# GENERAL MECHANICS EQUATION
# FOR THE MATERIAL SYSTEM
# WITH HOLONOMIC AND NONHOLONOMIC CONSTRAINTS

## *UDC 531.261+531.231:681.3.06(045)*

## Zbigniew Osiński

Warsaw University of Technology, Warsaw Poland

**Abstract**. *Author started from presentation of two classical ways of desibing of equation of motion of materials system. One way is the using of Newton's laws second is connected with Analytic Mechanics principles.*
*Author presents his concept of presentation of general mechanics equation. This presentation is based on the equivalency of generalized forces and generalized d'Alembert forces (inertial forces).*
*The presentation is valid for holonomic and nonholonomic systems. One presents also a method of computer aided derivation of equation of motive. They are given the examples.*

## 1. INTRODUCTION

When speaking about the general mechanics equation we may distinguish two directions. One is connected with the names of Galileo, Newton and Euler. The modern mechanics started from Galileo (1638) experiments. After Newton presented his laws. The second law is the law of motion. Speaking about general equation of motion one must take into consideration that Newton presented the law in descriptive way not giving the mathematical formula.

The known in this days mathematical formulation of second Newton's law as a relation between force and acceleration in form of differential equation was given by Euler.

Euler also presented the Newton's law in the form of impulse law (1750) and the momentum law (1775). These laws enable solving all mechanical problems. One must take into consideration that the laws presented as the laws are material point. That is a conception of Euler because Nowton spoke about the body in the directed linear motion. The using of mentioned law is possible for all mechanical systems if we take into consideration that every system may be considered as a set of material points.

In that way it is possible to express the equations of motion for rigid body, elastic

---

body, fluids and gases.

The second way to present the equations of motion is the result of the analytical mechanics. The idea of analytic mechanics is perform the mechanics laws as one general equation of motion which may described all the mechanical phenomena.

The founder of this conception was Lagrange who applicated also the concepts of d'Alembert and Bernoulli.

Langrange preparated (1788) the mathematical basis of analytical mechanics concepts.

As was giving the well known Lagrange's equation, he present also the d'Alemberts principle some concepts of d'Alembert, Bernoulli and Varignon.

The development of analytical mechanics is connected with the names of Jourdain, Gauss, Appel, Maggi, Hamilton and others.

Using the concepts of analytical mechanics one can present the equation of motion for all mechanical models. The way is independent of Newtons law.

The analytical mechanics gives several number of methods.

The target of this work is to present the one way prefered by the author. The autor gives also the concept of using the presented method as the base of computer aided derivation of equations of motion of the mechanical system.

## 2. GENERAL MECHANICS EQUATION OF THE SYSTEM OF MATERIAL POINTS.

We take into consideration a constrained system of n material points having masses $m_i$ and undergoing forces $\mathbf{F}_i$. The position of each point is described by the position vector:

$$\mathbf{r}_i = \mathbf{r}_i(t) \quad i = 1...n \tag{1}$$

The holonomic constraints equations are:

$$f_\alpha(\mathbf{r}_i, t) = 0 \quad \alpha = 1...a \tag{2}$$

The nonholonomic constraints linearly depending on velocities are:

$$\varphi_\beta = \sum_{v=1}^{n} \mathbf{A}_v^{(\beta)} \mathbf{v}_v + \beta_\beta = 0 \qquad \beta = 1...b \tag{3}$$

We introduce the generalised coordinates $q$. Using the holonomic constraints equations (2) we may describe the generalised coordinates as functions of initial coordinates:

$$q_j = q_j(\mathbf{r}_i) \quad j = 1...s \tag{4}$$

Then we introduced the quasi-coordinates $\lambda$. Using the nonholonomic constraints equations we may describe the generalised coordinates as functions of quasi coordinates

$$q_j = q_j(\lambda_r) \quad r = 1...p \quad p = s - b = n - a - b \tag{5}$$

This formulation is formal. Usually we choose the quasi-velocities and formulate the similar as (5) relation between the quasi-velocities and generalised velocities.

After using the d'Alemberts, Jourdain's or Gauss' principles [1] we obtain in the case of holonomic system the equations of motion as follows:

$$Q_j + Q_{Bj} = 0 \quad j = 1 \cdots s \tag{6}$$

where:

$$Q_j = \sum_{i=1}^{n} F_i C_{ij} \quad \text{(generalised force)} \tag{7}$$

$$Q_{Bj} = \sum_{i=1}^{n} F_{Bi} \, C_{ij} \quad \text{(generalised inertial force)} \tag{8}$$

$$F_{Bi} = -m_i p_i \tag{9}$$

$$C_{ij} = \frac{\partial r_i}{\partial q_j} = \frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial p_i}{\partial \ddot{q}_j} \tag{10}$$

In the case of nonholonomic system the equation of motion has the form:

$$\Phi_r + \Phi_{Br} = 0 \quad r = 1 \cdots p \tag{11}$$

where:

$$\Phi_r = \sum_{j=1}^{s} Q_j D_{jr} \tag{12}$$

(quasi – generalised force)

$$\Phi_{sr} = \sum_{j=1}^{s} Q_{Bj} D_{jr} \quad r = \cdots - p \tag{13}$$

(quasi-generalised inertial force)
and

$$D_{jr} = \frac{\partial q_q}{\partial \dot{\lambda}_r} = \frac{\partial q_j}{\partial \dot{\lambda}} = \frac{\partial \ddot{q}_j}{\partial \dddot{\lambda}_j} \tag{14}$$

### 3. THE GENERAL MECHANICS EQUATION FOR THE SYSTEM OF RIGID BODIES WITH HOLONOMIC AND NONHOLONOMIC CONSTRAINS.

We take into consideration the system of $k$ rigid bodies; which the movements are constrained by means of holonomic constraints and nonholonomic dependent linearly from velocity. The each body denoted by index $r$ changing from 1 to $k$, we take as the set of material points denoted by index $i$ changing from 1 do $n$; $n$ may be unlimited, than we replace the adequate sums through the integrals. The position of each point is described by the position vector $\mathbf{r}_{ir}$ extended of the origin 0 of the reference frame. On each point having the mass $m_{ir}$ acts the force $\boldsymbol{F}_{ir}$. The velocity of the point is $\boldsymbol{v}_{ir}$, and the acceleration is $\boldsymbol{p}_{ir}$.

The general mechanics equation we can obtain using one of the general mechanics principles. When the system has the nonholonomic constraints it is usefull to start from the Jourdains principle.

We can use than the quasivelocities instead of quasicoordinates which describing is difficult or impossible.

Using the Jourdains principle in the analysed system of rigid bodies we obtain

$$\sum_{r=1}^{k} \sum_{i=1}^{n} (\mathbf{F}_{ir} - m_{ir}\mathbf{p}_{ir})\delta\mathbf{v}_{ir} = 0 \tag{15}$$

The velocity of the arbitrary point of the body can be formulated as the sum of the velocity of arbitrary chosen point $A_r$ of the body $r$ and the velocity caused by the revolution of the body $r$ around the axis going through point $A_r$

$$\mathbf{v}_{ir} = \mathbf{v}_{Ar} + \boldsymbol{\omega}_r \mathrm{x} \boldsymbol{\rho}_i \tag{16}$$

The virtual increments of the velocities can be presented in the form

$$\delta\mathbf{v}_{ir} = \delta\mathbf{v}_{Ar} + \delta\boldsymbol{\omega}_r \mathrm{x} \boldsymbol{\rho}_i \tag{17}$$

Substitution of this expression into equation (15) leads to equation:

$$\sum_{r=1}^{k} \sum_{i=1}^{n} (\mathbf{F}_r - m_{ir}\mathbf{p}_{ir})(\delta\mathbf{v}_{Ar} + \delta\boldsymbol{\omega}_r \mathrm{x} \boldsymbol{\rho}_{ir}) = 0 \tag{18}$$

This equation we can transform to the form:

$$\sum_{r=1}^{k} \sum_{i=1}^{n} \{(\mathbf{F}_{ir} - m_{ir}\mathbf{p}_{ir})\delta\mathbf{v}_{Ar} + [\boldsymbol{\rho}_{ir} \times (\mathbf{F}_{ir} - m_{ir}\mathbf{p}_{ir})]\delta\boldsymbol{\omega}_r\} = 0 \tag{19}$$

We introduce the notation:

$$\sum_{i=1}^{n} \mathbf{F}_{ir} = \mathbf{S}_r \tag{20}$$

$$\sum_{i=1}^{n} (-m_{ir}\mathbf{p}_{ir}) = \mathbf{S}_{Br} \tag{21}$$

$$\sum_{i=1}^{n} (\boldsymbol{\rho}_{ir} \times \mathbf{F}_{ir}) = \mathbf{M}_r \tag{22}$$

$$\sum_{i=1}^{n} [\boldsymbol{\rho}_{ir} \times (-m_{ir}\mathbf{p}_{ir})] = \mathbf{M}_{Br} \tag{23}$$

Where:
$\mathbf{S}_r$ – the sum of forces acting on body $r$,
$\mathbf{S}_{Br}$ – the sum of inertial forces of the body $r$,
$\mathbf{M}_r$ – moment of the forces acting on body $r$ about origin 0,
$\mathbf{M}_{Br}$ – moment of the inertial forces of the body $r$ about origin 0.

The general mechanics equation we can present in the form

$$\sum_{r=1}^{k} [(\mathbf{S}_r + \mathbf{S}_{Br})\delta\mathbf{v}_{Ar} + (\mathbf{M}_r + \mathbf{M}_{Br})\delta\boldsymbol{\omega}_r] = 0 \tag{24}$$

The equations of holonomic constraints we express through velocities, after differentiation of the both sides of equations of geometric constraints.
After that, we obtain the holonomic constraints equation in the form:

$$\varphi_l(...\mathbf{v}_{AR}...,\boldsymbol{\omega}_r...) = 0 \quad\quad l = 1...a \tag{25}$$

The equations of the nonholonomic constraints linearly dependent on velocities are:

$$\vartheta_t = \sum_{r=1}^{k}(\mathbf{A}_r^{(t)}\mathbf{v}_{AR} + \mathbf{B}_r^{(t)}\boldsymbol{\omega}) + \beta_t = 0 \quad t = 1...b \tag{26}$$

We chose the system of generalised coordinates $q_j$ ($j = 1...s$).

After using the equation of motion we can express the velocities $\mathbf{v}_{AR}$ and $\omega_r$ as functions of generalized velocities

$$\mathbf{v}_{Ar} = \mathbf{v}_{Ar}(\dot{q}_1...\dot{q}_j...\dot{q}_s) \tag{27}$$

$$\boldsymbol{\omega}_r = \boldsymbol{\omega}_r(\dot{q}_1...\dot{q}_j...\dot{q}_s) \tag{28}$$

The virtual increments of velocities can be presented as follows:

$$\delta\mathbf{v}_{Ar} = \sum_{j=1}^{s}\frac{\partial\mathbf{v}_{Ar}}{\partial\dot{q}_j}\delta\dot{q}_j = \sum_{j=1}^{s}\mathbf{C}_{rj}\delta\dot{q}_j \tag{29}$$

$$\delta\boldsymbol{\omega}_r = \sum_{j=1}^{s}\frac{\partial\boldsymbol{\omega}_r}{\partial\dot{q}_j}\delta\dot{q}_j = \sum_{j=1}^{s}\mathbf{E}_{rj}\delta\dot{q}_j \tag{30}$$

The general equation of motion can be then presented in the form:

$$\sum_{r=1}^{k}\sum_{j=1}^{s}[(\mathbf{S}_r + \mathbf{S}_{Br})\mathbf{C}_{rj} + (\mathbf{M}_r + \mathbf{M}_{Br})\mathbf{E}_{rj}]\delta\dot{q}_j = 0 \tag{31}$$

We introduce the generalized forces and generalized inertial forces as follows

$$Q_j = \sum_{r=1}^{k}(\mathbf{S}_r\mathbf{C}_{rj} + \mathbf{M}_r\mathbf{E}_{rj}) \quad j = 1...s \tag{32}$$

$$Q_{Bj} = \sum_{r=1}^{k}(\mathbf{S}_{Br}\mathbf{C}_{rj} + \mathbf{M}_{Br}\mathbf{E}_{rj}) \quad j = 1...s \tag{33}$$

The general mechanics equation (31) has then the form:

$$\sum_{j=1}^{s}(Q_j + Q_{Bj})\delta\dot{q}_j = 0 \tag{34}$$

When we have the system with holonomic constrains only the generalized velocities are independent and the equation (34) is equivalent to the set of $s$ equations:

$$Q_j + Q_{bj} = 0 \quad j = 1...s \tag{35}$$

This is the simplest possible form of equations of motion of the holonomic system of rigid bodies.

When the constraints are nonholonomic the generalised velocities are dependent and the formulation (35) is not valid. The nonholonomic constraints equations (26) after introducing the general velocities and coordinates have the form:

$$\vartheta_t = \sum_{r=1}^{s}a_{st}\dot{q}_j + a_t = 0 \quad t = 1...b \tag{36}$$

We introduce the system of quazicoordinates $\lambda$ when the quazivelocities $\dot{\lambda}_m$ are the

arbitrary chosen functions of generalized velocities

$$\dot{\lambda}_m = \dot{\lambda}_m(\dot{q}_j) = \sum_{j=1}^{s} a_{mj}\dot{q}_j + a_{j0} \qquad m = 1...z \tag{37}$$

The coefficients $a_{mj}$ may be functions of generalised coordinates $q_j$. The number of degrees of freedom is:

$$z = a - b = n - a - b \tag{38}$$

The generalized velocities can expressed as functions of quazivelocities

$$\dot{q}_j = \dot{q}_j(\dot{\lambda},...\dot{\lambda}_m,...\dot{\lambda}_z) \qquad j = 1...s \tag{39}$$

The virtual increments of the generalised velocities may be presented in the form:

$$\delta\dot{q}_j = \sum_{m=1}^{z} \frac{\partial\dot{q}_j}{\partial\dot{\lambda}_m}\delta\dot{\lambda}_m = \sum_{m=1}^{z} D_{jm}\delta\dot{\lambda}_m \tag{40}$$

Formulating the general mechanics equation (31) in the system of quazicoordinates we come to expression:

$$\sum_{j=1}^{s}\sum_{m=1}^{z}(Q_j + Q_{Bj})D_{jm}\delta\dot{\lambda}_m \tag{41}$$

We introduce the quazigeneralized force that means the generalized force concerning the quazicoordinates and the adequate quazigeneralized inertial force

$$\Phi_m = \sum_{j=1}^{s} Q_j D_{jm} = \left[\sum_{r=1}^{k}\sum_{j=1}^{s}(\mathbf{S}_r\mathbf{C}_{rj} + \mathbf{M}_r\mathbf{E}_{rj})\right]D_{jm} \tag{42}$$

$$\Phi_{Bm} = \sum_{j=1}^{s} Q_{Bj} D_{jm} = \left[\sum_{r=1}^{k}\sum_{j=1}^{s}(\mathbf{S}_r\mathbf{C}_{rj} + \mathbf{M}_r\mathbf{E}_{rj})\right]D_{jm} \tag{43}$$

The general mechanics equation (41) we can write in the form:

$$\sum_{m=1}^{z}(\Phi_m + \Phi_{Bm})\delta\dot{\lambda}_m = 0 \tag{44}$$

Because the virtual increments of quazivelocities are independent, the equation (44) is equivalent to the set of equations:

$$\Phi_m + \Phi_{Bm} = 0 \qquad m = 1...z \tag{45}$$

That is the simplest possible form of the equations of motion for the system with holonomic and nonholonomic constraints.

### 4. COMPUTER AIDED DERIVATION OF EQUATION OF MOTION.

Obtaining equations of motion corresponding to a given system can be achieved by making use of MATHEMATICA package [2,3]. To make the task easier one should follow the given procedure containing the following steps:

1. Drawing a figure presenting the given mechanical system (made by operator).
2. Putting down a table of material points masses, coordinates and forces (operator).
3. Choosing a system of generalised coordinates (operator).
4. Formulating constraints equations (operator).
5. Solving the constraints equations and describing the initial coordinates as functions of generalised coordinates (MAT).
6. Calculating the second derivatives expressed in terms of generalised coordinates (MAT) from the formulas (8) and $\mathbf{C}_{rj}$ and $\mathbf{E}_{rj}$ from the formulas (29) and (30)
7. Calculating the coefficient $\mathbf{C}_{ij}$, $\mathbf{E}_{rj}$ and presenting into matrices (MAT).
8. Calculating the generalised forces and generalised inertial forces (MAT).
9. If the constraints are only holonomic we obtain the equation of motion of the system. When the constraints are nonholonomic we prepare the next steps (MAT).
10. We assume the system of quasi-velocities (operator).
11. Formulating the equations of nonholonomic constraints.
12. Solving the constraints equation and describing the generalised velocities as functions of quasi-velocities (MAT).
13. Calculating the second derivatives expressed in terms of quasi-coordinates (MAT).
14. Calculating the coefficient $D_{jr}$ from the formulas (41) and presenting into matrices (MAT).
15. Calculating the quasi-generalised forces and quasi-generalised inertial forces (MAT).
16. Deriving the equations of motion of nonholonomic system.

## 5. EXAMPLE 1

Two points move on the plain. There are connected with rigid, massless rode. The velocity of point 2 must have a direction along the rode, Fig. 1.
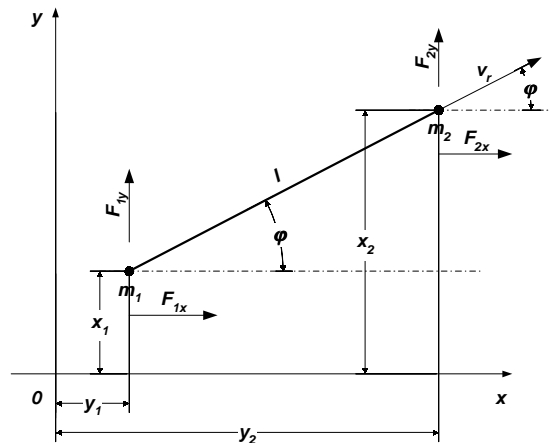


Fig. 1.

The derivation of the equation of motion goes as follow.

```
ClearAll[ab,"m*","x*","y*","z*","f*","q*"];
ab={{m1,x1,y1,0,F1x,F1y,0},{m2,x2,y2,0,F2x,F2y,0}};
n=.; n=Length[ab];
m=.; m=Array[0,n];
x=.; x=Array[0,n];
y=.; y=Array[0,n];
z=.; z=Array[0,n];
fx=.; fx=Array[0,n];
fy=.; fy=Array[0,n];
fz=.; fz=Array[0,n];
 For[k=1,k<=n,k++, m[[k]]=ab[[k,1]]; x[[k]]=ab[[k,2]];
     y[[k]]=ab[[k,3]]; z[[k]]=ab[[k,4]]; fx[[k]]=ab[[k,5]];
     fy[[k]]=ab[[k,6]]; fz[[k]]=ab[[k,7]]];
q=.; q={q1,q2,q3};
x1=q1; y1=q2; fi=q3;
rw1=(y2-y1)-L*Sin[fi];
rw2=(x2-x1)-L*Cos[fi];
SetAttributes[L,Constant];
rw1=Solve[rw1==0,y2];
y2=y2/.rw1;
rw2=Solve[rw2==0,x2];
x2=x2/.rw2;
l=.; l=Length[q];
cx=.;cx=Table[D[x[[i]],q[[j]]],{i,n},{j,l}];
cy=.;cy=Table[D[y[[i]],q[[j]]],{i,n},{j,l}];
cz=.;cz=Table[D[z[[i]],q[[j]]],{i,n},{j,l}];
qq=.;qq=Array[0,l];
For[k=1,k<=l,k++,
     qq[[k]]=Sum[fx[[i]]*cx[[i,k]],{i,n}]+
     Sum[fy[[i]]*cy[[i,k]],{i,n}]+
     Sum[fz[[i]]*cz[[i,k]],{i,n}]];
qqb=.;qqb=Array[0,l];
For[k=1,k<=l,k++,
     qqb[[k]]=Sum[m[[i]]*Dt[Dt[x[[i]]]]*cx[[i,k]],{i,n}]+
     Sum[m[[i]]*Dt[Dt[y[[i]]]]*cy[[i,k]],{i,n}]+
     Sum[m[[i]]*Dt[Dt[z[[i]]]]*cz[[i,k]],{i,n}]];
qq=Simplify[qq]; qqb=Simplify[qqb];
qqb=-qqb;
Print[""]
Print["Cxij","=",cx]
Print["Cyij","=",cy]
Print["Czij","=",cz]
Print[""]
Print["Qj","=",qq]
Print["Qbj","=",qqb]
8=.; 8={81,82};
```

```
q1/:Dt[q1]=81;
q3/:Dt[q3]=82;
rh1=Dt[x2]*Tan[fi]-Dt[y2];
rh1=Solve[rh1==0,Dt[q2]];
q2/:Dt[q2]=Dt[q2]/.rh1;
r=.; r=Length[8];
djr=.;djr=Table[D[Dt[q[[j]]],8[[i]]],{i,r},{j,l}];
djr=Simplify[djr];
Print[""]
Print["Djr","=",djr]
Mr=.; Mr=Array[0,r];
For[k=1,k<=r,k++,
      Mr[[k]]=Sum[qq[[j]]*djr[[k,j]],{j,l}]];
xMbr=.;xMbr=Array[0,r];
For[k=1,k<=r,k++,
      xMbr[[k]]=Sum[qqb[[j]]*djr[[k,j]],{j,l}]]
Mr=Simplify[Mr];
xMbr=Simplify[xMbr];
Print[""]
Print["Mr","=",Mr]
Print["Mbr","=",xMbr]
Print[""]
rr=.;rr=Array[0,r];
For[k=1,k<=r,k++,
      rr[[k]]=Mr[[k]]+xMbr[[k]];
      rr[[k]]=Simplify[rr[[k]]];
      Print["Equation ",k];
      Print["0=",rr[[k]]];Print[""]]
```

$Cxij=\{\{1,0,0\},\{\{1\},\{0\},\{-L\ Sin[q3]\}\}\}$
$Cyij=\{\{0,1,0\},\{\{0\},\{1\},\{L\ Cos[q3]\}\}\}$
$Czij=\{\{0,0,0\},\{0,0,0\}\}$

$Qj=\{\{F1x+F2x\},\{F1y+F2y\},\{L(F2yCos[q3]-F2xSin[q3])\}\}$
$Qbj=\{\{Lm2Cos[q3]Dt[q3]^2-(m1+m2)Dt[Dt[q1]]+Lm2Dt[Dt[q3]]Sin[q3]\},$
$\quad \{-(m1+m2)Dt[Dt[q2]]-Lm2(Cos[q3]Dt[Dt[q3]]-Dt[q3]^2Sin[q3])\},$
$\quad \{-Lm2(Cos[q3]Dt[Dt[q2]]+LDt[Dt[q3]]-Dt[Dt[q1]]\ Sin[q3])\}\}$
$Djr=\{\{1,\{Tan[q3]\},0\},\{0,\{-LSec[q3]\},1\}\}$

$Mr=\{\{F1x+F2x+(F1y+F2y)Tan[q3]\},\{L(F2yCos[q3]-(F1y+F2y)Sec[q3]-F2x\ Sin[q3])\}\}$
Mbr={{{-
$\frac{1}{2}Sec[q3]^3(2(m1+m2)Cos[q3]Dt[81]+82(-Lm182-2Lm282+Lm182Cos[2q3]+$
$\quad 2m181Sin[q3]+2m281Sin[q3])-L(m1+m2)Dt[82]Sin[2q3])\}\},$
$\{\{$
$\frac{1}{2}Lsec[q3]^3(-LCos[q3](2m1+m2-m2Cos[2q3]Dt[82]+82(2m1181+m281-$
$\quad m281Cos[2q3]-2Lm182Sin[q3]-2Lm282Sin[q3])+(m1+m2)Dt[81]Sin[2q3])\}\}\}$

**Equation 1**

$0=\{\{F1x+F2x-$

1 over $2Sec[q3]^3(2(m1 + m2)Cos[q3]Dt[81]+82(-Lm182-2Lm282+Lm182Cos[2q3]+$
$2m181Sin[q3]+2m281Sin[q3])-L(m1+m2)Dt[82]Sin[2q3])+(F1y+F2y)Tan[q3]\}\}$

**Equation 2**

$0=\{\{L(F2yCos[q3]-(F1y+F2y)Sec[q3]-F2xSin[q3])+$

1 over $2LSec[q3]^3(-LCos[q3](2m1+m2-$
$m2Cos[2q3])Dt[82]+82(2m181+m281-m281Cos[2q3]-2Lm182Sin[q3]-$
$2Lm282Sin[q3])+(m1 + m2)Dt[81]Sin[2q3])\}\}$

EXAMPLE 2

The sledge (rigid body) is moving on the plain. The velocity of the center of mass has the directing of the symetry of the slegde, Fig. 2.
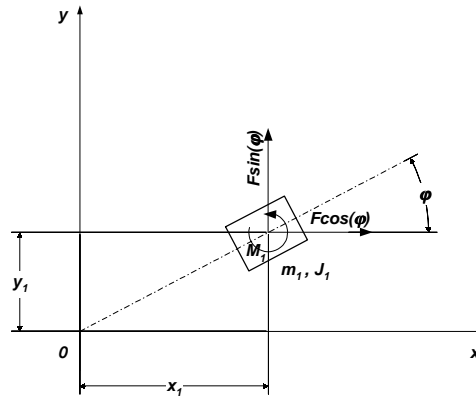


Fig. 2.

The derivation of. equations of motion goes as follows.
```
ClearAll[ab,"m*","x*","y*","f*","q*","J","M"];
ab={{m1,J1,x1,y1,F1*Cos[v],F1*Sin[v],M1}};
n=.; n=Length[ab];
m=.; m=Array[0,n];
J=.; J=Array[0,n];
M=.; M=Array[0,n];
x=.; x=Array[0,n];
y=.; y=Array[0,n];
z=.; z=Array[0,n];
fx=.; fx=Array[0,n];
fy=.; fy=Array[0,n];
For[k=1,k<=n,k++, m[[k]]=ab[[k,1]];J[[k]]=ab[[k,2]]; x[[k]]=ab[[k,3]];
    y[[k]]=ab[[k,4]]; fx[[k]]=ab[[k,5]];fy[[k]]=ab[[k,6]]; M[[k]]=ab[[k,7]]];
q=.; q={q1,q2,q3};
```

```
x1=q1; y1=q2; fi=q3;
l=.; l=Length[q];
cx=.; cx=Table[D[D[x[[i]]],D[q[[j]]]],{i,n},{j,l}];
cy=.; cy=Table[D[D[y[[i]]],D[q[[j]]]],{i,n},{j,l}];
ee=.; ee=Table[D[D[fi],D[q[[j]]]],{i,n},{j,l}];
qq=.; qq=Array[0,l];
For[k=1,k<=l,k++,
     qq[[k]]=Sum[fx[[j]]*cx[[j,k]],{j,n}]+
     Sum[fy[[j]]*cy[[j,k]],{j,n}]+
     Sum[M[[j]]*ee[[j,k]],{j,n}]];
qqb=.; qqb=Array[0,l];
For[k=1,k<=l,k++,
     qqb[[k]]=Sum[m[[j]]*Dt[Dt[x[[j]]]]*cx[[j,k]],{j,n}]+
     Sum[m[[j]]*Dt[Dt[y[[j]]]]*cy[[j,k]],{j,n}]+
     Sum[J[[j]]*Dt[Dt[fi]]*ee[[j,k]],{j,n}]];
qq=Simplify[qq]; qqb=Simplify[qqb];
qqb=-qqb;
Print[""]
Print["Cxrj","=",cx]
Print["Cyrj","=",cy]
Print["Erj","=",ee]
Print[""]
Print["Qj","=",qq]
Print["Qbj","=",qqb]
8=.; 8={81,82};
q1/:Dt[q1]=81;
q3/:Dt[q3]=82;
rw=Dt[x1]*Tan[fi]-Dt[y1];
rw=Solve[rw==0,Dt[y1]];
q2/:Dt[q2]=Dt[q2]/.rw;
r=.; r=Length[8];
drj=.; drj=Table[D[Dt[q[[j]]],8[[i]]],{i,r},{j,l}];
drj=Simplify[drj];
Print[""]
Print["Drj","=",drj]
M=.; Mr=Array[0,r];
For[k=1,k<=r,k++,
     Mr[[k]]=Sum[qq[[j]]*drj[[k,j]],{j,l}]];
xMbr=.; xMbr=Array[0,r];
For[k=1,k<=r,k++,
     xMbr[[k]]=Sum[qqb[[j]]*drj[[k,j]],{j,l}]]
Mr=Simplify[Mr];
xMbr=Simplify[xMbr];
Print[""]
Print["Mr","=",Mr]
Print["Mbr","=",xMbr]
```

```
Print[""]
rr=.; rr=Array[0,r];
For[k=1,k<=r,k++,
      rr[[k]]=Mr[[k]]+xMbr[[k]];
      rr[[k]]=Simplify[rr[[k]]];
      Print["Equation ",k];
      Print["0=",rr[[k]]];Print[""]]

Cxrj={{1,0,0}}
Cyrj={{0,1,0}}
Erj={{0,0,1}}

Qj={F1Cos[v],F1Sin[v],M1}
Qbj={-m1Dt[Dt[q1]],-m1Dt[Dt[q2]],-J1Dt[Dt[q3]]}

Drj={{1,{Tan[q3]},0},{0,{0},1}}

Mr={{F1(Cos[v]+Sin[v]Tan[q3])},{M1}}
Mbr={{-m1)Sec[q3]^3(Cos[q3]Dt[81]+8182Sin[q3]),{-J1Dt[82]}}
```

### Equation 1

$0=\{-m1Sec[q3]^3(Cos[q3]Dt[81]+8182Sin[q3])+F1(Cos[v]+Sin[v]Tan[q3])\}$

### Equation 2

$0=\{M1-J1Dt[82]\}$

The computer preparation of the examples made by J. Pankiewicz.

REFERENCES

1.  Z. Osiński., *Remarks concerning the general equation of mechanics* (in polish), Mechanika, T. 16,Z3,1997, Kraków
2.  Z. Osiński, J. Pankiewicz., *A method of computer aided derivation of the equations of motion of a system of material points*, Proc. Of the sixth Int. Congr. On Sound and vibration, 1999, Copenhagen.
3.  J.S. Wolfram, *The Mathematic Book*, Wolfram Media/Combridge Univesity Press 1996.

# OPŠTA JEDNAČINA ZA MATERIJALNI SISTEM SA HOLONOMNIM I NEHOLONOMNIM OGRANIČENJIMA

## Zbigniew Osiński

*Autor je počeo prezentacijom dva klasična načina opisivanja jednačine kretanja materijalnog sistema. Jedan način je korišćenje Newton-ovog zakona, drugi je u vezi sa principima anačitičke mehanike. Autor predstavlja svoj koncept predstavljanja opštih mehaničkih jednačina. Ovakvo predstavljanje se zasniva na ekvivalentnosti generalisanih sila i generalisanih D'Alambert-ovih sila (sila inercije). Predstavljanje važi za holonomne i neholonomne sisteme. Predstavlja se i metoda izvođenja jednačine kretanja uz računarsku podršku. Dati su i primeri.*