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A FIELD METHOD FOR SOLVING THE EQUATIONS OF MOTION OF EXCITED SYSTEMS

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Abstract. *In this paper, the field method for solving the equations of motion of holonomic nonconservative systems is extended to system under the action of excitation. Both external and parametric excitation are considered. The asymptotic solutions for these weakly non-linear systems are obtained by combining the field method with the perturbation technique of the dual time expansion.*

1. Introduction

Although the Hamilton Jacobi method is powerful procedure for solving the equations of motion of holonomic conservative systems, in the case of strictly nonconservative systems it fails to be applicable. Vujanovic [1] has proposed a field method as an extension of the Hamilton Jacobi method to holonomic systems with finite degrees of freedom and applied it in the study of weakly nonlinear vibrational problems [2,3], too. This application needs combining with the perturbation technique of the dual time expansion. Note also, that the field method has been extended to the nonholonomic systems [4].

In this paper, the field method is extended to excited one degree of freedom systems. Both parametric and external excitations are considered. The solutions in the first approximation are obtained. These are in the form usually given by the other asymptotic procedure, i.e. in the form of the first order differential equations for the amplitude and phase of motion. It means that the procedure given in the original version [1,2,3] is condensed and improved.

2. The basic concept of the field method

Consider the system governed by:

$$\dot{x}_i = X_i(t, x_1, ..., x_n),\ i = 1, ..., n, \tag{1}$$

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whose initial state is specified as:

$$x_i(0) = x_{i0}\,. \tag{2}$$

The basic assumption of the field method is that one of the states coordinates (generalized coordinate or momentum) can be considered as a field depending on the time and the rest of the coordinates:

$$x_1 = U(t, x_j), \quad j = 2,...,n. \tag{3}$$

By differentiating equation (3) with respect to the time and using the last (n-1) equations (1) we obtain, so called, the basic equation:

$$\frac{\partial U}{\partial t} + \sum_{j=2}^{n} \frac{\partial U}{\partial x_j} X_j(t,U,x_j) - X_1(t,U,x_j) = 0\,. \tag{4}$$

Finding a complete solution of equation (4) in the form depending on the time, variables x_j and arbitrary constants C_i:

$$x_1 = U(t, x_j, C_i), \tag{5}$$

and applying the initial conditions (3), one of the constants, say C_1, can be express by means of the initial values and the rest of constants. Then we obtain the conditioned form solution:

$$x_1 = \overline{U}(t, x_j, x_{i0}, \dot{x}_{i0}, C_j). \tag{6}$$

Vujanovic's theorem:

The solution of the system (1), (2), supposing that $\det(\partial^2 U/\partial x_\alpha \partial C_\beta) \neq 0$, $\alpha,\beta = 2,3,...,n$ can be obtained from the j equations:

$$\frac{\partial \overline{U}}{\partial C_j} = 0, \tag{7}$$

together with equation (6).

So, Vujanovic's method gives us possibility to find the solution of motion passing from the system of ordinary equations to one partial equation. We get the asked solution from the complete solution without any integration, i.e. by simple algebraic operations.

3. Application to the Nonlinear Theory of Vibrations

3.1. Parametric excitation

Firstly, we consider the system governed by a modified Mathieu equation:

$$\begin{aligned} \dot{x} &= p\,, \\ \dot{p} &= -(\delta + 2\varepsilon \cos 2t)x + \varepsilon f(x,p)\,, \end{aligned} \tag{8}$$

where ε is a small parameter, f is a nonlinear function of the state of the system. We restrict our analysis to the case of principal resonance, that is $\delta = \omega_0^2 \approx 1$.

Choosing the generalized coordinate for the field variable:

$$x = U(t, p), \tag{9}$$

we have the basic equation:

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}(-(\delta + 2\varepsilon \cos 2t) + \varepsilon f(x, p)) - p = 0. \tag{10}$$

To find its solution in the closed form is impossible. In order to accomplish the approximate asymptotic solution we will combine the basic concept of the field method with the method of two-time scale expansion [5].

So, we define "slow" and "fast" time: $\tau = \varepsilon t$, $T = t$.

We develop $U(t,p)$ and $p(t)$ in series with respect to the small parameter ε:

$$U(t, p, \varepsilon) = U_0(p_0, T, \tau) + \varepsilon U_1(p_1, T, \tau) + ..., \tag{11a}$$

$$p(t, \varepsilon) = p_0(T, \tau) + \varepsilon\, p_1(T, \tau) + ... \, . \tag{11b}$$

We also suppose that dependence of U_i on p_i is linear and unique: $\partial U/\partial p = \partial U_0/\partial p_0 = \partial U_1/\partial p_1$.

Consequently, the basic equation transforms, after equating terms by powers of ε, into the system of the following partial differential equations:

$$\frac{\partial U_0}{\partial T} - \omega_0^2 \frac{\partial U_0}{\partial p_0} U_0 - p_0 = 0, \tag{12a}$$

$$\frac{\partial U_1}{\partial T} - \omega_0^2 \frac{\partial U_1}{\partial p_1} U_1 - p_1 = -\frac{\partial \overline{U}_0}{\partial \tau} + 2\cos 2T \frac{\partial \overline{U}_0}{\partial p_0} U_0^* - \frac{\partial \overline{U}_0}{\partial p_0} f(U_0^*, p_0). \tag{12b}$$

The complete solution of (12a), according to [1,3], has the form:

$$\overline{U}_0 = \frac{p_0}{\omega_0^2} \operatorname{tg}\psi + \frac{A(\tau)\cos C - B(\tau)\sin C}{\cos\psi}, \quad \psi = \omega_0 T + C, \tag{13}$$

where C is a true constant and $A(\tau)$ and $B(\tau)$ are functions to be determined.

Applying the rule (7) we find:

$$p_0 = \omega_0 \cdot (-A \sin \omega_0 T + B \cos \omega_0 T), \tag{14}$$

and substituting this into (13) we get, so called, solution along trajectory:

$$U_0^* = A \sin \omega_0 T + B \cos \omega_0 T. \tag{15}$$

According to the similar form of the left sides of equations (12), we assume the complete solution of equation (12b) in the same form as (13):

$$U_1 = \frac{p_1}{\omega_0} \operatorname{tg}\psi + \frac{D_1(T, \tau)}{\cos\psi}, \quad \psi = \omega_0 T + C, \tag{16}$$

where D_1 is an unknown function.

Introducing it into the left side of (12b) and (13), (15) to the right one we obtain:

$$\frac{dD_1}{dT} = -\frac{dA}{d\tau}\cos C + \frac{dB}{d\tau}\sin C + \frac{2\cos 2T}{\omega_0^2}\sin\psi(A\sin\omega_0 T + B\cos\omega_0 T) - \frac{1}{\omega_0} f\sin\psi = 0. \quad (17)$$

To express the nearness of δ to 1 we let $1 = \omega_0 + \varepsilon\sigma$, where σ is a constant.

To eliminate the secular terms, we have to equal with zero all term containing cos C and sin C. Respectively, we find:

$$-\frac{dA}{d\tau} - \frac{1}{2\omega_0}A\sin 2\sigma\tau - \frac{1}{2\omega_0}B\cos 2\sigma\tau - \frac{1}{2\omega_0} f\sin\omega_0 T = 0, \quad (18a)$$

$$\frac{dB}{d\tau} + \frac{1}{2\omega_0}A\cos 2\sigma\tau - \frac{1}{2\omega_0}B\sin 2\sigma\tau - \frac{1}{2\omega_0} f\cos\omega_0 T = 0. \quad (18b)$$

By letting $A = a(\tau)\cos\beta(\tau)$, $B = -a(\tau)\sin\beta(\tau)$, and applying some manipulations, the solution for the system (8) is found in the form of the first order differential equations for the amplitude and phase:

$$x = a\cos(\omega_0 T + \beta), \quad (19a)$$

$$a' = -\frac{a}{2\omega_0}\sin(2\sigma\tau - 2\beta) - \frac{1}{2\pi\omega_0}\int_0^{2\pi} f\sin\phi\, d\phi, \quad \phi = \omega_0 T + \beta, \quad (19b)$$

$$a\beta' = \frac{a}{2\omega_0}\cos(2\sigma\tau - 2\beta) - \frac{1}{2\pi\omega_0}\int_0^{2\pi} f\cos\phi\, d\phi. \quad (19c)$$

This solution is completely in accordance with the solution presented in [6]. Note that at this point we returned to the usual point for the methods of slowly varying parameters and applied the averaging principle over a period 2π.

3. 2. External excitation

Here we consider the system:

$$\dot{x} = p, \qquad \dot{p} = -\omega_0^2 x + \varepsilon f(x,p) + \varepsilon k\cos\Omega t, \quad (20)$$

where ε is a small parameter, ω_0^2 and k are known constants. We assume that $\Omega = \omega_0 + \varepsilon\sigma$. It means that we are interested in the case of primary resonance only.

In analogy with the previously presented procedure we choose the variable x for the field, i.e. $x = U(t,p)$

The corresponding basic equation is:

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}(-\omega_0^2 x + \varepsilon f(x,p) + \varepsilon k\cos\Omega t) - p = 0. \quad (21)$$

Using the same scheme for the time scales and the series of U and p given by equations (11), the basic equations becomes:

$$\frac{\partial U_0}{\partial T}-\omega_0^2\frac{\partial U_0}{\partial p_0}U_0-p_0=0, \tag{22a}$$

$$\frac{\partial U_1}{\partial T}-\omega_0^2\frac{\partial U_1}{\partial p_1}U_1-p_1=-\frac{\partial \overline{U}_0}{\partial \tau}-\frac{\partial \overline{U}_0}{\partial p_0}f(U_0,p_0)-\frac{\partial \overline{U}_0}{\partial p_0}k\cos(\omega_0 T+\sigma\tau). \tag{22b}$$

The solution of equation (22a) is given by equation (13), while the solutions for the first component p_0 and solution along trajectory have the same form as, respectively, (14) and (15). Introducing these relations to the equation (22b) leads to:

$$-\frac{dA}{d\tau}-\frac{1}{\omega_0}f\sin\omega_0 T+\frac{k}{2\omega_0}\sin\sigma\tau=0, \tag{23a}$$

$$\frac{dB}{d\tau}-\frac{1}{\omega_0}f\cos\omega_0 T-\frac{k}{2\omega_0}\cos\sigma\tau=0. \tag{23b}$$

After substituting $A=a(\tau)\cos\beta(\tau), B=-a(\tau)\sin\beta(\tau)$ into (23) and applying the averaging principle we find the solution for the (19):

$$x=a\cos(\omega_0 t+\beta), \tag{24a}$$

$$a'=\frac{k}{2\omega_0}\sin(\sigma\tau-\beta)-\frac{1}{2\pi\omega_0}\int_0^{2\pi}f\sin\phi\, d\phi, \quad \phi=\omega_0 T+\beta, \tag{24b}$$

$$a\beta'=-\frac{k}{2\omega_0}\cos(\sigma\tau-\beta)-\frac{1}{2\pi\omega_0}\int_0^{2\pi}f\cos\phi\, d\phi. \tag{24c}$$

This solution is identical with the solution obtained by the method of multiple scales [6].

4. CONCLUSION

In the present paper approximate analytical solutions for weakly nonlinear ordinary differential equations have been found by a field method. The field method of the generalized coordinate has been applied to the case of parametric and external excitations. It is shown that this method gives solutions that are in good agreement with those obtained by means of the other methods.

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PRIMENA METODA POLJA U REŠAVANJU JEDNAČINA KRETANJA SISTEMA SA PRINUDOM

Ivana Kovačić

U ovom radu je izvršeno proširivanje metoda polja za rešavanje jednačina kretanja holonomnog nekonzervativnog sistema na sistem pod dejstvom prinude. Razmatrane su parametarska i spoljašnja prinuda. Asimptotsko rešenje za slabo nelinearni oscilatorni sistem je dobijeno kombinovanjem metoda polja sa perturbacionom tehnikom dvojne vremenske skale.