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DERIVATION OF GENERALIZED VARIATIONAL PRINCIPLES WITHOUT USING LAGRANGE MULTIPLIERS PART II: APPLICATIONS TO SOLID MECHANICS

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Abstract. *An alternative derivation of the well-known Hu-Washizu principle, Hellinger-Reissner in elasticity is provided. It is shown that the present approach, which is based on the semi-inverse method proposed by He, is simple and straightforward. The variational crisis and its removal are also discussed, it also reveals that the semi-inverse method is one of the best approach to eliminating the crisis.*

1. INTRODUCTION

Variational theory is the theoretical basis for the finite element techniques and other direct variational methods such as Ritz's, Trefftz's and Kantorovitch's methods [13]. Recent study [4] also reveals that variational theory is also the theoretical basis for meshfree particle method. So the importance of searching for a variational representation for the discussed problem must not be demonstrated in detail in this paper, however, it is very difficult to search for a variational representation directly from the field equations and boundary.

In this paper we will use the semi-inverse method [3~11] to re-deduce the various variational principles in elasticity.

2. MATHEMATICAL FORMULATION OF SMALL DISPLACEMENT PROBLEMS IN

Let τ be the volume of nonlinear elastibody subjected to the action of distributed body forces f_i (i = 1,2,3), Γ_σ be the portion of boundary surface subjected to the action of external forces $\bar{p}_i$, and Γ_u be the other portion of boundary surface where the displacements $\bar{u}_i$ are given. Under static equilibrium, the stresses σ_{ij}, strains e_{ij} and

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displacement u_i satisfy the following five sets of conditions, namely

1) Equilibrium conditions:

$$\sigma_{ij,j} + f_i = 0, \qquad (\text{in} \quad \tau), \tag{2.1}$$

in which $\sigma_{ij,j} = \partial\sigma_{ij} / \partial x_j$.

2) Stress-strain relations: For linear elasticity, we have

$$\sigma_{ij} = a_{ijkl} e_{kl}, \qquad (\text{in} \quad \tau), \tag{2.2a}$$

or

$$e_{ij} = b_{ijkl} \sigma_{kl}, \qquad (\text{in} \quad \tau), \tag{2.2b}$$

in which a_{ijkl}, b_{ijkl} represent elastic and compliance tensors respectively.

Let us now introduce the strain energy density A and complementary B. They are defined in general by

$$A = \int_0^e \sigma_{ij} de_{ij} = \frac{1}{2} e_{ij} a_{ijkl} e_{kl} \ \text{ or } \ \frac{\partial A}{\partial e_{ij}} = \sigma_{ij}, \tag{2.2c}$$

$$B = \int_0^\sigma e_{ij} \, \mathrm{d}\sigma_{ij} = \frac{1}{2} \sigma_{ij} b_{ijkl} \sigma_{kl} \ \text{ or } \ \frac{\partial B}{\partial \sigma_{ij}} = e_{ij}, \tag{2.2d}$$

and satisfy the following energy identity

$$A + B = e_{ij} \sigma_{ij} . \tag{2.2e}$$

3) Strain-displacement relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad (\text{in} \quad \tau). \tag{2.3}$$

4) Boundary conditions for given surface displacement

$$u_i = \bar{u}_i, \qquad (\text{on} \quad \Gamma_u). \tag{2.4}$$

5) Boundary conditions for given external force on boundary surface

$$\sigma_{ij} n_j = \bar{p}_i, \qquad (\text{on} \quad \Gamma_\sigma). \tag{2.5}$$

3. VARIATIONAL CRISIS AND THE SEMI-INVERSE METHOD

The traditional approach to establish generalized variational principles is the well-known Lagrange multiplier method. By such method the constraints of a known functional can be eliminated, in sometime, however, the variational crisis (some multipliers become zero) might occur during the derivation of generalized variational principles [3,7]. On the other hand, the Lagrange multiplier method is not valid to deduce a variational representation directly from the field equations and boundary/initial conditions. In this section we first illustrate the variational crisis and the ways to overcome them.

We have the following well-know Hellinger-Reissner principle

$$J(\sigma_{ij},u_i)=\iiint\{B+u_i(\sigma_{ij,j}+f_i)\}dV-\iint_{Su}\sigma_{ij}n_j\overline{u}_i dS-\iint_{S\sigma}u_i(\sigma_{ij}n_j-\overline{p}_i)dS\,, \quad (3.1)$$

It is well-known that the Hellinger-Reissner principle (3.1) is subject to the constraints of the stress-strain relations (2.2). Applying the Lagrange multipliers λ_{ij} to eliminate the constraints, we obtain

$$J(\sigma_{ij},e_{ij},u_i,\lambda_{ij})=\iiint\{B+u_i(\sigma_{ij,j}+f_i)+\lambda_{ij}(e_{ij}-a_{ijkl}\sigma_{kl})\}\mathrm{d}V+IB, \quad (3.2)$$

where the multipliers λ_{ij} are considered as independent variables according to traditional Lagrange multiplier theory.

The stationary conditions with respect to e_{ij} can be readily obtained

$$\lambda_{ij}=0\,. \quad (3.3)$$

The phenomenon is called variational crisis after Chien. In this section, the present author wants to propose a new view to explain the equation (3.3). In my own view, the equation (3.3) might reveal a field equation:

$$\lambda_{ij}=\alpha(e_{ij}-a_{ijkl}\sigma_{kl})=0, \quad (3.4a)$$

or

$$\lambda_{ij}=\beta(\sigma_{ij}-b_{ijkl}e_{kl})=0, \quad (3.4b)$$

or

$$\lambda_{ij}=\alpha_{ijkl}(e_{kl}-a_{klmn}\sigma_{mn})\,, \quad (3.4c)$$

where α, β and α_{ijkl} are nonzero constants, and the determinant $|\alpha_{ijkl}|$ is nonzero.

Substituting the identified multipliers into the functional (3.2) yields following generalized variational principles

$$J(\sigma_{ij},e_{ij},u_i)=\iiint\{B+u_i(\sigma_{ij,j}+f_i)+\alpha(e_{ij}-a_{ijkl}\sigma_{kl})(e_{ij}-a_{ijmn}\sigma_{mn})\}dV+IB\,, \quad (3.5a)$$

$$J(\sigma_{ij},e_{ij},u_i)=\iiint\{B+u_i(\sigma_{ij,j}+f_i)+\beta(\sigma_{ij}e_{ij}-A-B)\}dV+IB\,, \quad (3.5b)$$

$$J(\sigma_{ij},e_{ij},u_i)=\iiint\{B+u_i(\sigma_{ij,j}+f_i)\alpha_{ijkl}(e_{kl}-a_{klmn}\sigma_{mn})(e_{ij}-a_{ijmn}\sigma_{mn})\}dV+IB \quad (3.5c)$$

The equation (3.5c) transforms into the equation (3.5b) by setting $\alpha_{ijkl}=\alpha a_{ijkl}$.

It easy to prove that the stationary conditions of the above obtained functionals (3.5a~c) satisfy all the field equations and boundary conditions of elasticity. Setting $\beta=1$ in the functional (3.5b) and integrating by parts, we can obtain the well-known Hu-Washizu principle.

According to Liu [12], the crisis comes from the absence of e_{ij} in the Hellinger-Reissner principle, generally speaking, this means that, if any variable involved only in the constraint equation does not appear explicitly in the original functional, the corresponding Lagrange multiplier will inherently be zero, leading to $\lambda_{ij}=0$.

To eliminate the crisis we replace the term B by $\frac{1}{2}\sigma_{ij}e_{ij}$, so we can rewrite the functional (3.2) as follows

$$J(\sigma_{ij}, e_{ij}, u_i, \lambda_{ij}) = \iiint \{\tfrac{1}{2}\sigma_{ij}e_{ij} + u_i(\sigma_{ij,j} + f_i) + \lambda_{ij}(e_{ij} - a_{ijkl}\sigma_{kl})\}dV + IB \qquad (3.6)$$

The multipliers can be easily identified as

$$\lambda_{ij} = -\tfrac{1}{2}\sigma_{ij} . \qquad (3.7)$$

But the substitution of the identified multipliers into the functional (3.2) results in the original Hellinger-Reissner principle. This means that the constraints have not been eliminated by the multiplier method. The phenomenon is called the second-class of the variational crisis [7]. In my own view, this crisis comes from the Lagrange multiplier method itself. The multipliers are considered as independent variations during the process of variation, but after identification, they all become the functions of the original variables in the functional, that means that the multipliers are not independent variations at all. To overcome the contradiction, we can assume at *a prior* that the multipliers λ_{ij} can be expressed in the form

$$\lambda_{ij} = \lambda_{ij}(\sigma, e, u) = \lambda_{ij}(\sigma_{mn}, e_{mn}, u_m, \sigma_{mn,m}, e_{mn,m}, u_{m,n}) \qquad (3.8)$$

Accordingly, the functional (3.2) can be rewritten in the form

$$J(\sigma_{ij}, e_{ij}, u_i) = \iiint \{\tfrac{1}{2}\sigma_{ij}e_{ij} + u_i(\sigma_{ij,j} + f_i) + \lambda_{ij}(\sigma, e, u)(e_{ij} - a_{ijkl}\sigma_{kl})\}dV + IB . \qquad (3.9)$$

where the multipliers λ_{ij} are now not considered as independent variations.

The Euler equations of the above functional can be readily obtained

$$\tfrac{1}{2}e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i}) - a_{ijkl}\lambda_{kl} + \frac{\delta\lambda_{mn}}{\delta\sigma_{ij}}(e_{mn} - a_{mnkl}\sigma_{kl}) = 0 , \qquad (3.10a)$$

$$\tfrac{1}{2}\sigma_{ij} + \lambda_{ij} + \frac{\delta\lambda_{mn}}{\delta\gamma_{ij}}(e_{mn} - a_{mnkl}\sigma_{kl}) = 0 , \qquad (3.10b)$$

$$\sigma_{ij,j} + f_i + \frac{\delta\lambda_{mn}}{\delta u_i}(e_{mn} - a_{mnkl}\sigma_{kl}) = 0 , \qquad (3.10c)$$

where $\delta F/\delta\Psi = \partial F/\partial\Psi - (\partial F/\partial\Psi_{,i})_{,i}$ is called functional derivative, Ψ is an arbitrary variable.

It can see clearly that if $\lambda_{ij} = -\tfrac{1}{2}\sigma_{ij}$, the equation (3.10b) will be vanished completely leading to the second-class of variational crisis. A careful investigation of the equations (3.10a)~ (3.10c), which should satisfy the field equations of elasticity, the multipliers can be identified as follows

$$\lambda_{ij} = \tfrac{1}{2}\sigma_{ij} - b_{ijmn}e_{mn} . \qquad (3.11)$$

The substitution of the equation (3.11) into the functional (3.2) results in the following generalized variational principle of elasticity

$$\begin{aligned} J(\sigma_{ij}, e_{ij}, u_i) &= \iiint \{2\sigma_{ij}e_{ij} + u_i(\sigma_{ij,j} + f_i) - 2B - A\}dV + IB \\ &= \iiint \{-A + u_i(\sigma_{ij,j} + f_i) + +2(\sigma_{ij}e_{ij} - B - A)\}dV + IB . \end{aligned} \qquad (3.12)$$

To make the problem more simple, we can introduce a new function F defined as

$$F(\sigma, e, u) = \lambda_{ij}(\sigma, e, u)(e_{ij} - a_{ijkl}\sigma_{kl}) , \tag{3.13}$$

the functional (3.2), therefore, can be rewritten in the form

$$J(\sigma_{ij}, e_{ij}, u_i) = \iiint \{B + u_i(\sigma_{ij,j} + f_i) + F(\sigma, e, u)\} dV + IB \tag{3.14}$$

The functional (3.14) with an unknown function F is called trial-functional. The method to identify the unknown F is called the semi-inverse method [9].

Making the above trial-functional stationary yields the following trial-Euler equations

$$\frac{\partial B}{\partial \sigma_{ij}} - \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{\delta F}{\delta \sigma_{ij}} = 0 , \tag{3.15a}$$

$$\frac{\delta F}{\delta e_{ij}} = 0 , \tag{3.15b}$$

$$\sigma_{ij,j} + f_i + \frac{\delta F}{\delta u_i} = 0 . \tag{3.15c}$$

The above trial-Euler equations should satisfy three sets of the field equations. We set

$$\frac{\delta F}{\delta u_i} = 0 , \tag{3.16}$$

so that the equation (3.15c) satisfies the equilibrium equation of elasticity. Accordingly, the unknown F can be identified as

$$F = \lambda \sigma_{ij} e_{ij} + F_1 , \tag{3.17}$$

where λ is a nonzero constant, F_1 is a newly introduced unknown function which should be free from u_i.

Substituting the equation (3.17) into the trial-Euler equations (3.15a) and (3.15b), which should satisfy the stress-strain relations and strain-displacement relations, we can determine the unknown F_1 as follows

$$F_1 = -\lambda(A + B) . \tag{3.18}$$

So we obtain the following generalized variational principle

$$J(\sigma_{ij}, e_{ij}, u_i) = \iiint \{B + u_i(\sigma_{ij,j} + f_i) + \lambda(\sigma_{ij} e_{ij} - B - A)\} dV + IB . \tag{3.19}$$

The above functional is first obtained by Chien by the so-called high-order Lagrange multiplier method.

According to the semi-inverse method, the functional (3.6) can be re-written in the form

$$J(\sigma_{ij}, e_{ij}, u_i) = \iiint \{\tfrac{1}{2}\sigma_{ij} e_{ij} + u_i(\sigma_{ij,j} + f_i) + F(\sigma, e, u)\} dV + IB . \tag{3.20}$$

The trial-Euler equations of the above trial-functional read

$$\tfrac{1}{2}e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i}) + \frac{\delta F}{\delta \sigma_{ij}} = 0\,, \tag{3.21}$$

$$\tfrac{1}{2}\sigma_{ij} + \frac{\delta F}{\delta e_{ij}} = 0\,, \tag{3.21b}$$

$$\sigma_{ij,j} + f_i + \frac{\delta F}{\delta u_i} = 0\,. \tag{3.21c}$$

We set $F = (B\text{-}A)/2$, so that the above equations (3.21a~c) satisfy all the field equations, so we have following functional

$$J(\sigma_{ij}, e_{ij}, u_i) = \iiint \{\tfrac{1}{2}(\sigma_{ij}e_{ij} - A - B) + B + u_i(\sigma_{ij,j} + f_i)\}dV + IB\,. \tag{3.22}$$

4. Derivation of variational principles

We begin with the following trial-functional

$$J(\sigma_{ij}, e_{ij}, u_i) = \iiint L dV \tag{4.1a}$$

in which L is ***a trial-Lagrange function,*** and can be freely constructed, for example, we can write the following one

$$L = \sigma_{ij}e_{ij} + F \tag{4.1b}$$

Hereby σ_{ij}, e_{ij}, u_i are considered as independent variations, F is an unknown function.

There exist several ways to construct energy trial-functionals, details have been discussed in Refs. [1~9]. Here we will identify the unknown F step by step.

Step 1

Making the above trial-functional (4.1a) stationary with respective to σ_{ij}

$$\delta_\sigma J = \iiint \left\{ e_{ij} + \frac{\delta F}{\delta \sigma_{ij}} \right\} \delta\sigma_{ij} dV = 0 \tag{4.2}$$

yields following equations

$$\delta\sigma_{ij}:\ e_{ij} + \frac{\delta F}{\delta \sigma_{ij}} = 0 \tag{4.3}$$

We set

$$\frac{\delta F}{\delta \sigma_{ij}} = -e_{ij} = -\frac{1}{2}(u_{i,j} + u_{j,i})\,, \tag{4.4}$$

so the unknown F can be identified as

$$F = -\frac{1}{2}\sigma_{ij}(u_{i,j} + u_{j,i}) + F_1\,, \tag{4.5a}$$

or

$$F = u_i\sigma_{i,jj} + F_1\,, \tag{4.5b}$$

or in a more general form

$$F = -\frac{1}{2}\alpha\sigma_{ij}(u_{i,j} + u_{j,i}) + \beta u_i \sigma_{ij,j} + F_1 , \tag{4.5c}$$

where F_1 is a newly introduced unknown function free from σ_{ij}., α and β are constants, and it follows $\alpha + \beta = 1$.

Substituting (4.5c) into the trial-Lagrange function (4.1b) results in a renewed one:

$$L = \sigma_{ij}e_{ij} - \frac{1}{2}\alpha\sigma_{ij}(u_{i,j} + u_{j,i}) + \beta u_i \sigma_{ij,j} + F_1 . \tag{4.6}$$

Step 2

The trial-Euler equation for δe_{ij} reads

$$\delta e_{ij} : \ \sigma_{ij} + \frac{\delta F_2}{\delta e_{ij}} = 0 . \tag{4.7}$$

We set

$$F_1 = -A + F_2 , \tag{4.8}$$

so that the trial-Euler equation (4.7) satisfies the field equation (2.2c).

Step 3

The stationary condition with respect to u_i reads

$$\delta u_i : \ (\alpha + \beta)\sigma_{ij,j} + \frac{\delta F_2}{\delta u_i} = 0 \tag{4.9}$$

Supposing the above trial-Euler equation satisfies Eq. (2.1), we, therefore, have

$$F_2 = f_i u_i \tag{4.9}$$

Finally we have following functional

$$J(\sigma_{ij}, e_{ij}, u_i) = \iiint L dV \tag{4.10a}$$

where

$$L = \sigma_{ij}e_{ij} - \frac{1}{2}\alpha\sigma_{ij}(u_{i,j} + u_{j,i}) + \beta u_i \sigma_{ij,j} - A + f_i u_i . \tag{4.10b}$$

The parameters α, β can be chosen arbitrarily ($\alpha + \beta = 1$). The presence of the free parameters offers an opportunity for the systematic derivation of energy-balanced finite elements that combine displacement and stress assumptions, details can be found in Felippa's Ref. [2].

We can obtain some famous generalized functionals by prescribing the free parameters. By setting $\alpha = 1$, $\beta = 0$ the well-known Hu-Washizu principle can be deduced

$$\begin{aligned} J_{H-W}(\sigma_{ij}, e_{ij}, u_i) = & -\iiint \{A - f_i u_i - \sigma_{ij}e_{ij} + \tfrac{1}{2}(u_{i,j} + u_{j,i})\sigma_{ij}\} dV \\ & + \iint_{\Gamma_u} \{\sigma_{ij} n_j (u_i - \bar{u}_i)\} dS + \iint_{\Gamma_\sigma} \bar{p}_i u_i dS \end{aligned} \tag{4.11}$$

By setting $\alpha = 0$, $\beta = 1$, and using the relation $\sigma_{ij}e_{ij} - A = B$, the well-known Hellinger-Reissner principle can be arrived at.

5. CONCLUSION

Hereby a unified generalized variational principle with three free parameters, without using the Lagrange multiplier method, has been successfully established by the semi-inverse method, by specially setting the parameters, we can naturally obtain the well-known Hu-Washizu principle and Hellinger-Reissner principle.

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IZVOĐENJE UOPŠTENIH VARIJACIONIH PRINCIPA BEZ KORIŠĆENJA LAGRANGE-OVIH MNOŽILACA DEO II: PRIMENA NA MEHANIKU KRUTIH TELA

Ji-Huan He

U radu je prikazano alternativno izvođenje Hu-Washizu principa, Hellinger-Reissner princip elastičnosti je dat. Pokazano je da je sadašnji pristup, koji je zasnovan na polu-inverznoj metodi predloženoj od strane He-ija, jednostavan i direktan. Varijaciona kriza kao i njeno uklanjanje se takođe razmatraju u radu, i on takođe otkriva da je polu-inverzna metoda jedan od najboljih pristupa za eliminaciju krize.