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EXPLICIT ANALYTICAL SOLUTION FOR THE SHEARING AND RADIAL STRESSES IN COMPOSITE CURVED BEAMS SUBJECTED TO UNSYMMETRICAL BENDING

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Abstract. *The shearing stresses in composite curved beams subjected to unsymmetrical bending can be solved explicitly by differential and integral manipulations of an integral equation. The novel formulae for shearing and radial stresses in composite curved beams are also presented in this paper. These stress formulae, in special case, are reduced to those for composite curved beams subjected to symmetrical bending. The results of an example reveal that the shearing and radial stresses arrive at their maximum values slightly below the centriod of the beam's section.*

Key words: *composite curved beam, shearing stress, radial stress, integral equation, unsymmetrical bending*

1. INTRODUCTION

The formulae of normal stress for composite curved beams with general sectional shape under complicated loads, which lay a sound foundation for further study of shearing and radial stresses for composite curved beams, can be found in Ref.[1]. And the developments of shearing and radial stresses for composite curved beams are finally reduced to solving an integral equation. In the same way as shearing stresses for curved beams of one material studied in Ref. [2,3], the integral equation of shearing stress for composite curved beams is still inexplicit, leading to the inconvenience of numerical simulation. In this paper, we deduce a new equation of solving shearing stresses for composite curved beams with general section under complicated loads, which leads to an explicit solution. By differential and integral transformations, we can obtain formulae for shearing and radial stresses, which have not appeared in any literature.

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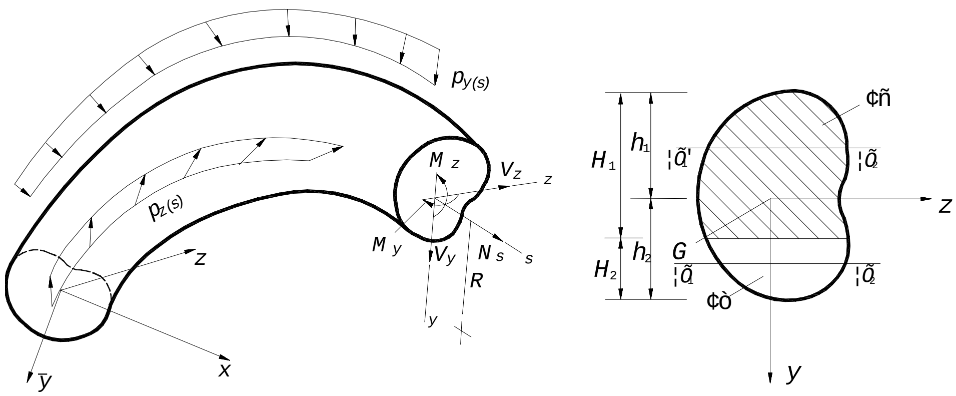
2. The loads and section of a composite curved beam

For the purpose of convenience of discussion, we take the example of a plane curved beam with the constant section combined with two different materials and suppose the two materials are connected with each other firmly, as shown in Fig.1(a), and the composite curved beam is subjected to complicated loads. We denote the y and z components of the external forces per unit length arc by $p_y(s)$ and $p_z(s)$, respectively, and assume that the twisting of any cross section is zero or negligible [3]. In the curvilinear system (s, y, z), s is the arc length measured along the geometric axis, y is a radial coordinate directed toward the center of curvature of this axis, and the direction of z is normal to the plan of the beam. The section of the beam with two different materials I and II is shown in Fig.1 (b). The interface between material I and II parallels to the z axis, and G is the centroid of the beam's section, where h_1, h_2 are the distances from centroid axis z of the section to the upper and lower edges, and H_1, H_2 are the depths of material I and II on the section, respectively.



(a) (b)

Fig. 1. Bending of composite curved-beams with general section shape

3. Shearing stresses in composite curved beams

Let us now examine the equilibrium of a slice of the element of the sectional area A'' for material II, as shown in the Fig.2. For simplicity, we assume that the dimension b' of A'' is the distance y from z axis and parallel to z axis. The normal stresses developed on A'' result in normal force

$$N_2^* = \int_{A''} \sigma_2 dA \tag{1}$$

Substituting the expression for σ_2, which is obtained in Ref. [1], into Eq. (1), we obtain

$$\begin{aligned} N_2^* = \frac{E_2}{H}\{&[(eg - f^2)N_s - (bg - df)M_z + (bf - de)M_y]\int_{A''}\frac{R}{R-y}dA - \\ &-[(bg - df)N_s - (ag - d^2)M_z + (af - bd)M_y]\int_{A''}\frac{Ry}{R-y}dA + \\ &+[(bf - de)N_s - (af - bd)M_z + (ae - b^2)M_y]\int_{A''}\frac{Rz}{R-y}dA\} \end{aligned} \tag{2}$$

in which $H = (aeg - af^2 - b^2 g + 2bdf - d^2 e)$

$$a = E_1\left(A_1 + \frac{1}{R}\int_{A_1} ydA + \frac{1}{R^2}J_{z1}\right) + E_2\left(A_2 + \frac{1}{R}\int_{A_2} ydA + \frac{1}{R^2}J_{z2}\right)$$

$$b = E_1\left(\int_{A_1} ydA + \frac{1}{R}J_{z1}\right) + E_2\left(\int_{A_2} ydA + \frac{1}{R}J_{z2}\right)$$

$$d = E_1\left(\int_{A_1} zdA + \frac{1}{R}J_{yz1}\right) + E_2\left(\int_{A_2} zdA + \frac{1}{R}J_{yz2}\right)$$

$$e = E_1 J_{z1} + E_2 J_{z2} \qquad f = E_1 J_{yz1} + E_2 J_{yz2} \qquad g = E_1 J_{y1} + E_2 J_{y2}$$

$$J_{y1} = \int_{A_1} \frac{z^2}{1 - y/R} dA \qquad J_{yz1} = \int_{A_1} \frac{yz}{1 - y/R} dA \qquad J_{z1} = \int_{A_1} \frac{y^2}{1 - y/R} dA$$

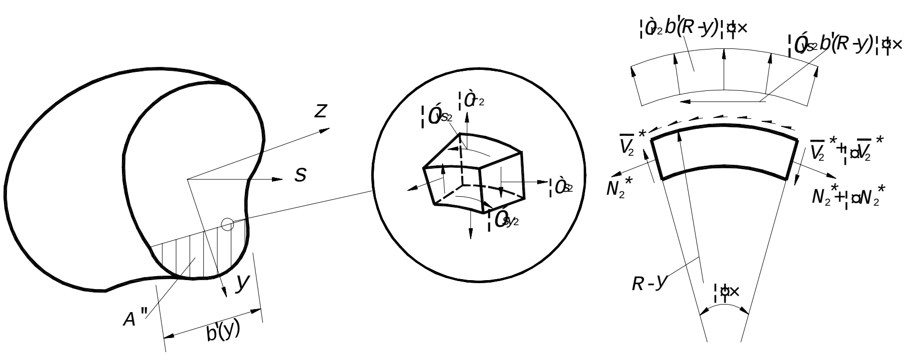
$$J_{y2} = \int_{A_2} \frac{z^2}{1 - y/R} dA \qquad J_{yz2} = \int_{A_2} \frac{yz}{1 - y/R} dA \qquad J_{z2} = \int_{A_2} \frac{y^2}{1 - y/R} dA$$

Hereby N_s, M_y, M_z are the normal force and the moments on any section, respectively. Assume that all the internal forces are in positive direction, as shown in Fig.1(a). E_1, E_2 are modulus of elasticity of material I and II, respectively.

Similarly, the shearing stresses developed on A'' result in a force V_2^* parallel to the cross section, where

$$V_2^* = \int_{A''} \tau_2 dA \tag{3}$$

where $\tau_{sy2} = \tau_{ys2} = \tau_2$. Summing the forces shown in Fig.2(b) in the horizontal direction gives



(a) (b)

Fig. 2. The equilibrium of a slice of the element

$$\tau_2 = \frac{R}{b'(R - y)}\left(\frac{\partial N_2^*}{\partial s} - \frac{1}{R}V_2^*\right) \tag{4}$$

Similarly, the shearing stresses on material I of the cross section can be written as

$$\tau_1 = \frac{R}{b^{'}(R-y)}\left(\frac{\partial N_1^*}{\partial s} - \frac{1}{R}V_1^*\right) \tag{5}$$

where

$$N_1^* = \int_{A_2} \sigma_2 dA + \int_{A^{'}} \sigma_1 dA \tag{6}$$

$$V_1^* = \int_{A_2} \tau_2 dA + \int_{A^{'}} \tau_1 dA \tag{7}$$

Here σ_1, τ_1 are normal and shearing stresses at every point of material I on the cross section, respectively. A_2 is the area of material II on the cross section, and A' is the sectional area of material I on the cross section.

The quantity V_2^* in Fq.(4) must be evaluated using Eq. (3). The integral in Eq. (3), however, involves the unknown function τ_2, thus, Eq.(4) is an integral equation in the dependent variable τ_2. Differentiating Eq.(2) with respect to s and substituting the result and Eq. (3) into Eq. (4), and by means of the equilibrium equation of a typical element [3]

$$\begin{aligned} &\frac{dN_s}{ds} = \frac{V_y}{R} && \frac{dV_z}{ds} = -p_z \\ &\frac{dV_y}{ds} = -p_y - \frac{N_s}{R} && \frac{dM_y}{ds} = V_z \\ &\frac{dM_z}{ds} = V_y \end{aligned} \tag{8}$$

Eq(4) can be further written into the form :

$$\begin{aligned} b^{'}(R-y)\tau_2 = \frac{RE_2}{H}\{&[(eg-f^2)\frac{V_y}{R} - (bg-df)V_y + (bf-de)V_z]\int_y^{h_2}\frac{Rb^{'}}{R-y}dy - \\ &[(bg-df)\frac{V_y}{R} - (ag-d^2)V_y + (af-bd)V_z]\int_y^{h_2}\frac{Rb^{'}y}{R-y}dy + \\ &[(bf-de)\frac{V_y}{R} - (af-bd)V_y + (ae-b^2)V_z]\int_y^{h_2}\frac{R(\int_{\phi_1(y)}^{\phi_2(y)} zdz)}{R-y}dy\} - \int_y^{h_2} b^{'}(y)\tau_2 dy \end{aligned}$$

where, $\phi_1(y)$ and $\phi_2(y)$ are shown in Fig. 1(b). Differentiating each side of the above equation with respect to y, then multiplying each side of the equation by factor $b'(R-y)$ and rearranging it, results in:

$$[b^{'}(R-y)^2\tau_2]^{'}_y = -\frac{R^2E_2V}{H}(\alpha_1 b^{'} - \alpha_2 b^{'} y - \alpha_3\int_{\phi_1(y)}^{\phi_2(y)} zdz)$$

where

$$\alpha_1 = \frac{1}{R}(eg-f^2) - (bg-df) + \mu(bf-de) \qquad \alpha_2 = \frac{1}{R}(bg-df) - (ag-d^2) + \mu(af-bd)$$

$$\alpha_3 = \frac{1}{R}(bf-de) - (af-bd) + \mu(ae-b^2) \qquad V = V_y = \frac{V_z}{\mu}$$

Integrating over y to each side of the above equation and identifying the integral constant by the boundary condition: $\tau_2(h_2) = 0$, we have

$$\tau_2 = \frac{R^2E_2V}{Hb'(R-y)^2}][\alpha_1\int_y^{h_2} b'dy - \alpha_2\int_y^{h_2} b'ydy + \alpha_3\int_y^{h_2} (\int_{\phi_1(y)}^{\phi_2(y)} zdz)dy] \qquad (h_2 - H_2 \le y \le h_2) \quad (9)$$

Similarly, we obtain

$$\begin{aligned}\tau_1 = &\frac{R^2E_1V}{Hb'(R-y)^2}[\alpha_1\int_y^{h_2-H_2} b'dy - \alpha_2\int_y^{h_2-H_2} b'ydy + \alpha_3\int_y^{h_2-H_2} (\int_{\phi_1'(y)}^{\phi_2'(y)} zdz)dy] + \\ &\frac{R^2E_2V}{Hb'(R-y)^2}[\alpha_1\int_{h_2-H_2}^{h_2} b'dy - \alpha_2\int_{h_2-H_2}^{h_2} b'ydy + \alpha_3\int_{h_2-H_2}^{h_2} (\int_{\phi_1(y)}^{\phi_2(y)} zdz)dy] \quad (h_1 \le y \le (h_2 - H_2))\end{aligned} \quad (10)$$

in which we use the condition that shearing stresses are equal at the interface between material I and II

$$\tau_1(h_2 - H_2) = \tau_2(h_2 - H_2)$$

4. Radial stresses in composite curved beams

Summing the forces shown in Fig.2(b) in the vertical direction gives

$$\sigma_{r2} = \frac{R}{b'(R-y)}(\frac{N_2^*}{R} + \frac{\partial V_2^*}{\partial s})$$

Substituting Eq.(9) into Eq. (3) and differentiating the result with respect to *s,* then substituting what is obtained and Eq. (2) into the above equation, we obtain

$$\begin{aligned}\sigma_{r2} = &\frac{RE_2N_s}{Hb'(R-y)}\left[\alpha_4\int_y^{h_2}\frac{b'dy}{R-y} - \alpha_5\int_y^{h_2}\frac{b'ydy}{R-y} + \alpha_6\int_y^{h_2}\frac{(\int_{\phi_1(y)}^{\phi_2(y)} zdz)}{R-y}dy\right] + \\ &\frac{R^3E_2}{Hb'(R-y)}\left\{\left[F_1(y) - F_3(h_2)\int_y^{h_2}\frac{dy}{(R-y)^2}\right]\left(p_y + \frac{N_s}{R}\right) + \left[F_2(y) - F_4(h_2)\int_y^{h_2}\frac{dy}{(R-y)^2}\right]p_z\right\}\end{aligned} \quad (11)$$

where

$$\alpha_4 = [(eg - f^2) - \mu_1(bg - df) + \mu_2(bf - de)]$$
$$\alpha_5 = [(bg - df) - \mu_1(ag - d^2) + \mu_2(af - bd)]$$
$$\alpha_6 = [(bf - de) - \mu_1(af - bd) + \mu_2(ae - b^2)]$$
$$\mu_1 = \frac{M_z}{N_s}, \qquad \mu_2 = \frac{M_y}{N_s}$$

$$F_1(y)=\beta_1\int_y^{h_2}\frac{(\int b^{'}dy)}{(R-y)^2}dy-\beta_2\int_y^{h_2}\frac{(\int b^{'}ydy)}{(R-y)^2}dy+\beta_3\int_y^{h_2}\frac{(\int(\int_{\phi_1(y)}^{\phi_2(y)}zdz)dy)}{(R-y)^2}dy$$

$$F_2(y)=\beta_1^{'}\int_y^{h_2}\frac{(\int b^{'}dy)}{(R-y)^2}dy-\beta_2^{'}\int_y^{h_2}\frac{(\int b^{'}ydy)}{(R-y)^2}dy+\beta_3^{'}\int_y^{h_2}\frac{(\int(\int_{\phi_1(y)}^{\phi_2(y)}zdz)dy)}{(R-y)^2}dy$$

$$F_3(y)=\beta_1\int b^{'}dy-\beta_2\int b^{'}ydy+\beta_3\int(\int_{\phi_1(y)}^{\phi_2(y)}zdz)dy$$

$$F_4(y)=\beta_1^{'}\int b^{'}dy-\beta_2^{'}\int b^{'}ydy+\beta_3^{'}\int(\int_{\phi_1(y)}^{\phi_2(y)}zdz)dy$$

$$\beta_1=\frac{1}{R}(eg-f^2)-(bg-df),\quad \beta_2=\frac{1}{R}(bg-df)-(ag-d^2),\quad \beta_3=\frac{1}{R}(bf-de)-(af-bd)$$

$$\beta_1^{'}=(bf-de),\qquad \beta_2^{'}=(af-bd),\qquad \beta_3^{'}=(ae-b^2)$$

in above equation, the valid region for coordinate y is $[(h_2-H_2),h_2]$.

Similarly, we have

$$\sigma_{r1}=\frac{R}{b^{'}(R-y)}\left(\frac{N_1^*}{R}+\frac{\partial V_1^*}{\partial s}\right)$$

Substituting Eqs. (9) and (10) into Eq. (7), and differentiating the result with respect to s, then substituting what is obtained and Eq. (6) into the above equation, and using Eq. (8), we obtain

$$\begin{aligned}\sigma_{r1}=&\frac{RE_2}{Hb^{'}(R-y)}\left(\alpha_4\int_{h_2-H_2}^{h_2}\frac{b^{'}dy}{R-y}-\alpha_5\int_{h_2-H_2}^{h_2}\frac{b^{'}ydy}{R-y}+\alpha_6\int_{h_2-H_2}^{h_2}\frac{(\int_{\phi_1(y)}^{\phi_2(y)}zdz)}{R-y}dy\right)+\\&\frac{RE_1}{Hb^{'}(R-y)}\left(\alpha_4\int_y^{h_2-H_2}\frac{b^{'}dy}{R-y}-\alpha_5\int_y^{h_2-H_2}\frac{b^{'}ydy}{R-y}+\alpha_6\int_y^{h_2-H_2}\frac{(\int_{\phi_1^{'}(y)}^{\phi_2^{'}(y)}zdz)}{R-y}dy\right)+\\&\frac{R^3E_2}{Hb^{'}(R-y)}\left[F_1(h_2-H_2)(p_y+\frac{N_s}{R})+F_2(h_2-H_2)p_z\right]+\\&+\frac{R^3E_1}{Hb^{'}(R-y)}\left[F_5(y)(p_y+\frac{N_s}{R})+F_6(y)p_z\right]-\frac{R^3(E_1-E_2)}{Hb^{'}(R-y)}\cdot\\&\cdot\left[F_3(h_2-H_2)(p_y+\frac{N_s}{R})+F_4(h_2-H_2)p_z\right]\int_y^{h_2-H_2}\frac{dy}{(R-y)^2}-\\&-\frac{R^3E_2}{Hb^{'}(R-y)}\left[F_3(h_2)(p_y+\frac{N_s}{R})+F_4(h_2)p_z\right]\left(\int_y^{h_2-H_2}\frac{dy}{(R-y)^2}+\int_{h_2-H_2}^{h_2}\frac{dy}{(R-y)^2}\right)\end{aligned}\tag{12}$$

where

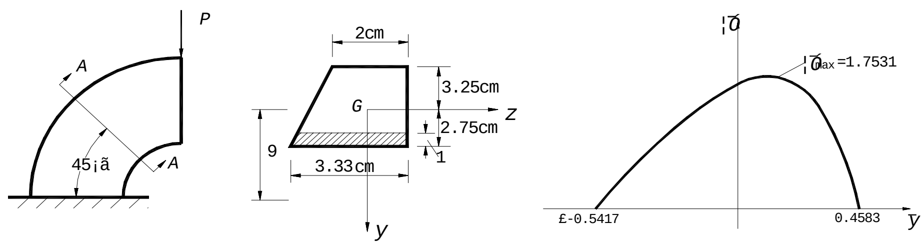
$$F_5(y)=\beta_1\int_y^{h_2-H_2}\frac{(\int b^{'}dy)}{(R-y)^2}dy-\beta_2\int_y^{h_2-H_2}\frac{(\int b^{'}ydy)}{(R-y)^2}dy+\beta_3\int_y^{h_2-H_2}\frac{(\int(\int_{\phi_1^{'}(y)}^{\phi_2^{'}(y)}zdz)dy)}{(R-y)^2}dy$$

$$F_5(y) = \beta_1' \int_y^{h_2 - H_2} \frac{(\int b' dy)}{(R-y)^2} dy - \beta_2' \int_y^{h_2 - H_2} \frac{(\int b' y dy)}{(R-y)^2} dy + \beta_3' \int_y^{h_2 - H_2} \frac{(\int (\int_{\phi_1'(y)}^{\phi_2'(y)} z dz) dy)}{(R-y)^2} dy$$

in which the valid region for coordinate y is $[-h_1, (h_2 - H_2)]$.

5. NUMERICAL EXAMPLE

A circular beam with trapezoid cross section, made of two materials, carries a concentrated load P at its free end, while leaves the other end fixed, as shown in Fig.3(a).



(a) (b)

Fig. 3. Shearing stress distribution and the maximum value of $\bar{\tau}_{max}$ at 45° section of the composite curved beam with trapezoid section

Supposing that the ratio of modulus of elasticity of material I and II is 3, we can obtain the formulae of shearing and radial stresses for the composite curved beam by simply equating $V_z = M_y = 0$ and $p_y = p_z = 0$ in Eqs.(9), (11) and (12). Section constants for the beam shown in Fig.3, after some lengthy integration, are

$$J_{y1} = 4.007 cm^4, \quad J_{z1} = 22.519 cm^4, \quad J_{yz1} = -2.499 cm^4$$
$$J_{y2} = 7.569 cm^4, \quad J_{z2} = 26.092 cm^4, \quad J_{yz2} = -2.889 cm^4$$

From Eqs.(9) and (10), we can compute the variation in shearing stresses over the section AA, which is plotted in Fig.3(b).

6. CONCLUSIONS

a. In this paper, the general formulae for shearing and radial stresses in composite curved beams have been obtained, which obviously satisfy the mechanical boundary conditions on the top and bottom of the beams and the continuity conditions of shearing and radial stresses at the interface between material I and II.
b. The results of the numerical example show that the shearing and radial stresses in the beam arrive at their maximum values slightly below the centroid of the section, and both of them are the same in value at every point on the 45° section, but the opposite in sign.

c. The explicit analytical solution presented in this paper are valid for shearing and radial stresses of composite curved beams with general section under complicated load, while the formulae obtained in Ref. [4] are valid only for composite curved beams subjected to symmetry bending.

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EKSPLICITNO ANALITIČKO REŠENJE ZA SMIČUĆE I RADIJALNE NAPONE U KOMPOZITNIM ZAKRIVLJENIM ŠTAPOVIMA IZLOŽENIM NESIMETRIČNOM SAVIJANJU

Ai-Min Yu

Smičući naponi u kompozitnim zakrivljenim štapovima izloženim nesimetričnom savijanju mogu biti rešeni eksplicitno diferencijalnim i integralnim transformacijama integralne jednačine. Nove formule za smičuće i radijalne napone u kompozitnim zakrivljenim štapovima su takođe prikazane u ovom radu. Ove naponske formule su u specijalnom slučaju redukovane na one za kompozitne zakrivljene štapove izložene simetričnom savijanju. Rezultati za jedan primer otkrivaju da smičući i radijalni naponi postižu maksimalne vrednosti malo (neznatno) ispod centrioda preseka grede.