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ASYMPTOTIC APPROACHES TO ANALYSIS OF STRONGLY NON-LINEAR AND NON-SMOOTH SYSTEMS

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Abstract. *Three important problems of mechanics are addressed. The first corresponds to analytical prediction of a suitable choice of a delay loop parameters in order to control a transitional (perturbed) behaviour of a periodic vibro-impact process modelled by one degree-of-freedom system. The second presents an application of the Melnikov's method to stick-slip chaos prediction in periodically driven self-excited oscillator. The third one illustrates an application of a so called "small δ method" to analysis of strong non-linear oscillations using example of a mathematical pendulum.*

1. INTRODUCTION

It appears that a good prognosis for a future of asymptotic approaches can be formulated. Many of researches realized that only numerical investigation is certainly not a panaceum for a solution to many fundamental problems including those taken from engineering science [1-13]. In recent years the asymptotical approaches have been highly developed from both qualitative and quantitative points of view. The most interesting directions refer to detection of new non-trivial small (perturbation) parameters (even in classical problems) and application of various methods (follow, for instance a recently increased applications of Padé approximants).

In this work only three points of application of perturbation oriented technique (supported by symbolic computations using *Mathematica* and verified by numerical simulations) are illustrated and discussed. However, they touch the most important directions of nowadays mechanics; i.e. control of discontinuous systems, stick-slip chaos prediction and analysis of strongly nonlinear systems.

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2. STABILITY IMPROVEMENT OF PERIODIC VIBRO-IMPACT PROCESSES [2]

The analysed one-degree-of-freedom vibro-impact system is governed by the following equation

$$\ddot{x} + c\dot{x} + \alpha^2 x = P_0 \cos\omega t + A[x(t) - x(t-T)] + B[\dot{x}(t) - \dot{x}(t-T)] \quad \text{for} \quad x < s, \tag{1}$$

$$\begin{cases} x_+ = x_- \\ \dot{x}_+ = -R_r \dot{x}_- \end{cases} \quad \text{for} \quad x \geq s,$$

where:

$$P_0 = \frac{y_0 k_2}{m}, \quad c = \frac{c_1}{m}, \quad \alpha^2 = \frac{(k_1 + k_2)}{m}, \quad A = \frac{k_2 a_1}{m}, \quad B = \frac{k_2 b_1}{m}$$

and $T = 2k\pi/\omega$ is the period of the considered periodic orbit being stabilised ($k = 1,2,...$ - number of periods of the excitating force between two successive impacts). Above y_0 and ω are the amplitude and the frequency of kinematic excitation; k_1, k_2 are the stiffnesses coefficients; c_1 is the damping coefficient; m is the mass; a_1, b_1 are control coefficients, $R_r \leq 1$ denotes the restitution coefficient.

A delay loop is switched off where perturbations are not present. In the case of perturbations the controller causes the perturbations to vanish more quickly than in the case without control. The problem of analytical estimation of the influence of control coefficients for periodic orbit stability cannot be solved in the standard way. Here we propose the following approach. Because in fact the differencies $x(t) - x(t-T)$ and $\dot{x}(t) - \dot{x}(t-T)$ are small, we express them by introducing the small parameter ε, which allows one then to apply the Krilov-Bogoljubov-Mitropolskij (KBM) method formally and next to take $\varepsilon = 1$.

We assume damping of the same order as ε, and from (1) we obtain

$$\ddot{x} + \alpha^2 x = P_0 \cos\omega t + \varepsilon A[x(t) - x(t-T)] + \varepsilon B\left[\left(1 - \frac{c}{B}\right)\dot{x}(t) - \dot{x}(t-T)\right]. \tag{2}$$

Introducing

$$x = z + \frac{P_0}{\alpha^2 - \omega^2}\cos\omega t, \tag{3}$$

we get

$$\ddot{z} + \alpha^2 z = \varepsilon f(a, \eta, \psi), \tag{4}$$

where:

$$\varepsilon f(a,\eta,\psi) = \varepsilon A\left[z + \frac{P_0}{\alpha^2 - \omega^2}\cos\omega t - z(t-T) - \frac{P_0}{\alpha^2 - \omega^2}\cos\omega(t-T)\right] +$$
$$+ \varepsilon B\left[\left(1 - \frac{c}{B}\right)\left(\dot{z} - \frac{P_0\omega}{\alpha^2 - \omega^2}\sin\omega t\right) - \dot{z}(t-T) + \frac{P_0\omega}{\alpha^2 - \omega^2}\sin\omega(t-T)\right],$$

$$\eta = \omega t, \quad \psi = \alpha t.$$

Using the KBM method we have truncated the ε series up to the order $O(\varepsilon)$ and we have obtained

$$z = a(t)\cos\psi(t)\,, \tag{5}$$

where:

$$\begin{aligned} \frac{da}{dt} &= \frac{Aa}{2\alpha}\sin\alpha T + \frac{1}{2}(B-c)a - \frac{Ba}{2}\cos\alpha T, \\ \frac{d\psi}{dt} &= \alpha - \frac{A}{2\alpha} + \frac{A}{2\alpha}\cos\alpha T + \frac{1}{2}B\sin\alpha T. \end{aligned} \tag{6}$$

For $A = B = 0$ we get the uncontrolled solution, which testifies the validity of our approach.

After integration of equations (6) we get

$$\begin{aligned} &a(t) = C_0 e^{Rt}, \quad \psi(t) = \alpha_0 t + \theta_0, \\ &R = \frac{1}{2\alpha}\{A\sin\alpha T - \alpha[c - B(1-\cos\alpha T)]\}, \\ &\alpha_0 = \alpha - \frac{A}{2\alpha}(1-\cos\alpha T) + \frac{B}{2}\sin\alpha T. \end{aligned} \tag{7}$$

Therefore, we analyse the following equivalent solution

$$x = \frac{P_0}{\alpha^2-\omega^2}\cos(\omega t+\varphi) + e^{Rt}(C\cos\alpha_0 t + D\sin\alpha_0 t) \quad \text{for} \quad x < s, \tag{8}$$

$$\begin{cases} x_+ = x_- \\ \dot{x}_+ = -R_r \dot{x}_- \end{cases} \quad \text{for} \ \ x \ge s,$$

where:

$$C = C_0\cos\theta_0, \quad D = -C_0\sin\theta_0, \quad C_0 = \sqrt{C^2+D^2}$$

and

$$C = s - F\cos\varphi, \quad D = \frac{C}{\sin 2\beta\lambda}(e^{\beta c} - \cos 2\beta\lambda),$$

$$\cos\varphi = \frac{s}{F} - \frac{(R_r+1)\dot{x}_-\sin 2\beta\lambda}{\lambda F(2\cos 2\beta\lambda - e^{-\beta c} - e^{\beta c})},$$

$$\sin\varphi = \frac{-0.5(R_r+1)c\sin 2\beta\lambda + \lambda[(R_r-1)\cos 2\beta\lambda + e^{-\beta c} - R_r e^{\beta c}}{\lambda F\omega(2\cos 2\beta\lambda - e^{-\beta c} - e^{\beta c})}\dot{x}_-,$$

$$F = \frac{P_0}{\sqrt{(\alpha_0^2-\omega^2)^2 + c^2\omega^2}}, \quad \lambda = \sqrt{\alpha^2-\omega^2}, \quad \beta = \frac{\pi k}{\omega}.$$

From equation (8) it is seen that when $R < 0$ the assumed solution is stabilised more quickly in comparison to the case of $R = 0$. However, a problem of stability investigation of the vibro-impact state is much more subtle. Before the impact number l, the mass possesses the velocity x_{l-}. This causes the following perturbation solution to occur

$$x + \delta x_l = e^{R_{\tau_l}}[(C+\delta C_l)\cos\alpha_0\tau_l + (D+\delta D_l)\sin\alpha_0\tau_l] + \frac{P_0}{\alpha^2-\omega^2}\cos(\omega\tau_l + \varphi + \delta\varphi_l). \tag{9}$$

A new time τ is measured from the l-th impact $\tau_l = \tau + \delta\tau_l$. For example, the next impact occurs for $\tau_{l+1} = (2\pi l/\omega) + \delta T_1$, where δT_1 denotes the period $T = 2\pi/\omega$ perturbation.

After some calculations we get

$$\delta x_l = e^{R\tau}[-C\alpha_0\delta\tau_l \sin\alpha_0\tau + \delta C_l \cos\alpha_0\tau + D\alpha_0\delta\tau_l \cos\alpha_0\tau + \delta D_l \sin\alpha_0\tau + \\ + RC\delta\tau_l \cos\alpha_0\tau + RD\delta\tau_l \sin\alpha_0\tau] - a\delta\varphi_l \sin(\omega\tau+\varphi) - a\omega\delta\tau_l \sin(\omega\tau+\varphi),$$

$$\delta\dot{x}_l = e^{R\tau}[-R\alpha_0\delta\tau_l(D\cos\alpha_0\tau - C\sin\alpha_0\tau) + R\delta C_l\cos\alpha_0\tau + R\delta D_l \sin\alpha_0\tau + \\ + R^2C\delta\tau_l\cos\alpha_0\tau + R^2D\delta\tau_l\sin\alpha_0\tau - \alpha_0^2\delta\tau_l(C\cos\alpha_0\tau - D\sin\alpha_0\tau) - \\ - \alpha_0\delta C_l\sin\alpha_0\tau + \alpha_0\delta D_l\cos\alpha_0\tau - R\delta\tau_l C\alpha_0\sin\alpha_0\tau + R\delta\tau_l D\alpha_0\cos\alpha_0\tau] - \\ - a\omega\delta\varphi_l\cos(\omega\tau+\varphi) - a\omega^2\delta\tau_l\cos(\omega\tau+\varphi),$$

where: $a = P_0/(\alpha^2 - \omega^2)$.

The following boundary conditions are introduced:

$$\begin{aligned} l: &\quad \tau = 0,\ \delta\tau_l = 0,\ \delta x_l = 0,\ \delta\dot{x}_l = \delta\dot{x}_{l+} = -R_r\delta\dot{x}_{l-}, \\ l+1: &\quad \tau = \frac{2\pi k}{\omega} + \delta\tau_l,\ \delta\tau_l = \delta T_l,\quad \delta x_l = 0,\quad \delta\dot{x}_l = \delta\dot{x}_{(l+1)_-}. \end{aligned} \tag{10}$$

Then we get six equations as follows:

$$\begin{aligned} &\delta C_l - a\delta\varphi_l \sin\varphi = 0, \\ &R\delta C_l + \delta D_l\alpha_0 - a\omega\delta\varphi_l\cos\varphi = \delta\dot{x}_{l+}, \\ &\delta C_{l+1} - a\delta\varphi_{l+1}\sin\varphi = 0, \\ &R\delta C_{l+1} + \delta D_{l+1}\alpha_0 - a\omega\delta\varphi_{l+1}\cos\varphi = \delta\dot{x}_{(l+1)+}, \end{aligned} \tag{11}$$

$$e^{(\frac{2\pi k}{\omega}+\delta T_l)R}\left\{\alpha_0\delta T_l\left[D\cos\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0 - C\sin\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0\right]+ \right. \\ + \delta C_l\cos\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0 + \delta D_l\sin\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0 + \\ \left. + R\delta T_l C\cos\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0 + R\delta T_l D\sin\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0\right\} - \\ - a\delta\varphi_l\sin\left[\left(\frac{2\pi k}{\omega}+\delta T_l\right)\omega+\varphi\right] - a\omega\delta T_l\sin\left[\left(\frac{2\pi k}{\omega}+\delta T_l\right)\omega+\varphi\right] = 0,$$

$$e^{(\frac{2\pi k}{\omega}+\delta T_l)R}\left\{R\alpha_0\delta T_l\left[D\cos\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0 - C\sin\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0\right]+ \right. \\ + R\delta C_l\cos\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0 + R\delta D_l\sin\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0 +$$

$$
\begin{aligned}
&+R^2\delta T_l C\cos\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0+R^2\delta T_l D\sin\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0-\\
&-\alpha_0^2\delta T_l\left[C\cos\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0+D\sin\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0\right]-\\
&-\delta C_l\alpha_0\sin\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0+\delta D_l\alpha_0\cos\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0-\\
&-R\delta T_l C\alpha_0\sin\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0+R\delta T_l D\alpha_0\cos\left(\frac{2\pi k}{\omega}+\delta T_l\right)\alpha_0\Big\}-\\
&-a\omega\delta\varphi_l\cos\left[\left(\frac{2\pi k}{\omega}+\delta T_l\right)\omega+\varphi\right]-a\omega^2\delta T_l\cos\left[\left(\frac{2\pi k}{\omega}+\delta T_l\right)\omega+\varphi\right]=\delta\dot{x}_{(l+1)-}.
\end{aligned}
$$

After some calculations we have obtained the following equations

$$
\begin{aligned}
&\delta C_l-a\delta\varphi_l\sin\varphi=0,\\
&e^{2\beta R}\{\delta C_l\cos 2\beta\alpha_0+\delta D_l\sin 2\beta\alpha_0+\\
&+\frac{1}{\omega}(\delta\varphi_{l+1}-\delta\varphi_l)[(\alpha_0 D+RC)\cos 2\beta\alpha_0+(RD-\alpha_0 C)\sin 2\beta\alpha_0]\}-\delta C_{l+1}=0,\\
&R_r e^{2\beta R}\{\delta C_l(R\cos 2\beta\alpha_0-\alpha_0\sin 2\beta\alpha_0)+\delta D_l(R\sin 2\beta\alpha_0+\alpha_0\cos 2\beta\alpha_0)+\\
&+\frac{1}{\omega}(\delta\varphi_{l+1}-\delta\varphi_l)[(R^2C-\alpha_0^2C+2R\alpha_0 D)\cos 2\beta\alpha_0+\\
&+(R^2D-\alpha_0^2D-2R\alpha_0 D)\sin 2\beta\alpha_0]\}+R\delta C_{l+1}+\alpha_0\delta D_{l+1}-(R_r+1)a\omega\delta\varphi_{l+1}\cos\varphi=0,
\end{aligned}
\tag{12}
$$

where: $\beta=\pi k/\omega$.

Assuming that

$$
\delta\varphi_l=\delta\varphi_0+\sum_{i=1}^{l}\omega\delta T_i, \tag{13}
$$

and introducing

$$
\delta C_l=a_1\gamma^l,\quad \delta D_l=a_2\gamma^l,\quad \delta\varphi_l=a_3\gamma^l \tag{14}
$$

we get the following characteristic equation

$$
b_2\gamma^2+b_1\gamma+b_0=0, \tag{15}
$$

where:

$$
\begin{aligned}
&b_2=\alpha_0\{a\sin\varphi-\frac{1}{\omega}e^{2\beta R}[(\alpha_0 D+RC)\cos 2\beta\alpha_0+(RD-\alpha_0 C)\sin 2\beta\alpha_0]\},\\
&b_1=e^{2\beta R}\{\frac{1}{\omega}\alpha_0[(RC+\alpha_0 D)(\cos 2\beta\alpha_0-R_r e^{2\beta R})+(RD-\alpha_0 C)\sin 2\beta\alpha_0]-\\
&-(R_r+1)a\omega\cos\varphi\sin 2\beta\alpha_0+a\sin\varphi[(R_r-1)\alpha_0\cos 2\beta\alpha_0+(R_r+1)R\sin 2\beta\alpha_0]\},\\
&b_0=R_r\alpha_0 e^{4\beta R}[\frac{1}{\omega}(RC+\alpha_0 D)-a\sin\varphi].
\end{aligned}
\tag{16}
$$

The problem of stability is reduced to consideration of the second order characteristic Eq. (15). If the roots of Eq. (15) are $|\gamma_{1,2}| < 1$ then according to the assumed solutions (14) δC_1, δD_1 and $\delta\varphi_1$ approach zero for $1 \to +\infty$, and the solutions will be asymptotically stable. We can easily estimate the stability regions, which are defined by the following inequalities

$$\left|\frac{b_0}{b_2}\right| < 1 \quad \text{and} \quad \left|\frac{b_1}{b_0 + b_2}\right| < 1. \tag{17}$$

Taking into account equations (15) it is easy now to find parameters of the system (or a delay loop) which fulfil inequalities (17). Additionally, because of some mechanical reasons, we have $x(t) \le s$. The validity of our analytical approach has been testified by numerical simulations [2, 3].

3. Prediction of stick-slip chaos [4]

3.1 Analysed system and Melnikov's method

The analysed mechanical system with friction is governed by the equation

$$m\ddot{x} - k_1 x + k_2 x^3 = \varepsilon[\Gamma \cos\omega t - \theta(\dot{x} - v^*)], \tag{18}$$

where:

$$\theta(\dot{x} - v_*) = \theta_0 \operatorname{sign}(\dot{x} - v_*) - A(\dot{x} - v_*) + B(\dot{x} - v_*)^3. \tag{19}$$

In Eqs. (18) and (19) $\varepsilon > 0$ is the perturbation parameter and v_* corresponds to the velocity of the belt on which the considered mechanical system lies. θ_0, A, B are friction coefficients k_1 and k_2 are stiffness coefficients, whereas Γ and ω are the amplitude and frequency of excitation, respectively.

The equation (18) can be written transformed to the following set of first order ODE's

$$\begin{gathered} \dot{v} = ax - bx^3 + \varepsilon[\gamma\cos\omega t - T(v - v_*)], \\ \dot{x} = v, \quad a = \frac{k_1}{m},\ b = \frac{k_2}{m},\ \gamma = \frac{\Gamma}{m},\ T = \frac{\theta}{m},\ T_0 = \frac{\theta_0}{m},\ \alpha = \frac{A}{m},\ \beta = \frac{B}{m} \end{gathered} \tag{20}$$

where:

$$a = \frac{k_1}{m},\ b = \frac{k_2}{m},\ \gamma = \frac{\Gamma}{m},\ T = \frac{\theta}{m},\ T_0 = \frac{\theta_0}{m},\ \alpha = \frac{A}{m},\ \beta = \frac{B}{m}.$$

For $\varepsilon = 0$ we have the unperturbed system with the following Hamiltonian

$$H = \frac{1}{2}\left(v^2 - ax^2 + \frac{1}{2}bx^4\right), \tag{21}$$

and there are three critical points for $a,b > 0$: two centres at $\left(\pm\sqrt{a/b},0\right)$ and a hyperbolic saddle at the origin with the homoclinic orbit satisfying the equation

$$\frac{dx}{dt} = x\sqrt{a - b\frac{x^2}{2}} = \sqrt{a}\,x\sqrt{1 - \frac{b}{a}\frac{x^2}{2}} \tag{22}$$

Integrating (22) in respect to t, we obtain the following homoclinic orbit

$$x_0(t) = \pm\sqrt{\frac{2a}{b}}\operatorname{sech}\sqrt{a}t, \ v_0(t) = a\sqrt{\frac{2}{b}}\operatorname{sech}\sqrt{a}t\tanh\sqrt{a}t. \tag{23}$$

The Mielnikov's function for our case reads

$$\begin{aligned}M(t_0) = &\int_{-\infty}^{\infty}\{\gamma v_0(t-t_0)\cos\omega t + \\ &- v_0(t-t_0)[T_0 \operatorname{sign}[v_0(t-t_0)-v_*] - \alpha(v_0(t-t_0)-v_*) + \beta(v_0(t-t_0)-v_*)^3]\}dt,\end{aligned} \tag{24}$$

which can be transformed to the form

$$\begin{aligned}M(t_0) = &-a\sqrt{\frac{2}{b}}\gamma\sin t_0\int_{-\infty}^{\infty}(\operatorname{sech}\sqrt{a}\tau\tanh\sqrt{a}\tau\sin\omega\tau)d\tau - T_0 a\sqrt{\frac{2}{b}}\left\{\int_{-\infty}^{\infty}\left[\operatorname{sech}\sqrt{a}\tau\tanh\sqrt{a}\tau d\tau\times\right.\right.\\ &\left.\left.\times\operatorname{sign}(\operatorname{sech}\sqrt{a}\tau\tanh\sqrt{a}\tau - v_*)\right]d\tau\right\} - 4\beta\frac{a^4}{b^2}\int_{-\infty}^{\infty}(\operatorname{sech}\sqrt{a}\tau\tanh\sqrt{a}\tau)^4 d\tau + \\ &+3\beta v_* a^3\left(\frac{2}{b}\right)^{3/2}\int_{-\infty}^{\infty}\operatorname{sech}^3\sqrt{a}\tau\tanh^3\sqrt{a}\tau d\tau + 2(\alpha-3\beta v_*^2)\frac{a^2}{b}\int_{-\infty}^{\infty}\operatorname{sech}^2\sqrt{a}\tau\tanh^2\sqrt{a}\tau d\tau + \\ &+(\alpha-\beta v_*^2)v_* a\sqrt{\frac{2}{b}}\int_{-\infty}^{\infty}\operatorname{sech}\sqrt{a}\tau\tanh\sqrt{a}\tau d\tau,\end{aligned} \tag{25}$$

where: $\tau = t - t_0$. The integrals of (25) can be calculated using the *Mathematica* symbolic package and we finally obtain

$$\begin{aligned}M(t_0) = &-\pi\gamma\omega\sqrt{\frac{2}{b}}\operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{a}}\right)\sin\omega t_0 - \frac{16}{35}\beta\frac{a^4}{\sqrt{a}b^2} + \frac{4}{3}(\alpha-3\beta v_*^2)\frac{a^2}{\sqrt{a}b} + \\ &-\begin{cases}2T_0\sqrt{\frac{2a}{b}}\left[\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{b}{2a^2}v_*^2}} - \sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{b}{2a^2}v_*^2}}\right] & \text{for} \quad v_* < \frac{a}{\sqrt{2b}}, \\ 0 & \text{for} \quad v_* \geq \frac{a}{\sqrt{2b}}.\end{cases}\end{aligned} \tag{26}$$

The obtained formula is applicable in both cases, i.e. for stick-slip and slip-slip motions. The latter one corresponds to a zero additional value given in the brackets. The Melnikov criterion for occurrence of chaos is given by equation

$$\begin{aligned}\pi\gamma\omega\operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{a}}\right) > &\frac{16}{35}\beta\frac{a^4}{b\sqrt{2ab}} - \frac{4}{3}(\alpha-3\beta v_*^2)\frac{a^2}{\sqrt{2ab}} + \\ &+\begin{cases}2T_0 a\left[\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{b}{2a^2}v_*^2}} - \sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{b}{2a^2}v_*^2}}\right] & \text{for} \quad v_* < \frac{a}{\sqrt{2b}}, \\ 0 & \text{for} \quad v_* \geq \frac{a}{\sqrt{2b}},\end{cases}\end{aligned} \tag{27}$$

which allows to estimate a chaotic threshold in a weakly forced stick-slip oscillator.

In order to check the validity of our results, we have taken $a = b = 1$, $\alpha = \beta = T_0 = 0.3$, $\omega = 2$, $v_* = 0.4$, and we have obtained $\gamma_c = 0.65$. The *Mathematica* program and the MATLAB-SIMULINK package have been used to simulate the analysed system. We are going now to check numerically the validity of the analytical prediction. For this aim we have fixed $a = b = 1$, $\alpha = \beta = T_0 = 0.3$, $\omega = 2$, $v_* = 0.4$ and we have used γ as the control parameter. For $\gamma = 0.6$ we have obtained a periodic orbit, which with an increase in γ doubles its period. Closely to $\gamma_c = 0.65$ (analytical prediction) a very "slight" chaotic behaviour has been observed. For $\gamma = 1.2$ two-well potential chaos is shown. In all of the phase portraits the cusps corresponding to a sign change of the relative velocity, as well as short horizontal parts corresponding to the sticks during motions are visible. The obtained results illustrate practically a good agreement with the analytical prediction (see Fig. 1).

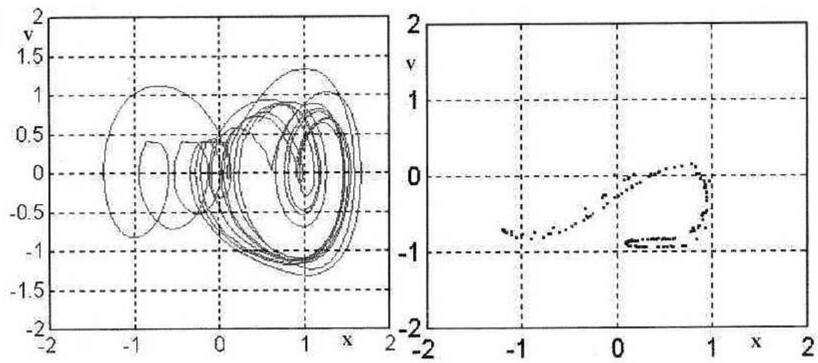

Fig. 1. Phase portrait and Poincaré map of a stick-slip chaos (see reference [4]).

Because we have found the cusp of the curve $\gamma(v_*)$ therefore we have tried to manipulate the parameters in such a way that the cusp can touch the horizontal coordinate. This corresponds to a special threshold for which we have $\gamma = 0$, i.e. an autonomous system. Then, an infinitely small periodic perturbation will lead the system to chaos. To achieve that we have used parameter the b. For example, for b = 5 there exist two following values of v_* (two cusps), for which $\gamma = 0$: $v_* = 0.310429$ and $v_* = 0.557204$ (see reference [4]).

The γ parameter represents the forcing amplitude and $\gamma = 0$ corresponds to a one-degree-of-freedom autonomous stick-slip system. Because according to the Poincaré-Bendixon theorem we cannot get chaos in an autonomous one-degree-of-freedom system, therefore we have found the autonomous system lying on the border of chaos.

4. STRONGLY NON-LINEAR SYSTEMS [5]

We consider the homogeneous nonlinear differential equation

$$\ddot{x} + x^n = 0\,, \tag{28}$$

and we are going to find a periodic solution with the following initial conditions

$$x(0) = 1, \qquad \dot{x}(0) = 0. \tag{29}$$

Using the small δ method, equation (28) is transformed into the following one

$$\ddot{x} + x^{1+2\delta} = 0. \tag{30}$$

Taking into account the following series

$$(x^2)^\delta = 1 + \delta \ln(x^2) + \frac{\delta^2}{2}[\ln(x^2)]^2 + ... \tag{31}$$

the solution to equation (30) is sought in the form

$$x = \sum_{n=0}^{\infty} x_n \delta^n . \tag{32}$$

Besides, we introduce a standard change of time following the formula

$$t = \tau / \omega, \tag{33}$$

and the introduced parameter ω is defined by the series

$$\omega^2 = 1 + \alpha_1 \delta + \alpha_2 \delta^2 + ... \tag{34}$$

Substituting expressions (32) - (34) into equation (30) and taking into account (31), we get (after splitting with respect to δ) the following recurrent system of equations

$$\ddot{x}_0 + x_0 = 0, \tag{35}$$

$$x_0(0) = 1, \quad \dot{x}_0(0) = 0; \tag{36}$$

$$\ddot{x}_1 + x_1 = -x_0 \ln(x_0)^2 - \alpha_1 \ddot{x}_0, \tag{37}$$

$$x_1(0) = \dot{x}_1(0) = 0 \tag{38}$$

$$\ddot{x}_2 + x_2 = -[x_1 \ln(x_0^2) + 2x_1] - x_0[\ln(x_0^2)]^2 - \alpha_2 \ddot{x}_0 - \alpha_1 \ddot{x}_1, \tag{39}$$

$$x_2(0) = \dot{x}_2(0) = 0. \tag{40}$$

One obtains from equations (35), (36) $x_0 = \cos t$,

Then from equation (37) we get

$$\ddot{x}_1 + x_1 = -\cos t \ln(\cos^2 t) + \alpha_1 \cos t. \tag{41}$$

The most popular method of the solution construction to equation (41) is related to the development of the Fourier series of the first term of the right-hand side equations in order to cancel the resonance behaviour by a proper choice of α_1. The averaging procedure applied to the right-hand side of equation (41) is related to the so called Lobatczevski function

$$L(x) = -\int_0^x \ln|\cos t|\, dt,$$

which has the following properties:

Symmetry and periodicity

$$L(x) = -L(-x) \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Pseudoperiodicity

$$L(x-\pi) = L(x) - \pi \ln 2, \qquad L(x+\pi) = L(x) + \pi \ln 2,$$

Series approximation

$$L(x) = x\ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin 2k_x}{k^2}.$$

Taking into account the two first properties, one obtains:

$$L(2\pi) = 2\pi \ln 2, \text{ and thus } \alpha_1 = 4\pi \ln 2.$$

Therefore, a period of the solution can be defined as follows

$$T \approx 2\pi[1 - 2\delta \ln 2]. \tag{42}$$

For $n = 3$ ($\delta = 1$) the formula (42) yields $T = 6.8070$, where the exact period value $T = 7.4164$ (the error doesn't exceed 8%). The next approximation gives practically exact value ($T = 7.5111$). For $n = 5$ the approximation formula (42) does not represent a real period value.

The small δ method allows for rather simple investigations of many problems, for which an application of the quasi-linear approach is difficult.

As an example we consider the mathematical pendulum governed by the equation

$$\ddot{x} + \sin x = 0\,. \tag{43}$$

The quasi-linear approach does not guarantee the required accuracy of the solution to boundary value problem (43).

Equation (43) can be transformed into the form

$$\ddot{x} + x - \frac{1}{3!}x^{1+2\delta} + \frac{1}{5!}x^{1+4\delta} + ... = 0, \tag{44}$$

and it will be solved using the small δ method with the boundary conditions (29).

Let us consider the expression

$$\Omega(x,\delta) = x - \frac{1}{3!}x^{1+2\delta} + \frac{1}{5!}x^{1+4\delta} - ...,$$

which reads as follows

$$\Omega(x,0) = x\left(1 - \frac{1}{3!} + \frac{1}{5!} - ...\right) = x\sin 1.$$

Defining

$$\omega^2 = \sin 1$$

we apply the series (31) to each of the term of the function $\Omega(x,\delta)$.

Therefore

$$\Omega(x,\delta) = x[\omega^2 + \delta \ln(x^2)\omega_1^2 + \delta^2[\ln(x^2)]^2 + ...],$$

where:

$$\omega_1^2 = \frac{1}{3!} - \frac{2}{5!} + \frac{3}{7!} - \frac{4}{9!} + \ldots$$

$$\omega_2^2 = 0.5\left(\frac{1}{3!} - \frac{2^2}{5!} + \frac{3^2}{7!} - \frac{4^2}{9!} + \ldots \right)$$

After splitting with respect to δ, we get the following recurrent equations

$$\ddot{x}_0 + \omega^2 x_0 = 0, \quad (45)$$

$$x_0(0) = 1, \quad \dot{x}_0(0) = 0; \quad (46)$$

$$\ddot{x}_1 + \omega^2 x_1 = \omega_1^2 x_0 \ln(x_0)^2 - \alpha_1 \ddot{x}_0, \quad (47)$$

$$x_1(0) = \dot{x}_1(0) = 0; \quad (48)$$

$$\ddot{x}_2 + \omega^2 x_2 = \omega_1^2 x_1 \ln(x_0^2) + \omega_2^2 x_0 [\ln(x_0^2)]^2 - \alpha_2 \ddot{x}_0 - \alpha_1 \ddot{x}_1, \quad (49)$$

$$x_2(0) = \dot{x}_2(0) = 0; \quad (50)$$

The zero order solution (initial value problem (45), (46)) has the form

$$x_0 = \cos \omega t .$$

From the first approximation equation (47), using a method of avoiding the secular terms, we get

$$\alpha_1 = -2(\omega_1 / \omega)^2 \ln 2.$$

Taking δ = 1, we get the formula for the period of oscillations

$$T = 2\pi[1 - (\omega_1 / \omega)^2 \frac{\ln 2}{\pi}]. \quad (51)$$

The next successive approximations can be obtained in a similar way.

The numerical computation of the period of equation (43) gives the value T = 6.6. Approximation of 0-order gives T = 6.8, and this approximation is better then usually using approximation of this order T = 2π. In the first approximation one obtains practically exact value T = 6.57 (error consists approximately of 0.5%).

Therefore, the method of small δ can be treated as an adequate one for the approximate integration of equations, which do not include explicitly small parameters.

5. CONCLUSIONS

In this paper three examples of application of an asymptotic technique are given. The illustrated approaches indicate a powerful of the analytical asymptotical analysis which can be applicable successfully for both discontinuous mechanical system (impacts, friction) and strongly non-linear systems. However, these are only first steps which wait for further development (for instance, for extension of the presented methods to either multibody dynamical or continuous mechanical systems).

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ASIMPTOTSKI PRISTUPI ANALIZI STROGO NELINEARNIH SISTEMA I NEGLATKIH SISTEMA

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Rad se odnosi na tri važna problema mehanike. Prvi odgovara analitičkom predviđanju odgovarajućeg izbora parametara kola kašnjenja da bi se upravljalo prelaznim (poremećenim) ponašanjem periodičnog procesa vibro-sudara modeliranog sistemom sa jednim stepenom slobode. Drugi predstavlja primenu Melnikov-ljeve metode na stick-slip predikciju haosa u periodično pobuđenom samopobudnom oscilatoru. Treći ilustruje primenu takozvanog "malog δ metoda" na analizu strogo nelinearnih oscilacija primenom primera matematičkog klatna.