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THE MODIFICATION OF GODEL METRIC

UDC 531

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Abstract. *We present the modification of Gödel metric. This modification is performed in order to find out some other perfect fluid solutions. The perfect fluid solution like in the Gödel case is obtained, and it is shown that another type solutions are not defined.*

We shall try to perform a modification of Gödel metric in order to find another perfect fluid solution. Gödel metric is presented in the form

$$ds^2 = a^2\left\{(dx^1)^2 + \frac{e^{2x^1}}{2}(dx^2)^2 + (dx^3)^2 - (e^{x^1}dx^2 + dx^4)^2\right\} \tag{1}$$

Time part of the metric is the summ of two differentials, but it is not an exact differential. If we add a term $x^2e^{x^1}dx^1$ we shall obtain an expression which represents an exact differential:

$$x^2e^{x^1}dx^1 + e^{x^1}dx^2 + dx^4 = d(x^2e^{x^1} + x^4)$$

This is the first step in the modification we propose. Furthremore, we shal change the g_{11} component. Namely, we propose the metric

$$ds^2 = \frac{(x^2)^2e^{2x^1}}{2}(dx^1)^2 + \frac{e^{2x^1}}{2}(dx^2)^2 + (dx^3)^2 - (x^2e^{x^1}dx^1 + e^{x^1}dx^2 + dx^4)^2 \tag{2}$$

The Riemann-Christoffel tensor of this metric is zero, so the metric represents a flat space.

Next, we shall try to use a sort of "inverse" procedure in order to find out a metric which could represent a perfect fluid solution, like the original one. If we drop the $e^{x^1}dx^2$ term which appears in the time part of the metric and make the g_{22} component to be 1, we obtaine the metric:

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$$ds^2 = \frac{(x^2)^2 e^{2x^1}}{2}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (x^2 e^{x^1} dx^1 + dx^4)^2$$

However, this metric could not represent the perfect fluid solution.

We shall make another, a more general assumption:

$$ds^2 = (k + n^2)\xi^2\psi^2(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (n\xi\psi dx^1 + dx^4)^2$$

This metric is represnted by two functions, ξ and ψ, and two parameters, k and n. The function ξ depends on x^1 only, and ψ depends on x^2 only. We introduced $\xi(x^1)$ instead of e^{x1} appearing in (2), and $\psi(x^2)$ instead of linear function x^2.

First, we caculate the Ricci tensor components and we persent only those different from zero. Primes denote the differentiation with respect to x^2.

$$\begin{aligned}
R_{11} &= \frac{k\xi^2}{2(k+n^2)}\{2(k+n^2)\psi\psi'' + n^2(\psi')^2\} \\
R_{14} &= \frac{n\xi}{2(k+n^2)\psi}\{(k+n^2)\psi\psi'' - k(\psi')^2\} \\
R_{22} &= \frac{1}{2(k+n^2)\psi^2}[2(k+n^2)\psi\psi'' - n^2(\psi')^2] \\
R_{44} &= \frac{-n^2}{2(k+n^2)}\frac{(\psi')^2}{\psi^2}
\end{aligned} \tag{*}$$

We present the Ricci scalar curvature also:

$$R = \frac{1}{2(k+n^2)}\left\{4(k+n^2)\frac{\psi''}{\psi} - n^2\frac{(\psi')^2}{\psi^2}\right\}$$

In order to make the considered metric represening the perfect fliud solution, the necessary condition is that R_{22} becomes zero, ie

$$2(k+n^2)\psi\psi'' - n^2(\psi)^2 = 0$$

Solving this equation we get

$$\psi^\alpha d\psi = dx^2, \alpha = -\frac{n^2}{2(k+n^2)}$$

The form of the solution depends on α. If $\alpha = -1$, ψ is exponential, otherwise it is polynomial. In the case of exponential type

$$k = -\frac{n^2}{2}, \quad \psi = e^{Ax^2}, \quad A = const.$$

We put this solution in (*) and calculate the Ricci tensor components

$$R_{11} = -A^2 n^2 \xi^2 e^{2Ax^2}$$
$$R_{14} = A^2 n \xi e^{Ax^2}$$
$$R_{44} = -A^2$$

This form of the Ricci tensor components enables us to interpret the proposed metric as if it represents a perfect fluid solution with the density

$\rho = A^2$, and the fourveloxity $u_\alpha = \{-n\xi e^{Ax^2}, 0, 0, 1\}$.

There is no restriction on function $\xi(x^1)$. The Ricci scalar curvature is constant, as in [1].

And so, we obtained the metric which represents a perfect fluid solution

$$ds^2 = \frac{n^2}{2}\xi^2 e^{2Ax^2}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (n\xi e^{Ax^2} dx^1 + dx^4)^2$$

the metric which is good for "every" function $\xi(x^1)$.

The condition R_{22} we have just used to obtain this metric provides one more solution type. There is a polynomial solution, but it does not provide the energy-momentum tensor of the perfect fluid or any other satisfactory interpretation.

Furthermore, we could impose another condition on the components of the Ricci tensor (*). Having in mind that R_{11}, R_{14} and R_{44} cuold represent $(u_1)^2$, $u_1 u_4$ and $(u_4)^2$ we put the condition

$$R_{11} R_{44} = (R_{14})^2$$

This condition implies

$$\psi\psi'' = \sqrt{\frac{-k}{k+n^2}}(\psi')^2, \quad k \in (-n^2, 0)$$

One solution of this equation is exponential and it is just the one we have obtained.

Another type is polynomial solution

$$\psi = M(x^2 + x_0^2)^{\frac{1}{1-\alpha}}$$

where M and x_0^2 are the constants of integration, and $\alpha = \sqrt{-k/(k+n^2)}$. We kept the constant x_0^2 to prevent $\psi(0) = 0$.

As a next step we put the obtained solution in (*) and we get

$$R_{11} = \frac{k[2\alpha(k+n^2)+n^2]}{2(k+n^2)(1-\alpha)^2} M^2 \xi^2 [x^2 + x_0^2]^{\frac{2\alpha}{1-\alpha}}$$

$$R_{14} = \frac{n[\alpha(k+n^2)-k]}{2(k+n^2)(1-\alpha)^2} M\xi [x^2 + x_0^2]^{\frac{2\alpha-1}{1-\alpha}}$$

$$R_{22} = \frac{2\alpha(k+n^2)-n^2}{2(k+n^2)(1-\alpha)^2} [x^2 + x_0^2]^{\frac{2\alpha-2}{1-\alpha}}$$

$$R_{44} = \frac{-n^2}{2(k+n^2)(1-\alpha)^2}[x^2 + x_0^2]^{\frac{2\alpha-2}{1-\alpha}}$$

We interpret R_{11}, R_{14} and R_{44} as to be $(u_1)^2$, u_1u_4 and $(u_4)^2$ and we take R_{22} to be $(v_2)^2$ of the spacelike vector $v_\alpha = \{0,v_2,0,0\}$.

In such a way we could obtain the metric which represents the Lichnerowicz type of the perfect fluid solution [3] (p.14). But the fourvelocity is the time-like vector and we have to test whether our choice of u_α satisfies that condition. In this case

$$g_{\alpha\beta}u_\alpha u_\beta = \frac{\alpha}{(1-\alpha)^2}[x^2 + x_0^2]^{\frac{2\alpha-2}{1-\alpha}} = -1 .$$

We need $\alpha = 1$ to make $[x^2 + x_0^2]^{\frac{2\alpha-2}{1-\alpha}}$ constant, but then $\frac{\alpha}{(1-\alpha)^2}$ is not defined.

So the polynomial solution has to be descarded.

The modification we proposed provides only one type of solution:

$$ds^2 = (k+n^2)\xi^2 e^{2Ax^2}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (n\xi e^{Ax^2}dx^1 + dx^4)^2$$

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MODIFIKACIJA GEDELOVE METRIKE

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Ova modifikacija Gedelove metrike izvedena je prvo u vremenskom delu metrike sa namerom da se dobiju druga rešenja istog tipa, odnosno da se dobije model idealnog fluida. Dobijeno je rešenje koje predstavlja idealni fluid kao i u slučaju Gedelove metrike, a u kratkoj analizi je pokazano da rešenja drugog tipa nisu definisana.