$$

This requires substantial calculation to check because it involves finding the eigenvalues of both K and G.

Inman Conjecture [10]

The system is stable if

$$G^T G + 4K - \varepsilon I > 0$$

where 2ε *is the maximum eigenvalue of the negative matrix* G^2.

Unfortunately, Inman's proof is not rigorous as pointed out in [11]. However, there is no counter example for this condition. In [12], with the help of Lyapunov's direct method, it was proved that in Inman's condition, ε can be chosen in a way that this condition guarantees stability (ε is equal to the double difference of the maximal and minimal eigenvalues of the negative matrix K).

Theorem 5 (Bulatović [13])

The system is stable if

$$G^T G + 2K - 2k_M I > 0$$

where k_M *is the maximum eigenvalue of the negative matrix K.*

It is follows from $G^T G \geq g_m I$ and $K \geq -k_M I$ that $G^T G + 2K - 2k_M \geq (g_m - 4k_M)I$. Hence, this theorem extends and improves Theorem 4.

Theorem 6 (Lancaster [6])

The system is unstable if

$$\| G \|_2^2 + 2Tr(K) \leq 0,$$

where $Tr(K)$ *is the trace of K and* $\|G\|_2$ *is the Euclidean matrix norm of G.*

Theorem 7 (Kozlov [14])

If G is nonsingular and

$$\| G^{-1} \| . \| (-K)^{1/2} \| < \frac{1}{2}$$

where $\|.\|$ *denotes a matrix norm, then the system is stable.*

For spectral norm, which is the best choice, we have $\| G^{-1} \| = (\lambda_{\max}(-G^{-2}))^{1/2}$ = $(\lambda_{\min}(-G^2))^{-1/2} = (g_m)^{-1/2}$ and $\| (-K)^{1/2} \| = (\lambda_{\max}(-K))^{1/2} = \sqrt{k_M}$, and hence this theorem coincides with Theorem 4.

At the cost of complexity, a sufficient stability condition which is sharper than Theorem 5 is given next.

Theorem 8

The system is stable if

$$G^T G + K - k_M I + G^T (k_M I - K)^{-1} KG > 0$$

Proof. According to [13], we introduce auxiliary function of the form

$$V(x, \dot{x}) = x^T (K^2 - k_M K)x + 2x^T KGx + \dot{x}^T (G^T G + K - k_M I)\dot{x} .$$

It is clear that $K^2 - k_M K > 0$. Setting

$$y = (K^2 - k_M K)^{1/2} x + (K^2 - k_M K)^{-1/2} KG\dot{x}$$

we can rewrite V as

$$V = y^T y + \dot{x}^T (G^T G + K - k_M I + G^T (k_M I - K)^{-1} KG\dot{x} .$$

Consequently, the function V is positive definite if and only if

$$G^T G + K - k_M I + G^T (k_M I - K)^{-1} KG > 0 .$$

On the other hand, the time derivative of V along every solution of equation (2) becomes $\dot{V} = 0$. Hence, the Theorem follows from Lyapunov's stability theorem.

The above presented theorems provide stability conditions directly in terms of the coefficient matrices, and they involve no undetermined parameters (zero-parameter criteria). In order to obtain sharper conditions several criteria which involve one or more undetermined parameters have been established. These criteria are as follows:

Theorem 9 (Barkwell-Lancaster [7])

The system is stable if

$$(G^T G)^{1/2} - \varepsilon I - \varepsilon^{-1} K > 0$$

for some $\varepsilon > 0$.

Theorem 10 (Walker [11])
The system is stable if
$K(K-\varepsilon I)>0$
and
$K-\varepsilon I+\varepsilon G^T(K-\varepsilon I)^{-1}G>0$
for some real scalar ε.

Theorem 11 (Walker [11])
The system is unstable if
$I-\varepsilon K>0$
and
$G^T(I-\varepsilon K)^{-1}G+4K(I-\varepsilon K)^{-1}<0$
for some real scalar ε.

Theorem 12 (Seyranian-Stoustrup-Kliem [15])
Suppose the matrix K *is diagonal and suppose* Δ *is a certain diagonal positive definite matrix. If* Δ *commutes with* G *and*
$K-\Delta+G^TG-G^T(I-\Delta K^{-1})^{-1}G>0$,
the system is stable.

It is important to note that the one and multi-parameter theorems lead to very difficult mathematical expressions. It has been discussed in [16]. There is no systematic algorithm for verifying these conditions (or otherwise), and it is an open question of some interest.

EXAMPLE

A standard example used in illustrating stability results for conservative gyroscopic systems (see [10], [12]) is that of a simplified model of a disk mounted on a noncircular weightless rotating shaft which is also subjected to a constant axial compression force. By using a reference frame rotating with the shaft the problem can readily be formulated, resulting in an equation of the form (2) with the matrices

$$G=2\xi\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \text{ and } K=\begin{pmatrix}c_1-\xi^2-\eta & 0\\ 0 & c_2-\xi^2-\eta\end{pmatrix},$$

where ξ is the shaft angular velocity, η is the axial force, and c_1, c_2 are elastic rigidities in the two principle directions. We fix the values of ξ^2 and η as those in [10], $\xi^2 = 4$, $\eta = 3$ and we assume that $0 < c_1 < 7$ and $0 < c_2 < 7$, i. e., K is a negative definite.

By means of spectral analysis, one can show that the system of this example is stable if and only if $c_1+c_2+2-2\sqrt{(c_1-7)(c_2-7)}>0$. Figure 1 shows the exact region of stability with respect to the stiffnesses c_1 and c_2.

Each of the zero-parameter criteria presented an derived in the previous section are next applied to the above system. The stability regions predicted by these criteria are displayed in Fig. 2 which provides a visual comparison of the various stability results. Each region is defined by inequalities in the two parameters c_1 and c_2 as indicated in the following:

Theorem 1. The system is unstable if $c_1 < 3$ and $c_2 < 3$. This corresponds to the square in the lower left corner of the Fig. 2.

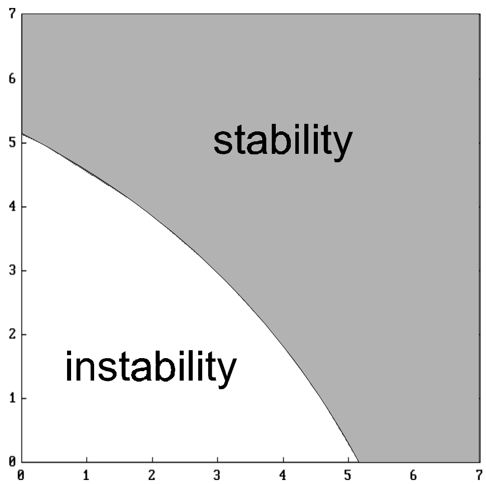

Fig. 1. Exact region of stability

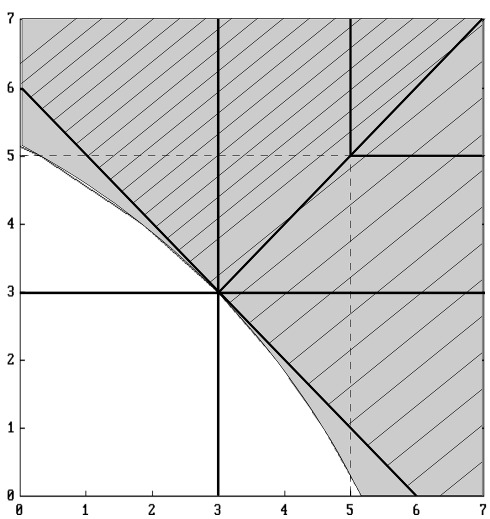

Fig. 2. Stability region predicted by various zero-parameter criteria

Theorem 2. The system is stable if $c_1 = c_2 > 3$. This result is represented by the line segment between (3,3) and (7,7).

Theorem 3. The system is stable if and only if $\sqrt{7-c_1}+\sqrt{7-c_2}<4$. This yields the exact region of stability, which is the shaded area in the figure.

Theorems 4, 5 and 7 yield the same result: the system is stable if $c_1 > 3$ and $c_2 > 3$. This corresponds to the larger square in the upper right corner of the figure.

Inman Conjecture. The system is stable if $c_1 > 5$ and $c_2 > 5$. This corresponds to the smaller square in the upper right corner of the figure.

Theorem 6. The system is unstable if $c_1 + c_2 \leq -2$, and this inequality yields us nothing.

Theorem 8. The system is stable if $c_1 + c_2 > 6$, which is the hatched area in the figure.

Let us notice in conclusion that all theorems except Theorem 6 are able to give a result of the example. Only Theorem 3 gives the exact result, but for $n > 2$ this theorem relates to a restricted class of the system. Theorem 5 is generally sharper than theorems 4 and 7, although they yield the same result for the example. Moreover, this theorem is certainly much easier to use than the other sufficient stability conditions. The best stability zero-parameter criterion is Theorem 8, but it is not as simple as Theorem 5. However, Theorem 8 is quite easy to use on the example where K is diagonal.

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O KRITERIJUMU STABILNOSTI ZA GIROSKOPSKE SISTEME SA NEGATIVNO DEFINITNOM KRUTOŠĆU

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Prikazan je kritički pregled publikovanih kriterijuma - izraženih svojstvima matrica sistema- za ispitivanje stabilnost linearnih konzervativnih giroskopskih sistema sa negativno definitnom matricom krutosti. Sem toga, izveden je i dovoljan uslov stabilnosti istog tipa. Za ilustrovanje primene urađen je jednostavni primer sistema sa dva stepena slobode.