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A GEOMETRICAL FORMULATION OF DEFORMABILE MEDIA KINEMATICS

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Abstract. *The new vector methods in the linear homogeneous deformation kinematics including their topological and algebraic characterization are formulated.*

1. Introduction

In approximation of globally averaged final linear homogenous standard contiinual medium, the geometrical aspect of kinematic description of the material is rigorously analised.. The approximation itself is conservative broadening of the term "global" rigid body. This model can be abandoned, if necessary. As a result of the basic Helmholtz theorem for kinematics, in this case new vector products for dilatation and shearing are defined after the analogy with rotatory motion. The primary idea is to base new laws of dynamics for the motions on the basis of the fact that rotatory motion follows the basic law of dynamics.The cost of this idea is: abandoning of the present Euclidean geometry and introduction of the global affine geometry.This is achieved by employing critical analysis of the objectivity principle. In this way Galileo's group of motions is also broadened. Mathematical innovation consists of the introduction of new vector products, and with this also the introduction of non-Lie's (non-quantum) algebras, groups and (differential) geometries. Cited references suggest that this is the new method. This paper is an introductory character.

2. Average global linear approximation of the final continual medium

The basic prerequisites for establishing this approximation are: 1) The space with the observer is a vector one, three-dimensional and three exists Descartes coordinate system. So all Latin indices an on bottom places; 2) the material observed is standard. Special rheological models are obtained, by assumption, on the basis of this model by weakening

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or setting particular requirements. In Euler's description, used in this paper, relevant functions of the position and time are continually differential sufficient number of times; 3) The body is observed during the two moments of time, so the discussion is of a geometrical type. In Euler's method $\boldsymbol{X}(\boldsymbol{x},t)$, $\boldsymbol{X}$ and $\boldsymbol{x}$ represent vectors of the continual medium particle position, i. e. of spatial points in relation to outer Descartes coordinate system. Two points A and B are observed (A ≠ B in every moment of time) of the continual medium during two different moments $t = t_0$ and $t > t_0$ (A_0 and B_0, i. e. A_t and B_t). If $\mathbf{e}_i$ $i = 1, 2, 3$ are unit vectors of coordinate axes, then $\mathbf{X} = X_i\,\mathbf{e}_i$, is valid, i.e. $\mathbf{x} = x_i\,\mathbf{e}_i$ ($\mathbf{E}_i = \mathbf{e}_i$). The opposite case $\mathbf{E}_i \neq \mathbf{e}_i$, two outer Descartes coordinate referentiall systems connected, for example, with Euclid's group, do not represent, in principle, a different case. The following differences can be defined in this way: $\Delta X_{i\,disA} = X_{i\,At} - X_{i\,A0}$, $\Delta X_{i\,disB} = X_{i\,Bt} - X_{i\,B0}$, $\Delta X_{i\,AB} = \Delta X_{i\,disA} - \Delta X_{i\,disB} + \Delta x_{i\,AB0}$. The first and the second formulae describe vectors of displacements of arbitrary points A and B (translation of the pole A, i.e. of the body-point B), while the third one expresses components of the vector which connects the points A and B during the moment of the time t. Total displacement is $(\Delta X_{i\ tot})_{A0} = (\Delta X_{i\,AB} - \Delta x_{i\,AB0})$, if translation of the pole A_0 is excluded. Developing into Taylor's order $(\Delta X_{i\,tot})_{A0}$ after $\Delta x_{j\,AB0}$ for an arbitrary continual body, linear approximation is used in three cases: 1) The body is of small dimensions ($\Delta x_{i\ AB0}$) and $(\Delta X_{i\ tot})_{A0}$ of a small size, while the values of higher derivatives are final in accordance with the approximation; 2) The values of higher derivatives are small while the body is of final dimensions under the same condition; 3) Both derivatives and body dimensions are small. The first and the third case do not allow final rotations of the body, while the second one allows it. From the point of view of natural broadening of the final rigid motion, the first and the third cases are useless. Special problem is connected with the approximation dependence on the points A_0 and B_0 .The idea is to include somehow the influence of other members of the Taylor's order is used. The total result would be dependence of gradient deformation average value on time. Taylor's development explicitly put down:

$$(\Delta X_{i\ \ tot})_{A0} = \left(\frac{\partial X_{i,\ AB}}{\partial x_j} \Bigg|_{A0} \Delta x_{j\ \ AB0} \right) \text{Res} = m'_{ijA0}\,(t)\quad \Delta x_{j\,AB0}\,\text{Res}. \tag{1}$$

It is averaged by means of expression m'_{ijA0} Res, which exists by assumption. Explicitly given, it is:

$$< m'_{ijA0}\,\text{Res} > (t) = \frac{1}{V_{A0}V_{B0}} \iint\limits_{V_{A0}V_{B0}} (m'_{ijA0}\,\text{Res}(\Delta\mathbf{x}_{AB0}, t))\,d\mathbf{x}_{A0}\,d\mathbf{x}_{B0}. \tag{2}$$

Concisely written down:

$$< (\Delta X_{i\ \ tot}) > (t) =< m_{ij}(t) > \quad \Delta x_{j\ \ AB0}. \tag{3}$$

The mistake of the method is checked up by dispersion defined in a standard way.

If the body translation is also introduced, Helmholtz theorem can be expressed in the following way.

Theorem 1. In the approximation stated every elementary displacement of the continual medium can be expressed in the form of elementary motions: of spatial translation $< dX_{i,\ \ disA} > \; = \; < dX_{i,\,tr} >$, real rotation, shearing and dilatation. Elementary rotation $< dX_{i,\,rot} >$ corresponds to the anti - symmetrical part $< dm_{ij} >_A (t_0)$ of the matrix $< dm_{ij} > (t_0)$

of the deformation averaged gradient. To the elementary shearing $< \mathrm{d}X_{i,sh} >$ corresponds extra diagonal part, while to the dilatation $< \mathrm{d}X_{i,dil} >$ corresponds the diagonal part of the symmetrical part $< \mathrm{d}m_{ij} >_S (t_0)$ of the matrix cited. Configurationally manifoldness is twelve-dimensional. The written form and structure of this statement are evident:

$$< dX_{i\ tot} > (t_0) = < dX_{i\ tr} > + < dX_{i\ rot} > + < dX_{i\ dil} > + < dX_{i\ sh} >, \tag{4}$$

$$\begin{gathered} < dX_{i\ tot} > (t_0) = < \Delta X_{i\ ABt} - \Delta x_{i\ AB0} > + < dX_{i\ tr} > = < dX_{i\ tot} >, \\ < dm_{ij} >_S (t_0) = \frac{1}{2}(< dm_{ij} > (t_0) + < dm_{ji} > (t_0)), \\ < dm_{ij} >_A (t_0) = \frac{1}{2}(< dm_{ij} > (t_0) - < dm_{ji} > (t_0)), \\ t = t_0 + dt. \end{gathered} \tag{5}$$

Where the relations for the corresponding values are:

Some non – translatory members in the equation (4) are obtained by means of adequate application of the second and third expressions in the equation (5) on any $\Delta \mathbf{x}_{j,Ab0}$. Along with that it is considered that nine elements of the matrix $< \mathrm{d}m_{ij} > (t_0)$ are insufficient. Due to the stated process of averaging this procedure is also realized for every two moments t and t + dt in the time interval observed. Rigid elementary rotation is obtained in case that $< m_{ij} >_A (t)$ $(< m_{ij} >_S (t) \equiv 0)$, and having in mind that it is Lie's algebras so (3,**R**) of anti - symmetrical matrices the use of exponential function of the parameterized anti - symmetrical matrix generates final rotations (local parameterization of the Lie's group SO (3,**R**)). Three parameters – rotation angles make associated vector for the anti - symmetrical matrices stated. In this way vector product for of two vectors is defined (which also satisfies the relations of Lie's algebra, as well as commutators of the associated matrices). Remembering the theorem on polar composition of the regular tensor, in case of using exponential functions $< M_{ij} >_{\mathrm{rot}} (t) = \exp (< m_{ij} >_{\mathrm{A}} (t))$, $t \in [t_0, \infty)$ represents orthogonal matrix and describes relations. So, this approach realizes theoretical existence of the term final rigid body in a natural way.

$$\begin{aligned} \theta_{i\ rot}(t) &= \frac{1}{2}\varepsilon_{ijk} < m_{jk} > (t), \\ \theta_{i\ dil}(t) &= \delta_{ijk} < m_{jk} > (t), \\ \theta_{i\ sh}(t) &= \frac{1}{2}|\varepsilon_{ijk}| < m_{jk} > (t). \end{aligned} \tag{6}$$

The associatively process of three-indices values of deformation is carried out in this way:

ε_{ijk} are known as Levi- Civita symbols, while δ_{ijk} are suggested by the author of this paper (they are equal to one for $i = j = k$ and to zero in all other cases). Tensor nature of all these values is easily defined.

Lemma 1. The process of association is one and sole.

The proof results out of relations:

$$\varepsilon_{ilm}\varepsilon_{jlm} = |\ \varepsilon_{ilm}\ \|\ \varepsilon_{jlm}\ | = 2\delta_{ij},$$
$$\delta_{ilm}\delta_{jlm} = \delta_{ij}. \tag{7}$$

This lemma establishes isomorphism among the corresponding three-indices values and matrices. Now, the equation (4) can be given in the following form:

$$< dX_{i\ \ tot} > (t_0) =$$
$$< dX_{i\ \ tr} > + \varepsilon_{ijk} d\theta_{j\ \ rot} \Delta x_{k\ \ AB0} + \delta_{ijk} d\theta_{j\ \ dil} \Delta x_{k\ \ AB0} + |\ \varepsilon_{ijk}\ | d\theta_{j\ \ sh} \Delta x_{k\ \ AB0}. \tag{8}$$

$$\mathbf{a} \times_1 \mathbf{b} = \varepsilon_{ijk} \mathbf{e}_i a_j b_k \quad , \quad \mathbf{a} \times_2 \mathbf{b} = \delta_{ijk} \mathbf{e}_i a_j b_k,$$
$$\mathbf{a} \times_3 \mathbf{b} = |\ \varepsilon_{ijk}\ |\ \mathbf{e}_i a_j b_k. \tag{9}$$

All products of the two vectors $\mathbf{a} = a_i\mathbf{e}_i$ i $\mathbf{b} = b_i\mathbf{e}_i$ can be expressed in the following form:

The last two products have not the same directions and intensity in various coordinate systems. An example can be the system S'obtained by rotation of a certain system S around the given axis (for example, axis z – coordinate system is fixed in this way without lowering of the generality) for a certain angle θ. During the association of the vector to the anti - symmetrical part of the deformation tensor corresponding vector product is also introduced. So, $\mathbf{A} \leftrightarrow \mathbf{a}$ i.e. $\mathbf{c} = \mathbf{a} \times_1 \mathbf{b} = \mathbf{A}\ \mathbf{b}$. If rotation is carried out being defined with matrix $\mathbf{R}$, it will be: $\mathbf{c'} = \mathbf{Rc} = \mathbf{R}\ (\mathbf{a} \times_1 \mathbf{b}) = \mathbf{R}\ \mathbf{A}\ \mathbf{b} = \mathbf{R}\ \mathbf{A}\ \mathbf{R}^{\mathbf{T}}\mathbf{R}\ \mathbf{b} = \mathbf{A'b'} = \mathbf{a'} \times_1 \mathbf{b'}$. In fact, the situation is as follows: Relation $\mathbf{a'} = \mathbf{R}\ \mathbf{a}$ should correspond to $\mathbf{A'} = \mathbf{RAR}^{\mathrm{T}}$ ($\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}$). Let axis z be that one around which the rotation is carried out for a certain angle (det $\mathbf{R} = 1$). For an arbitrary vector $\mathbf{a}$ and vector product "$\times_1$" the process of association satisfies the previous rotations. For the diagonal tensor and the vector associated with it, the angle θ must satisfy the relation sinθ cosθ = 0. The discussion is similar for the extra-diagonal symmetrical part of the tensor, too.

Theorem 2. Three-indices values connected with final dilatation and shearing do not represent vector values and do not represent objective physical values in Euclid's geometry.

The possibility left is – global affined geometry. It is characterized by the fact that the basic objective relation maintains valid - it is the relation of vector parallelism. In this affined geometry all stated three-indices values are objective vectors.

The last statement is the result of transformational linearity; global affined geometry is accepted to be the basic geometry in this paper. The author is of the opinion that the work in practice will show the necessity of this solution. Values $\theta_{i\ rot}$, $\theta_{i\ dil}$, $\theta_{i\ sh}$ will be called vectors of generalized angles.

Theorem 3. The first vector product is anticommutative and non-associative, while the other two are commutative. The second vector product is associative, while the third one is not.

The author's idea is to introduce anticommutators for the vector products connected with dilatation and shearing and to define their features (the structure of non Lie's algebras). In this way quantum Lie's algebras cannot be obtained [5]. On the basis of relations [6] the basic reason for this is that the multiplicity observed is locally diffeomorphic towards affined space. The features of these anticommutators are expressed in the following way:

Theorem 4. Vector products, which are connected with deformation motion, satisfy the following relations:

$$\begin{aligned}
&\mathbf{a}\times_{2,3}\mathbf{b}+\mathbf{b}\times_{2,3}\mathbf{a}=[\mathbf{a},\ \mathbf{b}]_{2,3},\\
&[[\mathbf{a},\ \mathbf{b}]_2,\ \mathbf{c}]_2=[\mathbf{a},\ [\mathbf{b},\ \mathbf{c}]_2]_2,\\
&[[\mathbf{a},\ \mathbf{b}]_{2,3},\ \mathbf{c}]_{2,3}+[[\mathbf{c},\ \mathbf{a}]_{2,3},\ \mathbf{b}]_{2,3}+[[\mathbf{b},\ \mathbf{c}]_{2,3},\ \mathbf{a}]_{2,3}-\\
&-([[\mathbf{a},\ \mathbf{c}]_{2,3},\ \mathbf{b}]_{2,3}+[[\mathbf{b},\ \mathbf{a}]_{2,3},\ \mathbf{c}]_{2,3}+[[\mathbf{c},\ \mathbf{b}]_{2,3},\ \mathbf{a}]_{2,3})=\mathbf{0}.
\end{aligned} \tag{10}$$

The problem of non-Lie's group's structures, which correspond to deformations, is set. We shall assume that for every algebraic element, in all cases only one curve should exist over the group so to transform, by means of an exponential function certain zero environment of an algebra on to certain environment of the group unit in a diffeomorphic way (local description). In this way defined deformation motions should be better understood and, through kinematics, the way towards dynamics would be paved. In case of use of non-linear members of the Taylor's order the situation is the same referring to the process of averaging and vector products. The proof for this statement is not given in this paper for the moment.

CONCLUSION

The next step is global affine broadening of the Galileo's group (kinematics). Naturally, a number of questions are set in relation to the behaviour of various physical values in case of differential affined geometry and a series of these new non-linear geometries. All the more so as the references for these cases are insufficient.

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GEOMETRIJSKA FORMULACIJA KINEMATIKE DEFORMABILNIH SREDINA

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Nove vektorske metode u kinematike linearne homogene deformacije, uključujući i njene topološke i algebarske karakterizacije su formulisane.