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SENSITIVITY MODEL FOR THE VARIABLE STRUCTURE SYSTEMS WITH VECTOR CONTROL ACTION

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Č. Milosavljević, S. Lj. Rančić

University of Niš, Faculty of Electronic Engineering, 18000 Niš, Yugoslavia

Abstract. *In this paper we discuss sensitivity for a special class of automatic control systems so-called variable structure systems. The sensitivity model is defined for the multivariable input systems and the connection with existing conclusions is outlined about the invariance conditions in the sliding mode.*

1. INTRODUCTION

The main subject of the sensitivity theory is the influence of the some system parameter variations, on their dynamics behavior. Sensitivity analysis is considered with how to define and determine sensitivity function or, adequately, sensitivity model (Tomović and Vukobratović [5]). The problem of determining sensitivity functions in many ways depends on what kind of system is to be analyzed and which parameters are to be studied. Therefore we need to develop different analytic or simulation methods for determining the sensitivity function.

The variable structure systems (VSS) are nonlinear. The control action is discontinuous in VSS and from (1) the state space derivatives are discontinuous too. By definition of the sensitivity coefficients it is clear that they are discontinuous to. A general problem of the coefficient discontinuity is presented and discussed in (Tomović and Vukobratović [5]) and (Bingulac [1]). In (Matić [4]) defined sensitivity model for VSS with a scalar control action. In this paper we will tray to determine the sensitivity model for VSS with a vector control action. The significant interest has been made on the sliding mode.

2. THE PROBLEM SETTING

The multiinput systems are described with matrix differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\,, \tag{1}$$

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where: $\mathbf{x} \in \Re^n$ - n state vector-column (it is understood that all its elements are available for forming a control action); $\mathbf{A} = \{a_{ij}\}$ $n \times n$ - matrix of systems coefficients; $\mathbf{B} = \{b_{ij}\}$ $n \times m$ - matrix, with linearly independent columns $\mathbf{b}_i$; : $\mathbf{u} \in \Re^m$ - m control vector-column, with linearly independent elements, called control functions.

The motion of such system can be represented most conveniently in n dimensional Euclidean space E^n. If the control action is suitable chosen, the phase trajectories can be attained in the vicinity of the hyperplane G, oriented toward it. The phase point which reaches such a hypersurface cannot leave it again, but it keeps shifting from one of its sides to another, moving on the average along the trajectory belonging G. The described motion is called the sliding mode and G is the sliding hypersurface. For multiinput systems considered in this paper, it is possible to realize the sliding mode both on certain hypersurfaces and their intersections. There are different methods to organize sliding modes (Hang and Gao [3]). One of this methods is so-called hierarchical method in which the control function u_1 provides the sliding mode on G_1, the control function u_2 on $G_1 \cap G_2$, etc. and finally, u_m guarantees the sliding mode on

$$G_1 \cap G_2 \cap \ldots \cap G_m \,.$$

The most acceptable hypersurfaces are the hyperplanes passing through the origin defined by equations:

$$g_i = \mathbf{c}_i^{\mathrm{T}} \mathbf{x} = 0 \,,$$

where $\mathbf{c}_i$ is a constant n row vector and g_i is a scalar called the switching function. When the final stage of motion the phase point moves in the sliding mode along the intersections of all m hypersurfaces G_i, the system phase trajectories belong to the subspace E^{n-m} defined by equation:

$$\mathbf{Cx} = \mathbf{0}, \tag{2}$$

where $\mathbf{C}$ is an $(m \times n)$ constant matrix with linearly independent rows $\mathbf{c}_i$. The motion in E^{n-m} will be considered as the main part of the motion, and the motionpreceding in the sliding mode in E^{n-m} as the preliminary part. The conditions the control function has to fulfill to provide the sliding mode are given in (Draženović [2]). The control vector is defined in the following way:

$$\mathbf{u} = \mathbf{\Psi x} \,,$$

where $\mathbf{\Psi}$ is a matrix of switching coefficients with elements:

$$\psi_{ij} = \begin{cases} \psi_{ij1}, & \text{for } g_i x_j < 0, \\ \psi_{ij2}, & \text{for } g_i x_j > 0. \end{cases}$$

If we denote $\mathbf{\Psi}_i$ to be the i-th row of matrix $\mathbf{\Psi}$ then, the control functions have the form

$$u_i = \mathbf{\Psi}_i^{\mathrm{T}} \mathbf{x} \,. \tag{3}$$

If the sliding conditions are satisfied, then for sliding on hypersurface, we can define a system sensitivity model. The vector of sensitivity coefficients is defined as:

$$\mathbf{s}^{ij} = \frac{\partial \mathbf{x}}{\partial a_{ij}}$$

where $\mathbf{s}^{ij} = \mathbf{s}^{ij}(s_1^{ij},...,s_n^{ij})$ is a vector of sensitivity coefficients with respect to the parameter a_{ij} (later in this paper we will omit index ij).

3. SENSITIVITY MODEL

By applying the partial differentiation to equation (1) with respect to the parameter a_{ef} (an element of matrix **A**), we can obtain the sensitivity equation. The continuity of the state space is a necessary condition for the partial derivative, with respect to the parameter of differentiation to be defined. In the real systems this condition is satisfied almost always. The sensitivity equation has the form:

$$\dot{\mathbf{s}} = \mathbf{A}^*\mathbf{s} + \mathbf{Q}^*\mathbf{x}^* + \mathbf{B}\frac{\partial \mathbf{u}}{\partial a_{ef}} \tag{4}$$

with

$$\mathbf{Q}^* = \frac{\partial \mathbf{A}}{\partial a_{ef}} \text{ and } \frac{\partial \mathbf{u}}{\partial a_{ef}} = \begin{bmatrix} \frac{\partial u_1}{\partial a_{ef}} & \frac{\partial u_2}{\partial a_{ef}} & \cdots & \frac{\partial u_n}{\partial a_{ef}} \end{bmatrix}^{\mathrm{T}}.$$

The symbol * in the exponent denotes altering structure of matrices **A** and **Q** in the preliminary part. We are considered our system from the moment t_1, when the sliding mode along G_1 has begun provided by control function u_1. With t_i we will define the moment when the sliding mode along G_i has begun provided by the control function u_i. This is a moment when the switching function g_i changes its sign. In the interval $[t_i, t_{i+1}]$, the switching function does not change its sign, but it is possible for the state space coordinates to change their signs. Therefore, if we describe the function sgn(*) using step function,

$$\operatorname{sgn}(*) = -1 + 2h(*),$$

with

$$\frac{\partial \operatorname{sgn} g}{\partial a_{ef}} = \frac{\partial \operatorname{sgn} g}{\partial g}\frac{\partial g}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial a_{ef}},$$

we can obtain:

$$\frac{\partial \operatorname{sgn} g_i}{\partial a_{ef}} = 2\delta_g \sum_{k=1}^{n} c_{ik},$$

where δ_g is Dirac impulse. In the same way we obtain:

$$\frac{\partial \operatorname{sgn} x_i}{\partial a_{ef}} = 2\delta_{x_i} s_i .$$

By differentiation of equation (3) we get:

$$\frac{\partial s_i}{\partial a} = \delta_g + f_i + \delta_{x_i} ,$$

where the functions of the right hand side are

$$f_i = \sum_{k=1}^{n} \psi_{ik}^{o} s_k + \sum_{k=1}^{n} \psi_{ik}^{\Delta} s_k \operatorname{sgn} x_k \operatorname{sgn} g_i ,$$

- function with a stepwise discontinuity,

$$\delta g_i = 2\delta_{g_{ii}} \sum_{k=1}^{m} \psi_{ki}^{\Delta} |x_k| \sum_{j=1}^{n} c_j s_j$$

- function with an impulswise discontinuity,

$$\delta_{x_i} = \sum_{l=1}^{p} 2\delta_{x_l} \sum_{k=1}^{n} \psi_{ki}^{\Delta} s_k x_k ,$$

- function with an impulswise discontinuity, and

$$\psi_{ik}^{o} = \frac{\psi_{ik1} + \psi_{ik2}}{2}; \quad \psi_{ik}^{o} = \frac{\psi_{ik1} - \psi_{ik2}}{2} .$$

Here, p denotes a number of coordinates changing their sign in the interval $[t_i, t_{i+1}]$, $\delta_{g_{ii}}$ is an impulse in the moment t_i, and δ_{x_l} is an impulse in the moment, when some of state space coordinates change their signs. Matrices **A*** and **Q*** have modified their structures every time when some sign changing is performed, and have decreased order by one (Draženović [2]), when the switching function is switched. Their explicit structures are not defined hire. In this context it is not so important, because we are interested just in what type of discontinuity the sensitivity coefficients have, in the so called preliminary part.

When in the final stage of motion the phase point moves in the sliding mode along the intersections of all of m hyperplanes G_i the system phase trajectories belong to the subspace E^{n-m} defined with (2). Since in the sliding mode the phase point does not leave the subspace E^{n-m}, it is accepted that the phase velocity also belongs to E^{n-m}, that is:

$$\mathbf{C}\dot{\mathbf{x}} = \mathbf{0} . \tag{5}$$

By substitution of value $\dot{\mathbf{x}}$ from (1), if the matrix **CB** is nonsingular, u is determined in an unique manner from (5),

$$\mathbf{u} = -(\mathbf{CB})^{-1}\mathbf{CAx} ,$$

and the system model has the form:

$$\dot{\mathbf{x}}^m = \mathbf{A}^m \mathbf{x}^m , \tag{6}$$

where $\mathbf{A}^m$ denotes system matrix after the elimination of the control vector. It is clear that the vector, denoted with $\mathbf{x}^m$, has (n-m) coordinates.

We can get sensitivity model in sliding mode by applying partial differentiation on equation (6) with respect to the parameter a_{ef}, in the form

$$\dot{\mathbf{s}}^m = \mathbf{A}^m \mathbf{s}^m + \frac{\partial \mathbf{A}^m}{\partial a_{ef}} \mathbf{x}^m . \tag{7}$$

To calculate partial derivative on the right side of equation (7) we have to find the explicit form of the matrix $\mathbf{A}^m$. First we introduce a transformation matrix $\mathbf{T}$ which maps the subspace E^{n-m} in to E^n, i.e.,

$$\mathbf{x} = \mathbf{T}\mathbf{x}^m .$$

The matrix $\mathbf{T}$ is a matrix of the basis vectors of the subspace E^{n-m} and it can be written in a general form like:

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ t_{n-m+1,1} & t_{n-m+1,2} & \dots & t_{n-m+1,n-m-1} & t_{n-m+1,n-m} \\ . & . & . & . & . \\ t_{n,1} & t_{n,2} & \dots & t_{n,n-m-1} & t_{n,n-m} \end{bmatrix}$$

where the elements t_{ij}, for $i = n - m + 1,...,n$ and $j = 1,...,n - m$, depend on matrix $\mathbf{C}$. With the introduced transformation, we can write system's model in the sliding mode in the following form:

$$\mathbf{T}\dot{\mathbf{x}}^m = (\mathbf{I} - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C})\mathbf{A}\mathbf{T}\mathbf{x}^m ,$$

i.e.,

$$\mathbf{T}\dot{\mathbf{x}}^m = \mathbf{R}\mathbf{A}\mathbf{T}\mathbf{x}^m , \tag{8}$$

where the matrix $\mathbf{R} = (\mathbf{I} - \mathbf{B}(\mathbf{CB})^{-1}\mathbf{C})$ do not depend on the elements of the matrix $\mathbf{A}^m$**.**

We can define an $(n - m) \times n$ matrix $\mathbf{T}_L$ such that

$$\mathbf{T}_L = [\mathbf{I}_{n-m};\ \mathbf{O}_{n-m,m}]$$

and

$$\mathbf{T}_L\mathbf{T} = \mathbf{I} \tag{9}$$

The matrix $\mathbf{T}_L$ satisfying Eq. (9) is called the left inverse of $\mathbf{T}$. After a multiplication Eq. (8) by the matrix $\mathbf{T}_L$ we get an explicit form of the matrix $\mathbf{A}^m$, i.e. the system model in the form:

$$\dot{\mathbf{x}}^m = \mathbf{T}_L\mathbf{R}\mathbf{T}\mathbf{x}^m . \tag{10}$$

Applying Eq. (10) we get a sensitivity model in the sliding mode like:

$$\dot{\mathbf{s}}^m = \mathbf{A}^m\mathbf{s}^m + \mathbf{T}_L\mathbf{R}\mathbf{Q}_{ef}\mathbf{T}\mathbf{x}^m , \tag{11}$$

where the matrix $\mathbf{Q}_{ef}$ has the form:

$$\mathbf{Q}_{ef} = \{q_{ij}\} = \begin{cases} 0 & i \neq e,\ j \neq f, \\ 1 & i = e,\ j = f. \end{cases}$$

The matrix denoted by $\frac{\partial \mathbf{A}^m}{\partial a_{ef}}$ is then:

$$\frac{\partial \mathbf{A}^m}{\partial a_{ef}} = \mathbf{T}_L \mathbf{R} \mathbf{Q}_{ef} \mathbf{T} .$$

After some matrix multiplication, because of their specific form, we obtain:

$$\frac{\partial \mathbf{A}^m}{\partial a_{ef}} = \begin{bmatrix} r_{1e} \\ r_{2e} \\ \dots \\ r_{(n-m)e} \end{bmatrix} [t_{f1} \quad t_{f2} \quad \dots \quad t_{f(n-m)}] .$$

The sensitivity model for the VSS in sliding mode motion, applied on the system with scalar control action reduces to

$$\dot{\mathbf{s}}^1 = \mathbf{A}^1 \mathbf{s}^1 + \frac{\partial \mathbf{A}^1}{\partial a_{ef}} \mathbf{x}^1 , \tag{12}$$

where the matrix $\frac{\partial \mathbf{A}^1}{\partial a_{ef}}$ is given by

$$\frac{\partial \mathbf{A}^1}{\partial a_{ef}} = \begin{bmatrix} -\frac{b_1 c_e}{(b,c)} \\ -\frac{b_2 c_e}{(b,c)} \\ 1-\frac{b_e c_e}{(b,c)} \\ \dots \\ -\frac{b_{n-1} c_e}{(b,c)} \end{bmatrix} [t_{f1} \quad t_{f2} \quad \dots \quad t_{f(n-m)}],$$

with transformation matrix **T** in the form

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots \\ . & . & . & . \\ -c_1 & -c_2 & \dots & -c_{n-1} \end{bmatrix} .$$

The model (12) corresponds to the model defined in (Mati} [4]). We can discuss the obtained model with respect to the invariance conditions in VSS. From (11) it is clear that the system is invariant with respect to the perturbation parameter (a_{ef}) if the condition

$$\mathbf{T}_L \mathbf{R} \mathbf{Q}_{ef} \mathbf{T} = \mathbf{0}$$

is satisfied. The latest condition can be written in the form

$$\mathbf{T}_L\mathbf{B}(\mathbf{CB})^{-1}\mathbf{CQ}_{ef}\mathbf{Tx}^m = \mathbf{T}_L\mathbf{Q}_{ef}\mathbf{Tx}^m \,.$$

The matrix equation is defined if

$$\mathbf{BM} = \mathbf{Q}_{ef}\mathbf{Tx}^m \,.$$

The vector $\mathbf{Tx}^m$ can be represented as a sum of vectors:

$$\mathbf{Tx}^m = \sum_{i=1}^{n-m} \mathbf{t}_i x_i^m \,,$$

where $\mathbf{t}_i$ is i-th column of matrix **T**. Equations system (12) is linear and by the rule of linear systems superposition

$$\mathbf{BM} = \mathbf{Q}_{ef}\mathbf{t}_i x_i^m \,,$$

we get solution for the complete system. A necessary and sufficient condition for a linear system to have solution is given by Kronecker-Capelly theorem

$$rank[\mathbf{B}, \mathbf{Q}_{ef}\mathbf{t}_i] = rank\mathbf{B} \,. \tag{13}$$

The latest condition can be interpreted in such way that it is necessary that the vector $\mathbf{Q}_{ef}\,\mathbf{t}_i$ is a linear combination of the columns of the matrix **B.** Conclusion which has been made here based on the sensitivity model is the same as conclusion made in [Dra`enovi} [2]) for the case when the one parameter of the matrix **A** is changed.

4. Conclusion

In this paper is presented sensitivity model for variable structure systems with vector control action. a special consideration has been made on so-called sliding mode. Until the moment of switching first structure, sensitivity model is the same as sensitivity model for linear systems. In the moments of structure switching discontinuity control functions are those with the same index as nonzero rows of the matrix **B**. Sensitivity model is then defined by equation (4). After the sliding mode is established, the sensitivity model has been given by equation (7). Equation (13) gives the conditions for parametric invariant systems in sliding mode motion, which are same as those in (Dra`enovi} [2]). For the scalar control systems one can obtain the same results as in the paper (Mati} [4]). If we consider varying parameter as disturbances, then the conclusion is: the necessary and sufficient conditions for a parametric invariant system in sliding mode is that the vector **b** is collinear with the disturbance vector (f-th column of matrix $\mathbf{Q}_{ef}$).

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MODEL OSETLJIVOSTI ZA SISTEME PROMENLJIVE STRUKTURE S VEKTORSKIM UPRAVLJANJEM

Č. Milosavljević, S. Lj. Rančić

U radu se razmatra osetljivost posebne klase sistema automatskog upravljanja - sistema promenljive strukture. Definisan je model osetljivosti za sisteme s vektorskim upravljanjem i ustanovljene su korelacije s poznatim rezultatima u pogledu uslova invarijantnosti u kliznom režimu.