FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol.3, N° 11, 2001, pp. 5 - 12

DERIVATION OF GENERALIZED VARIATIONAL PRINCIPLES WITHOUT USING LAGRANGE MULTIPLIERS PART I: APPLICATIONS TO FLUID MECHANICS

UDC 517.93:532(045)

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Abstract. *A systematic approach to derivation of variational principles directly from the partial differential equations of fluid mechanics is suggested herein. Based upon the semi-inverse method proposed by He, various variational principles can be readily obtained without using Lagrange multiplier method.*

1. INTRODUCTION

Generally speaking, there exist two basic ways to describe a physical problem: 1) by partial differential equations (PDEs) with boundary or initial conditions (BC or IC); 2) by variational principles (VPs). PDE model requires strong local differentiability (smoothness) of the physical field, while its VP partner requires weaker local smoothness or only local integrability. For discontinuous field, the PDE model is no longer valid, while its VP partner is powerfully applied. Moreover the VP model has many advantages over its PDE partner: simple and compact in form while comprehensive in content, encompassing implicitly almost all information characterizing the problem under considerationPDEs and natural BC/IC; capable of hinting naturally how the boundary/initial value problem should be properly posed. Applying variational principle with variable-domain [8,9,16,17], we can powerfully deal with discontinuities such as free surface, shock. It is also a sound theoretical foundation of the finite element method (FEM) [16], other modern numerical techniques such as meshfree particle method [10], and other direct variational methods such as Ritz's, Trefftz's, and Kantorovitch's methods.

It is well known that, in general, it is extremely difficult to deduce a generalized variational principle directly from its governing equations and boundary conditions or initial conditions. Much attention has been put on the existence and uniqueness for the inverse problem of calculus of variations and ways to search for its variational principle

Received October 02, 2000
Presented at 5th YUSNM Niš 2000, Nonlinear Sciences at the Threshold of the Third Millenium, October 2-5, 2000, Faculty of Mechanical Engineering University of Niš, Invited Plenary Lecture, Part I

of a physical problem. According to Vainberg's theorem, the VPs for a physical problem exist and can be constructed formally, if the differential operators in the PDE-formulation are symmetric. Such a requirement is overly restrictive, and it is important that we remove it if possible.

The general approach to establishment of a generalized variational principle is the Lagrange multiplier method. However, for some physical problems, no known variational principle is at hand. The Lagrange multiplier method, therefore, loses its power in such a case. Moreover, in using Lagrange multiplier method to arrive at a GVP, one may always come across variational crisis [1] (some of Lagrange multipliers become zero, and thus fail to reach its aim), which was found by Chien [1] in elasticity, and Liu [14] and He [11] in fluid mechanics. Various methods have been proposed to eliminate the crisis, for example, high-order Lagrange multiplier method by Chien [1], preconditioned method by Liu [14], and semi-inverse method by He [11].

2. Outline of the semi-inverse method

In order to best illustrate the basic idea of the proposed semi-inverse method [2~13], we consider the 2-D incompressible inviscid potential flow. The equations for incompressible potential flow can be written as:

$$\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}=0\ , \tag{2.1}$$

$$\frac{\partial \Phi}{\partial x}=D\ ,\ \frac{\partial \Phi}{\partial y}=C\ . \tag{2.2a,b}$$

the boundary conditions are:

on inlet $$\Gamma_{in}:\ \mathbf{q}\cdot\mathbf{n}=q_0 \tag{2.3}$$

on outlet $$\Gamma_{out}\ :\ \mathbf{q}\cdot\mathbf{n}=q_1 \tag{2.4}$$

where $\mathbf{q}=u\mathbf{i}+v\mathbf{j}$.

2.1 The First Line: Derivation of Variational Principles from PDE & BC

To establish a generalized variational principle with three independent variables (Φ,u,v), we can construct an energy-like integral like this

$$J(\Phi,u,v)=\iint\left\{u\frac{\partial \Phi}{\partial x}+v\frac{\partial \Phi}{\partial y}+F(u,v)\right\}dxdy+\int_{\Gamma_{in}}GdS+\int_{\Gamma_{out}}HdS\ , \tag{2.5}$$

where F , G and H are unknown functions.

The advantages of the above trial-functional is that the stationary condition(Euler equation) with respect to Φ is Eq.(2.1). Calculating variation with respect to Φ

$$\delta_\Phi J(\Phi,u,v)=\iint\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right\}\delta\Phi dxdy+\delta\int_{\Gamma_{in}}GdS+\delta\int_{\Gamma_{out}}HdS=0\ , \tag{2.6}$$

we immediately obtain Eq. (2.1) as Euler equation.

Now the other two Euler equations (called often trial-Euler equations) with respect to u and v read

$$\delta u: \ \frac{\partial \Phi}{\partial x} + \frac{\partial F}{\partial u} = 0 , \tag{2.7}$$

$$\delta v: \ \frac{\partial \Phi}{\partial y} + \frac{\partial F}{\partial v} = 0 . \tag{2.8}$$

The above two equations should satisfy two of the field equations (2.2a,b). So we can set

$$\frac{\partial F}{\partial u} = -u , \tag{2.9}$$

$$\frac{\partial F}{\partial v} = -v . \tag{2.10}$$

From the above relations, Eqs. (2.9) and (2.10), we can readily identify the unknown function F as follows

$$F = -\frac{1}{2}(u^2 + v^2) = -\frac{1}{2} q^2 \tag{2.11}$$

Using the Green theorem, we can obtain stationary conditions on the boundary.

On inlet:

$$\delta \Phi: \ \mathbf{q} \cdot \mathbf{n} + \frac{\partial G}{\partial \Phi} = 0 , \tag{2.12}$$

which should satisfy the boundary condition on inlet, Eq. (2.3), so we can set

$$\frac{\partial G}{\partial \Phi} = -\mathbf{q} \cdot \mathbf{n} = -q_0 \tag{2.13}$$

which leads to

$$G = -q_0 \Phi \tag{2.14}$$

Similarly we can determine the unknown H as follows:

$$H = -q_1 \Phi \tag{2.15}$$

So we can deduce the following generalized variational principle with three independent variables:

$$J(\Phi, u, v) = \iint \left\{ u \frac{\partial \Phi}{\partial x} + v \frac{\partial \Phi}{\partial y} - \frac{1}{2} q^2 \right\} dxdy - \int_{\Gamma_{in}} q_0 \Phi dS - \int_{\Gamma_{out}} q_1 \Phi dS . \tag{2.16}$$

2.2 The Second Line: Derivation of a generalized VP from a Known VP

We can also deduce a generalized variational principle with multi-variables from a known variational principle with one or fewer independent variables. Supposing there exists the following variational principle with a single independent variable:

$$J(\Phi) = \iint \frac{1}{2}(u^2 + v^2) dxdy , \tag{2.17}$$

which is subject to Eqs (1.2a,b).

Using Lagrange multipliers λ_1 and λ_2 to eliminate the constraints, we obtain

$$J(\Phi,u,v,\lambda_1,\lambda_2)=\iint\left\{\frac{1}{2}(u^2+v^2)+\lambda_1(\frac{\partial\Phi}{\partial x}-u)+\lambda_2(\frac{\partial\Phi}{\partial y}-v)\right\}dxdy\ . \tag{2.18}$$

In this simple problem, the multipliers can be easily determined: $\lambda_1 = u$ and $\lambda_2 = v$. For a complex problem, however, the multiplier method might fail due to the variational crisis[11]. In any cases, a multiplier can be expressed in the form

$$\lambda_i=\lambda_i(u,v,\Phi)\ ,\ (i=1,2). \tag{2.19}$$

So before the identification of the multipliers, we can introduce a new function F defined as

$$F(u,v,\Phi)=\lambda_1\left(\frac{\partial\Phi}{\partial x}-u\right)+\lambda_2\left(\frac{\partial\Phi}{\partial y}-v\right), \tag{2.20}$$

We, therefore, construct the following trial-functional

$$J(\Phi,u,v)=\iint\left\{\frac{1}{2}(u^2+v^2)+F(u,v,\Phi)\right\}dxdy\ . \tag{2.21}$$

The trial-Euler equations can be readily obtained:

$$\delta u:\ u+\frac{\partial F}{\partial u}=0 \tag{2.22}$$

$$\delta v:\ v+\frac{\partial F}{\partial v}=0 \tag{2.23}$$

$$\delta\Phi:\ \frac{\partial F}{\partial\Phi}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial\Phi_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial\Phi_y}\right)=0\ . \tag{2.24}$$

The above three equations should satisfy the field equations (2.1), (2.2a,b), so we can determine the unknown function F as follows

$$F=-u\frac{\partial\Phi}{\partial x}-v\frac{\partial\Phi}{\partial y}\ . \tag{2.25}$$

Substituting the identified F into Eq.(21) yields the required generalized variational principle.

2.3 The 3rd Line: From an Energy Integrate to GVP

The 3rd line is to construct an arbitrary trial-functional with energy form, for example, the following trial-functional can be constructed for the incompressible potential flow:

$$J(\Phi,u,v)=\iint\left\{u\frac{\partial\Phi}{\partial x}+F(\Phi,u,v)\right\}dxdy\ . \tag{2.26}$$

It is easy to prove the above integrate has the form of energy.

Making the above trial-functional, Eq. (2.26), stationary with respect to u, we obtain the following trial-Euler equation:

$$\frac{\partial \Phi}{\partial x}+\frac{\partial F}{\partial u}=0\ , \tag{2.27}$$

We set

$$\frac{\partial F}{\partial u}=-u\ , \tag{2.28}$$

so that Eq.(2.27) becomes Eq.(2.1a). We, therefore, identify the unknown function F as follows:

$$F=-\frac{1}{2}u^2+f(v,\Phi)\,, \tag{2.29}$$

where f is a newly introduced unknown function of v and Φ. By similar operation, we can determine the unknown function f step by step. Finally we obtain the following functional:

$$J(\Phi,u,v)=\iint\left\{-\frac{1}{2}(u^2+v^2)+u\Phi_x+v\Phi_y\right\}dxdy\ . \tag{2.30}$$

2.4 Derivation of Various VPs from a Known GVP

It is a quite a straightforward way to deduce various variational principles from a known generalized variational principle by constraining the obtained functional by selectively enforcing field equations (2.1), (2.1a) or (2.1b). For example, constraining the functional (2.30) by the field equations (2.1a) , we obtain

$$J(\Phi,v)=\iint\left\{\frac{1}{2}(\Phi_x^2-v^2)+v\Phi_y\right\}dxdy\ . \tag{2.31}$$

Further enforcing the functional (2.31) by Eq.(2.1b) leads to

$$J(\Phi)=\iint\left\{\frac{1}{2}(\Phi_x^2+\Phi_y^2\right\}dxdy\ . \tag{2.32}$$

3. Application

Let's consider the 1-D unsteady compressible fluid in a flexible tube of varying cross-sectional area $A(x,t)$.The governing equations are

1) *Continuity equation*

$$\frac{\partial(\rho A)}{\partial t}+\frac{\partial(\rho A u)}{\partial x}=0\ . \tag{3.1}$$

2) *Momentum equation*

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial P}{\partial x}\ . \tag{3.2}$$

3) *Pressure-density relation*

$$P=\sum_{n=0}a_n\rho^n\ , \tag{3.3}$$

where a_n are constants.

Using the relation (3.3), Eq.(3.2) can be rewritten in the following conservative form

$$\frac{\partial u}{\partial t}+\frac{\partial H}{\partial x}=0\ , \tag{3.4}$$

where H is defined as

$$H=\frac{1}{2}u^2+a_1\ln\rho+\sum_{n=2}na_n\rho^{n-1}\ . \tag{3.5}$$

We introduce two general functions: path function Ψ and potential function Φ, which are defined respectively as

$$\frac{\partial \Psi}{\partial t}=-\rho A u\ , \tag{3.6}$$

$$\frac{\partial \Psi}{\partial x}=\rho A\ , \tag{3.7}$$

and

$$\frac{\partial \Phi}{\partial t}=-H\ , \tag{3.8}$$

$$\frac{\partial \Phi}{\partial x}=u\ . \tag{3.9}$$

We can construct various trial-functionals, one of which reads

$$J(\Phi,\rho,u)=\iint\left\{\rho A\frac{\partial \Phi}{\partial t}+\rho A u\frac{\partial \Phi}{\partial x}+F(\rho,u)\right\}\mathrm{d}t\,\mathrm{d}x\ , \tag{3.10}$$

where Φ, ρ and u are all independent variables, F is an unknown function of ρ and u .

The stationary conditions with respect to u and ρ can be written respectively in the following forms:

$$\delta u:\ \rho A\frac{\partial \Phi}{\partial x}+\frac{\partial F}{\partial u}=0\ , \tag{3.11}$$

$$\delta\rho:\ A\frac{\partial \Phi}{\partial t}+Au\frac{\partial \Phi}{\partial x}+\frac{\partial F}{\partial \rho}=0\ . \tag{3.12}$$

We search for such an F, so that Eqs.(3.11) and (3.12) satisfy Eqs.(3.8) and(3.9) respectively. Accordingly we can set

$$\frac{\partial F}{\partial u}=-\rho A u\ , \tag{3.13}$$

$$\frac{\partial F}{\partial \rho}=A\left(\frac{1}{2}u^2+a_1\ln\rho+\sum_{n=2}na_n\rho^{n-1}\right)-Au^2=-\frac{1}{2}Au^2+Aa_1\ln\rho+A\sum_{n=2}na_n\rho^{n-1}\ . \tag{3.14}$$

From (3.13) and (3.14), the unknown F can be readily identified

$$F=-\frac{1}{2}A\rho u^2+Aa_1\rho(\ln\rho-1)+A\sum_{n=2}a_n\rho^n\ . \tag{3.15}$$

We, therefore, obtain the following functional

$$J(\Phi,\rho,u)=\iint\left\{\rho A\frac{\partial\Phi}{\partial t}+\rho Au\frac{\partial\Phi}{\partial x}-\frac{1}{2}A\rho u^2\right\}\mathrm{d}\,t\,\mathrm{d}\,x+ \\ +\iint\left\{Aa_1\rho(\ln\rho-1)+A\sum_{n=2}a_n\rho^n\right\}\mathrm{d}\,t\,\mathrm{d}\,x \tag{3.16}$$

It is very easy to deduce various variational principles from a known generalized variational principle. Constraining the functional (3.16) by selectively enforcing field equations yields various sub-generalized variational principles. For example, substituting equation(3.9) into the functional (3.16) yields the following functional :

$$J(\Phi,\rho)=\begin{matrix}\iint\left\{\rho A\frac{\partial\Phi}{\partial t}+\frac{1}{2}A\rho(\frac{\partial\Phi}{\partial x})^2\right\}\mathrm{d}\,t\,\mathrm{d}\,x \\ +\iint\left\{Aa_1\rho(\ln\rho-1)+A\sum_{n=2}a_n\rho^n\right\}\mathrm{d}\,t\,\mathrm{d}\,x\end{matrix}+L_\Phi\,, \tag{3.17}$$

which is subject to equation (3.9).

Further constraining the functional (3.17) by the equation (3.8), we have

$$\begin{aligned}J(u)&=\iint\left\{\rho A(-\frac{1}{2}u^2-a_1\ln\rho-\sum_{n=2}na_n\rho^{n-1})+\frac{1}{2}A\rho u^2\right\}\mathrm{d}\,t\,\mathrm{d}\,x \\ &\quad+\iint\left\{Aa_1\rho(\ln\rho-1)+A\sum_{n=2}a_n\rho^n\right\}\mathrm{d}\,t\,\mathrm{d}\,x+L_\Phi \\ &=\iint\left\{-Aa_1\rho-A\sum_{n=2}(n-1)a_n\rho^n\right\}\mathrm{d}\,t\,\mathrm{d}\,x+L_\Phi\,,\end{aligned} \tag{3.18}$$

which is a functional under the constraints of equations (3.8) and (3.9).

We can also establish a variational principle with independent variables Ψ, ρ and u. The trial-functional can be constructed as follows

$$J(\Psi,\rho,u)=\iint\left\{u\frac{\partial\Psi}{\partial t}+H\frac{\partial\Psi}{\partial x}+F(\rho,u)\right\}\mathrm{d}\,t\,\mathrm{d}\,x \tag{3.19}$$

We search for such F , so that the stationary conditions of the above trial-functional satisfy the field equations (3.4), (3.6) and (3.7). By the same manipulation, we can obtain the following functional:

$$J(\Psi,\rho,u)=\iint\left\{u\frac{\partial\Psi}{\partial t}+H\frac{\partial\Psi}{\partial x}-a_1\rho A+A\sum_{n=2}(n-1)\rho^n\right\}\mathrm{d}\,t\,\mathrm{d}\,x \tag{3.20}$$

4. CONCLUSION

It is obvious that the semi-inverse method is an effective approach to searching for various variational principles for fluid mechanics without using Lagrange multipliers.

Acknowledgment. *The author should thank Prof. Katica (Stevanovic) Hedrih, chairman of the conference, for her kind invitation and help. The support of the National Key Basic Research Special Fund of China (No. G1998020318) is gratefully acknowledged.*

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IZVOĐENJE UOPŠTENIH VARIJACIONIH PRINCIPA BEZ KORIŠĆENJA LAGRANGE-OVIH MNOŽILACA DEO I: PRIMENA NA MEHANIKU FLUIDA

Ji-Huan He

U radu je sugerisan sistematski pristup izvođenju diferencijalnih principa direktno iz parcijalnih diferencijalnih jednačina mehanike fluida. Zasnovan na polu-inverznoj metodi predloženoj od strane He-ija, različiti varijacioni principi mogu se lako dobiti bez korišćenja Lagrange-ove metode množioca.