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MODELING OF MATERIALS SENSITIVE TO THE TYPE OF RHEOLOGICAL PROCESSES

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Abstract. *Applying time discretization of the rheological process and considering every time step of rheological change as a combination of creep and relaxation a procedure for using constitutive equations with different material rheological characteristics of creep and relaxation is proposed. This procedure is illustrated on the simple numerical example.*

I. INTRODUCTION

In the last decades, the new technology of the structural materials is directed towards production of high-density materials, serving to decrease the structural weight. As usual this kind of materials possess a physical non-linearity and non-elasticity [1]. In many cases these materials also demonstrate a special type of rheological properties. Using experimental data for creep and relaxation [2] and after taking an appropriate rheological model, different rheological characteristics could be obtained, respectively. The usual explanations of this fact are:

(1) The proposed model is not sufficiently adequate for describing the material rheological behavior.
(2) The rheological characteristics obtained by creep data are fundamental, because the relaxation experiments are not sufficiently precise;

We agree with these possible explanations, which could be adequate in many cases. Nevertheless another explanation could also exist:

(3) The material possesses sensitivity to the type of rheological process (creep or relaxation) due to different micro-mechanisms of structural changes during these processes.

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On the base of point (3) one comes to the following important question: How to use constitutive equations with different material rheological characteristics for creep and for relaxation, respectively. We will propose some issue applying time discretization of the rheological process, considering every time step as a combination of creep and relaxation. If the significant difference between

When the significant difference between the theoretical creep curves, obtained using different moduli for the constitutive law, arises, then we will call this behavior a sensitivity of the material to the type of rheological process. This sensitivity is analogous to the type loading or type deformational sensitivity [3].

2. NON-LINEAR MAXWELL RHEOLOGICAL MODELS WITH PROCESS TYPE SENSITIVITY

We turn our attention to the group of Maxwell type nonlinear rheological models, based on Boltzmann's statistical thermo-fluctuational approach [4]. The model assumptions are:

(a) Strain rate is composed by elastic and non-elastic parts:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^a_{ij}, \ (i, j = 1,2,3) \tag{1}$$

(b) The elastic part of strain rate is connected with the stress-rate by Hook's law:

$$\dot{\varepsilon}^e_{ij} = H_{ijkl}\dot{\sigma}_{kl}, \ \ (i, j, k, l = 1,2,3) \tag{2}$$

where H_{ijkl} is the fourth-order tensor of the elastic compliances.

(c) The non-elastic part of the strain rate obeys the non-linear Maxwell rule in a concordance with the Boltzmann's statistical law:

$$\dot{\varepsilon}^a_{ij} = \frac{1}{\eta_C}\exp\left(-\frac{U_{aJ}}{k_B T}\right)\sigma_{ij}, \ \ (J = C, R), \tag{3}$$

where $\{\sigma_{ij}\}$ is the stress-tensor; $J = C$ means that the corresponding rheological characteristic is obtained by creep experimental data; $J = R$ means that the corresponding rheological characteristic is obtained by relaxation experimental data; η_J is the viscous coefficient; U_{aJ} is the activation energy of the corresponding process: for creep ($J = C$) and for relaxation ($J = R$); T is the absolute temperature; k_B is the Boltzmann's constant [4].

We propose the next calculation procedure to explain the sensitivity of constitutive law to the type of the rheological process. The process is divided in time steps $\Delta t_n = t_{n+1} - t_n$, $n = 1,2,...N$ for every body point $x_i \in \Omega_0$; Ω_0 is the region occupied by the body in the beginning of the process $t_1 = 0$, as a region in three-dimensional Euclidean space R_3 i.e. $\Omega_0 \subset R_3$. The process change during Δt_n is divided in two successive step processes:

(1) Relaxation (fictitious process past from t_{n+1} to t_n). For $\zeta \in |\zeta_0,\zeta_f|$, $\zeta_0 = t_{n+1}$, $\zeta_f = t_n$, $d\zeta = -dt$, the strain tensor is fixed i.e. $\{\varepsilon_{ij}\} = const$ for every $\zeta \in |\zeta_0,\zeta_f|$. The stress increment is $\Delta\sigma_{ij} = \sigma_{ij}(t_{n+1}) - \sigma_{ij}(t_n)$ and we calculate it by the equation:

$$H_{ijkl}\frac{d\sigma_{kl}}{d\zeta} = \frac{1}{\eta_R}\exp\left(-\frac{U_{aR}(\zeta)}{k_B T_0}\right)\sigma_{ij}(\zeta), \text{ if } T = T_0 = const \tag{4}$$

with the initial condition: for $\zeta = \zeta_0$, $\sigma_{ij}(\zeta_0) = \sigma_{ij}(t_n)$ - known from the process solution for the preliminary time step. We can solve Eq. (4) using f.e. Runge-Kutta numerical procedure.

(2) Creep with $\{\sigma_{ij}(t_{n+1})\} = const$ in time interval $[t_n, t_{n+1}]$. We introduce $\xi \in |\xi_0, \xi_f|$, $\xi_0 = t_n$, $\xi_f = t_{n+1}$. The stress increment $\Delta\varepsilon_{ij} = \varepsilon_{ij}(t_{n+1}) - \varepsilon_{ij}(t_n)$ will be obtained by the equation:

$$\frac{d\,\varepsilon_{kl}}{d\xi} = \frac{1}{\eta_C}\exp\left(-\frac{U_{aC}(\xi)}{k_B T_0}\right)\sigma_{ij}(t_{n+1}), \quad \xi \in [\xi_o, \xi_f] \tag{5}$$

with the following initial condition: for $\xi = \xi_0$, $\varepsilon_{ij}(\xi_0) = \varepsilon_{ij}(t_n)$ - these values are known from the process solution for the preliminary time step.

The solution of the process for every $x_i \in \Omega_0$ will be obtained step by step solving successively the equations (4) and (5) with initial process condition – pure elastic behavior at the moment $t_1 = 0$.

3. One-dimensional Maxwell-Gourevich model

3.1. Constitutive equations

We will examine structural material – polyethylene with high density. To describe the non-linear rheological behavior of the polyethylene we apply the model of Maxwell-Gourevich [5]. This model is a well known representative one from the group model from Section 2. On the base of some one-dimensional experiments in creep and relaxation regimes [2] we will provide an identification procedure for the model parameters [1].

The determination of the Maxwell-Gourevich model parameters will be provided in two stages on the base of the experimental data for polyethylene of high-density [2]. In the first stage, the Maxwell-Gourevich model parameters will be determined on condition of non-linear creep, while in the second one, the determination will be provided on condition of stress relaxation of the considered material.

Bellow, the brief description of uniaxial Maxwell-Gourevich equation [5], describing the non-linear creep and stress relaxation of polymer is given.

Let us consider the case of constant tension along the axis Ox: $\sigma_x = const$, where σ_x is the stress on the boundary of a specimen, defined by the external load.

The differential equation of Maxwell-Gourevich for creep in the uniaxal case is simplified and has a following form:

$$\frac{d\varepsilon_x}{dt} = \frac{1}{\eta_0^{\bullet}}\exp\left(\frac{\gamma^{*}\sigma_x}{3m^{*}}\right) f_{xx}^{*}\exp\left|\frac{f_{xx}^{*}}{m^{*}}\right| \tag{6}$$

where

$$f_{xx}^{\ *} = \sigma_x - E_\infty \varepsilon_x^{*} = (1 + E_\infty / E)\sigma_x - E_\infty \varepsilon_x$$

The initial condition is:

$$\varepsilon_x = \varepsilon_{x,0} = \sigma_x / E + \varepsilon_{x,0}^{*} \quad \text{at} \quad t = 0$$

If at the beginning of the loading, the non-linear strain value $\varepsilon_{x,0}^{*}$ is not big, then the following expression for elastic strain is valid:

$$\frac{df_{xx}^*}{dt} = -E_\infty \frac{d\varepsilon_x^*}{dt} = -E_\infty \frac{d\varepsilon_x}{dt} \tag{7}$$

Substituting (7) into (6), we obtain:

$$\frac{df_{xx}^*}{dt} = -\frac{E_\infty}{\eta_0^*} \exp\left(\frac{\gamma^* \sigma_x}{3m^*}\right) f_{xx}^* \exp\left|\frac{f_{xx}^*}{m^*}\right| \tag{8}$$

where $\varepsilon_x = \varepsilon_{x,0}$ and $f_{xx}^* = f_{xx,0}$ at $t = 0$.

Note, that (8) changes the sign in the case of compressive loading. Let us introduce the new variable ζ^*, as follows:

$$\zeta^* = \left|\frac{f_{xx}^*}{m^*}\right| = \left|(1 + E_\infty / E)\sigma_x - E_\infty \varepsilon_x\right| \frac{1}{m^*}$$

The Eq. (8) takes the form:

$$\frac{d\zeta^*}{dt} = -\frac{E_\infty}{\eta_0^*} \exp\left(\frac{\gamma^* \sigma_x}{3m^*}\right) \zeta^* \exp \zeta^* \tag{9}$$

at $t = 0 \quad \zeta^* = \zeta_0^* = | f_{xx,0}^* / m^* |$

The integral of the Eq. (9) is:

$$t = \frac{\eta_0^*}{E_\infty} \exp\left(-\frac{\gamma^* \sigma_x}{3m^*}\right) [-Ei(-\zeta^*) + Ei(-\zeta_0^*)] \tag{10}$$

where $-Ei(-x) = -\int_{-\infty}^{x} \frac{\exp(-x)}{x} dx$

If the Eq. (10) describes the bounded creep, then its value has to be determined by the relation:

$$\sigma_x = E_\infty \varepsilon_x^* = E_\infty^* \varepsilon_x \quad \text{at} \quad t \to \infty, \quad E_\infty^* = E_\infty / (1 + E_\infty / E)$$

For this limit case, the following expressions for strain rate and strain are used:

$$v_{\varepsilon x} = \frac{1}{\eta_0^*} \exp\left(\frac{\gamma^* \sigma_x}{3m^*}\right) \sigma_x \exp\left|\frac{\sigma_x}{m^*}\right| \tag{11}$$

$$\varepsilon_\varepsilon = \varepsilon_{x,0} + \frac{t}{\eta_0^*} \exp\left(\frac{\gamma^* \sigma_x}{3m^*}\right) \sigma_x \exp\left|\frac{\sigma_x}{m^*}\right| \tag{12}$$

In semi-logarithmic coordinate system (ε_x,lg t), the equation (10) takes the form:

$$\lg(t / t_0) = -\frac{\gamma^* \sigma_x}{3m^* \ln 10} + \lg \frac{\eta_0^*}{E_\infty t_0} + \lg[-Ei(-\zeta^*) + Ei(-\zeta_0^*)] \tag{13}$$

For the second stage, i.e., in the case of stress relaxation, we have: $\varepsilon_x = const = \varepsilon_e$.

The differential equation of Maxwell-Gourevich for the uniaxial stress relaxation can be simplified and has the following form:

$$\frac{d\sigma_x}{dt} = -\frac{E}{\eta_0^*} f_{xx}^* \exp\left\{\left[\frac{\gamma^*\sigma_x}{3} + |f_{xx}^*|\right]\frac{1}{m^*}\right\} \tag{14}$$

The initial condition is:

$$\sigma_x = \sigma_{x,0} \quad \text{at} \quad t = 0$$

and:

$$\frac{df_{xx}^*}{dt} = -\frac{E + E_\infty}{\eta_0^*} \exp(b^*\varepsilon_e) f_{xx}^* \exp\left|\frac{f_{xx}^*}{m_0^*}\right| \tag{15}$$

where:

$$m_0^* = m^* \Big/ \left[1 + \frac{\gamma^* sign\sigma_x}{3(1 + E_\infty / E)}\right], \quad b^* = \frac{E_\infty^* \gamma^*}{3m^*}$$

The integral of the Eq. (15) is:

$$t = \frac{\eta_0^*}{E + E_\infty} \exp(-b^*\varepsilon_x)[-Ei(-\overline{\zeta}^*) + Ei(-\overline{\zeta}_0^*)] \tag{16}$$

where

$$\overline{\zeta}^* = |f_{xx}^* / m_0^*| = |(1 + E_\infty / E)\sigma_x - E_\infty \varepsilon_e| \Big/ m_0^* \tag{17}$$

In semi-logarithmic coordinate system (ε_x,lg t), the equation (16) takes the form:

$$\lg(t / t_0) = -\frac{b^*\varepsilon_e}{\ln 10} + \lg\frac{\eta_0^*}{(E + E_\infty)t_0} + \lg[-Ei(-\overline{\zeta}^*) + Ei(-\overline{\zeta}_0^*)] \tag{18}$$

The following model parameters have to be determined: γ^* - the volumetric modulus coefficient, η_0^* - the initial viscosity modulus, E_∞ - the equilibrium viscoelastic modulus, E_∞ / m^* (m^* - the logarithmic modulus).

3.2. Methodology for determination of the constants from experimental data

The experimental curve $\varepsilon_x(\lg t)$ has an inflexion point. The value of the angle coefficient $|k| = \Delta\varepsilon_x/\Delta\lg(t/t_0)$ of the tangent to the inflexion point as well as the value of $\varepsilon_{x,p} = \xi_p / (E_\infty / m^*)$, where $\varepsilon_{x,p}$ is the ordinate of this point can be determined from the experimental curve. The first approximation of the value E_∞ / m^* can be calculated by the formula:

$$\frac{E_\infty}{m^*} = \frac{1}{|k|}\frac{1}{1 + 1/\xi_p^*} = \frac{1}{|k|}\left[1 - \frac{|\bar{k}|}{|\varepsilon_{x,p}|}\right]; \quad \bar{k} = k / \ln 10$$

To justify the parameter E_∞ / m^*, the new value $\xi_0^* = \varepsilon_{x,0} / (E_\infty / m^*)$ is found for solving the equation $[Ei(-\xi^* p) + Ei(-\xi_0^*)](1 + \xi_p^*) = 1$ with respect to ξ_p^*. This last value is used to determine the new values of E_∞ / m^* and $\varepsilon_{x,p}$. We repeat this way and stop when the new values reach the accuracy about 5–10%, satisfying the accuracy of the experimental data.

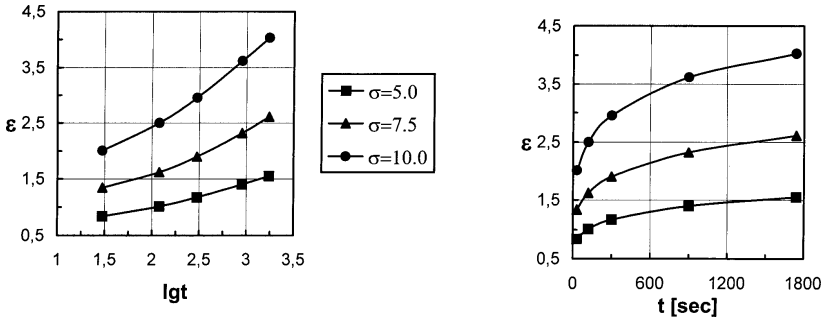
After the justification of the parameter E_∞ / m^*, the theoretical curve for some arbitrary value of $\eta_0^*/(E_\infty t_0)$ can be calculated from the following equation:

$$\lg(t/t_0) = \lg[\eta_0^*/(E_\infty t_0)] + \lg[-Ei(-\xi^*) + Ei(-\xi_0^*)]$$

Using this equation, the abscissa of the inflexion point of the theoretical curve can be determined. The distance Δ on abscissa between the respective theoretical and experimental inflexion points is taken into account for removing the theoretical curve up to coincidence the experimental one. Adding Δ to the arbitrary value of $\lg[\eta_0^*/(E_\infty t_0)]$, the real values of $\lg[\eta_0^*/(E_\infty t_0)]$ and η_0^*/E_∞ can be calculated.

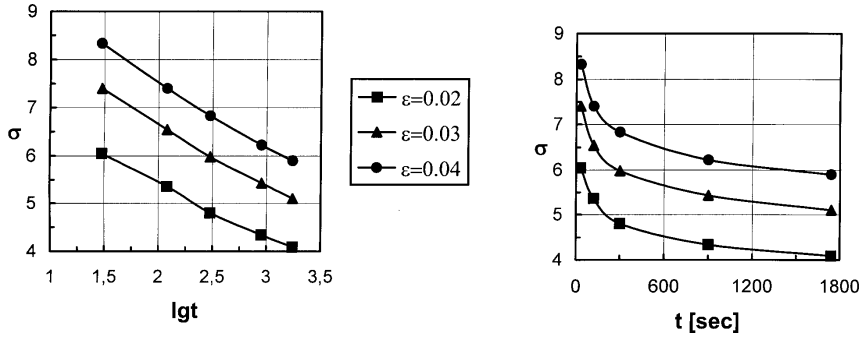
In the case of stress relaxation, the analogous procedure for determining the model parameters γ^*, η_0^* is used. The parameter E_∞ is obtained from $E_\infty = \sigma_{\min}/\varepsilon^*_{\max}$, where $\varepsilon^*_{\max} = \varepsilon_e - \overline{\omega}_{\min}/E$. The value of the parameter $\overline{m}_0^*$ is calculated from the following formula: $\overline{m}_0^{\bullet} = |\overline{k}|(1+1/\overline{\zeta}_p^*)$, $\overline{k} = k/\ln 10$, $k = \Delta\sigma_x / \Delta\lg(t/t_0)$.

The experimental data [2] for the case of non-linear creep and stress relaxation of the polyethylene with high density (E_0 = 867[N/mm^2], ν = 0.35) are given on Figs. 1, 2, where $\varepsilon = \varepsilon_x$, $\sigma = \sigma_x$. The determined Maxwell-Gourevich parameters by the above mentioned procedure are shown on Tables 1, 2:



(a) Non-linear creep

Fig. 1. Non-linear creep



(b) Stress relaxation

Fig. 2. Stress relaxation

Table 1. Rheological Characteristics at non-linear creep

E_∞ [N/mm^2]	m^* [N/mm^2]	η_0^* [N.sec/mm^2]	γ^*
430,4	4,777	$4{,}6.10^4$	0.0

Table 2. Rheological characteristics at stress relaxation

E_∞ [N/mm^2]	m^* [N/mm^2]	η_0^* [N.sec/mm^2]	γ^*
327,4	2.18	$3{,}3.10^5$	0.0

As a result of identification we have already obtained the case in which the considered material is sensitive to the type of rheological process. The degree of correspondence between the experimental and theoretical curves is evaluated by the relative square mean error measure of the range about 10%. This a relatively good approximation in rheology [1].

4. NUMERICAL PROCEDURE AND EXAMPLE

According to the procedure described in Section 2 we divide the process in time step Δt_i , $i = 1,2,...N$. During the time interval $[t_i, t_{i+1}]$ we realize the calculation of the next two successive steps of the rheological process:

(1) Relaxation in fictitious interval $[\zeta_0,\zeta_f]$, $\zeta_0 = t_{i+1}$, $\zeta_f = t_i$, $d\zeta = -dt$, $\varepsilon_x(t_i) = const$. The governing constitutive equation is:

$$\frac{d\sigma_x}{d\zeta} = \frac{E_0}{\eta_R}\exp(\beta_R\sigma_x(\zeta)).[\alpha_E^R\sigma_x(\zeta) - E_\infty^R\varepsilon_x(t_i)].\exp\left|\frac{\alpha_E^R\sigma_x(\zeta) - E_\infty^R\varepsilon_x(t_i)}{m_R}\right| \qquad (19)$$

with the initial condition: for $\zeta = \zeta_0$, $\sigma_x(\zeta_0) = \sigma_x(t_i)$ - known from the calculation in the preliminary step; $\beta_R = \dfrac{\gamma_R}{3m_R}$; $\alpha_E^R = 1 + \dfrac{E_\infty^R}{E_0}$.

(2) Creep with $\sigma_x(t_{i+1}) = const$, $\xi \in |\xi_0,\xi_f|$, $\xi_0 = t_i$, $\xi_f = t_{i+1}$. The governing constitutive equation is:

$$\frac{d\varepsilon_x}{d\xi} = \frac{1}{\eta_C}\exp(\beta_C\sigma_x(t_{i+1})).[\alpha_E^C\sigma_x(t_{i+1}) - E_\infty^C\varepsilon_x(\xi)].\exp\left|\frac{\alpha_E^C\sigma_x(t_{i+1}) - E_\infty^C\varepsilon_x(\xi)}{m_C}\right| \qquad (20)$$

with the following initial condition: for $\xi = \xi_0$, $\varepsilon_x(\xi_0) = \varepsilon_x(t_i)$ - known from the calculation in the preliminary time step.

The calculation scheme during the rheological process consists in successive calculation of Eq. (19) and Eq. (20), starting by pure elastic solution at the moment $t_1 = 0$ (Fig. 3).

To give the application of the proposed procedure, we consider a simple example for cantilever beam with the following geometric characteristics: $L = 100\ mm$, $b = 10\ mm$, $h = 20\ mm$ (Fig. 4). The beam is subjected to the time-depended moment $M_z(t) = M_0(1 + t^2)$, where $M_0 = const = 10$ [N.mm]. The initial conditions at $t = 0$ are taken from the corresponding elastic solution:

$$\sigma_x(y,t)=\frac{M_z(t)}{J_z}y\,,\quad \varepsilon_x(y,t)=\frac{\sigma_x(y,t)}{E_0}\,,\quad J_z=\frac{bh^3}{12}\,,\quad y=\frac{h}{2}\;M_z(t)=M_0(1+t^2)$$

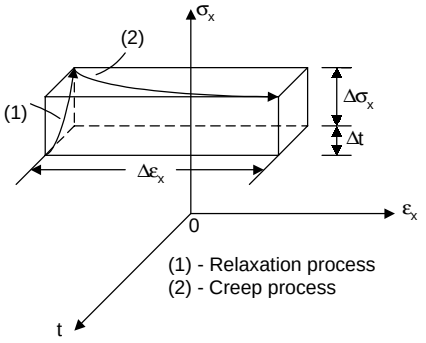


Fig. 3. Calculation scheme

The characteristics of the considered material (polyethylene with high density) are given in the previous section. The results for strain v/s time, obtained by the proposed procedure are shown on Fig. 5, where the curve 1 (ε_x) corresponds to the case of material sensitive to the type of rheological process and curve 2 ($\bar{\varepsilon}_x$) – to the case of material without this sensitivity. The error between two curves

$$r=\sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(\frac{\varepsilon_x(t_i)}{\bar{\varepsilon}_x(t_i)}-1\right)^2}.100\%\quad \text{is } 41{,}4\%\,.$$

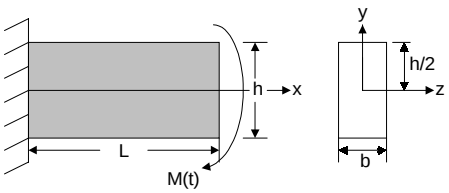


Fig. 4. Scheme of cantilever beam

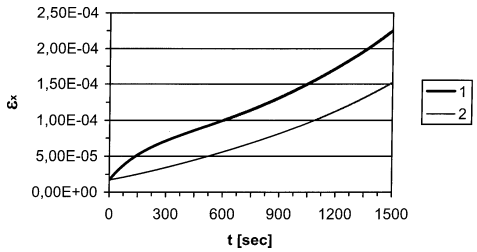


Fig. 5. Strain-time diagram

5. CONCLUSIONS

In the paper a procedure for using constitutive equations with different material rheological characteristics for creep and relaxation is proposed. The obtained numerical results prove the reason to take into account the sensitivity of the constitutive law to the type of rheological processes.

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MODELIRANJE MATERIJALA OSETLJIVIH NA ODREĐENI TIP REOLOŠKIH PROCESA

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Primenjujući vremensku diskretizaciju reološkog procesa i posmatrajući reološke promene u svakom periodu odabiranja kao kombinaciju puzanja i otpuštanja, predložen je postupak za korišćenje konstitutivnih jednačina sa različitim reološkim karakteristikama puzanja i otpuštanja materijala. Ovaj postupak ilustrovan je jednostavnim numeričkim primerom.