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QUASILINEAR LONGITUDINAL STABILITY OF AN AIRCRAFT

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Abstract. *The equations of the longitudinal motion are usually solved in the literature, using the method of linearization about a mean state of steady flight, to obtain the frequency and damping of the phugoid and short-period modes.*
In the present paper the quasilinear differential equations are obtained for constant air density and constant thrust along the flight path. The exact balance of lift, drag, weight, and inertia force is taken, together with the pitching moment equation, under the assumptions of flight at low Mach number. Three numerical examples are given: of a statically stable fighter, a statically unstable fighter and a large transport aircraft.

Nomenclature

b - mean aerodynamic chord, m,
c_{x0} - drag coefficient at zero angle of attack,
c_{y0} - lift coefficient at zero angle of attack,
c_y^α - slope of lift coefficient, rad^{-1},
g - acceleration of gravity, 9,81 m/s^2,
I_z - transverse moment of inertia, kg m^2,
k - coefficient of lift-induced drag,
l - wing span, m,
m - mass, kg,
m_z - pitching moment coefficient,
m_{z0} - pitching moment coefficient at zero angle of attack,
m_z^α - slope of pitching moment coefficient,rad^{-1},
M_z - pitching moment,
P - thrust, kN,
S - wing area, m^2,
V_0 - airspeed, m/s,
v - perturbation of velocity,
X - drag force,
Y - lift force,
α - perturbation of angle of attack, rad,
θ - perturbation of flight-path angle, rad,
ω_z - angular velocity of pitch, s^{-1},
M - Mach number,
H - altitude, m,
ρ - density, kg/m^3.

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1. INTRODUCTION

The longitudinal stability of an aircraft is examined beyond the well know linearization about a steady mean state, leading to the phugoid and short-period modes. All the variables in the equations of motion are considered by a reference value plus a perturbation or disturbance, i.e., it is assumed that the motion of the airplane consists of small deviations from a reference condition of steady flight [1], [3], [4], [6], [8], [9].

In this paper the exact equations of longitudinal motion of an aircraft are considered, i.e., balance of longitudinal and transversal force (with outside force) and balance of pitching moment. The drag terms included are friction and lift-induced drag. The mass density is taken as a constant, as well as thrust along the flight path. Using the trigonometric series, the quasilinear differential equations are obtained retaining nonlinear terms up to third-order.

2. DIFFERENTIAL EQUATIONS OF MOTION

In the general case the equations of the longitudinal motion are [4]:

$$F_x - mg\sin\theta = m\left(\frac{dV_x}{dt} - \omega_z V_y\right),$$

$$F_y - mg\cos\theta = m\left(\frac{dV_y}{dt} - \omega_z V_x\right), \tag{1}$$

$$M_z = I_z\frac{d\omega_z}{dt},$$

where $V_x = V\cos\alpha$, $V_y = -V\sin\alpha$, $F_x = P - X$ and $F_Y = Y$. Then Eqs. (1) is written in the form:

$$P - X - mg\sin\theta = m\frac{dV}{dt}\cos\alpha - mV\frac{d\alpha}{dt}\sin\alpha + mV\omega_z\sin\alpha, \tag{2a}$$

$$Y - mg\cos\theta = -m\frac{dV}{dt}\sin\alpha - mV\frac{d\alpha}{dt}\cos\alpha + mV\omega_z\cos\alpha, \tag{2b}$$

$$M_z = I_z\frac{d\omega_z}{dt}, \tag{2c}$$

From Eq.(2) we express dV/dt and obtain Eq. (3)

$$\frac{dV}{dt} = \frac{P}{m\cos\alpha} - \frac{X}{m\cos\alpha} - g\frac{\sin\theta}{\cos\alpha} - \frac{d\alpha}{dt}\frac{V\sin\alpha}{\cos\alpha} + \frac{V\omega_z\sin\alpha}{\cos\alpha}, \tag{3}$$

and from Eq.(2b) we express $d\alpha/dt$ and obtain Eq.(4)

$$\frac{d\alpha}{dt} = -\frac{dV}{dt}\frac{\sin\alpha}{V\cos\alpha} + \omega_z - \frac{Y}{mV\cos\alpha} + g\frac{\cos\theta}{V\cos\alpha}. \tag{4}$$

The drag force, lift force and pitching moment are

$$X = \frac{1}{2}\rho S V^2 c_x(\alpha), \tag{5a}$$

$$Y = \frac{1}{2}\rho S V^2 c_y(\alpha), \tag{5b}$$

$$M_z = \frac{1}{2}\rho S b V^2 m_z(\alpha), \tag{5c}$$

where we assume that

1) the lift coefficient is a linear function of the angle of attack

$$c_y(\alpha) = c_{y0} + \alpha c_y^{\alpha}, \tag{6}$$

2) the drag coefficient is

$$c_x(\alpha) = c_{x0} + k c_y(\alpha)^2, \tag{7}$$

3) the pitching moment coefficient is again a linear function of the angle of attack

$$m_z(\alpha) = m_{z0} + m_z^{\alpha}\alpha. \tag{8}$$

Substituting Eq.(7) into Eq.(5a), Eq.(5a) obtains the form (9)

$$X = \frac{1}{2}\rho S(c_{x0} + k c_{y0}^2)V^2 + \rho S k c_{y0} c_y^{\alpha} V^2 \alpha + \frac{1}{2}\rho S k (c_y^{\alpha})^2 V^2 \alpha^2. \tag{9}$$

Substituting Eq.(6) into Eq.(5b) and Eq.(8) into Eq.(5c) for lift force and pitching moment we obtain the following expressions

$$Y = \frac{1}{2}\rho S c_{y0} V^2 + \frac{1}{2}\rho S c_y^{\alpha} V^2 \alpha, \tag{10}$$

$$M_z = \frac{1}{2}\rho S b m_{z0} V^2 + \frac{1}{2}\rho S b m_z^{\alpha} V^2 \alpha. \tag{11}$$

Substituting Eq. (10) into Eq. (4) and the so obtained expression for $d\alpha/dt$ we substitute into Eq. (2a). Analogous we put Eq. (11) into Eq. (3) and the obtained expression for dV/dt is substituted into Eq. (2b). Substitute Eq. (11) into Eq. (2c) and finally obtain the full system of nonlinear differential equations, describing the longitudinal motion of an aircraft.

$$\begin{aligned} \frac{dV}{dt} = {} & \frac{P}{m}\cos\alpha - \frac{1}{2m}\rho S(c_{x0} + k c_{y0}^2)V^2 \cos\alpha - \frac{1}{m} S k c_{y0} c_y^{\alpha} V^2 \alpha \cos\alpha - \\ & - \frac{1}{2m}\rho S k (c_y^{\alpha})^2 V^2 \alpha^2 \cos\alpha - g\cos\alpha\sin\theta - \frac{1}{2m}\rho S c_{y0} V^2 \sin\alpha - \\ & - \frac{1}{2m}\rho S c_y^{\alpha} V^2 \alpha \sin\alpha + g\sin\alpha\cos\theta, \end{aligned} \tag{12}$$

$$\begin{aligned}\frac{d\alpha}{dt} &= \omega_z \cos^2\alpha - \frac{1}{2m}\rho S c_{y0} V\cos\alpha - \frac{1}{2m}\rho S c_y^\alpha V\alpha\cos\alpha + g\frac{\cos\alpha\cos\theta}{V} - \\ &-\frac{P}{m}\frac{\sin\alpha}{V} + \frac{1}{2m}\rho S(c_{x0}+kc_{y0}^2)V\sin\alpha + \frac{1}{m}Skc_{y0}c_y^\alpha V\alpha\sin\alpha + \\ &+\frac{1}{2m}\rho Sk(c_y^\alpha)^2 V\alpha^2\alpha\sin\alpha + g\frac{\sin\alpha\sin\theta}{V} + \omega_z \sin^2\alpha, \\ \frac{d\omega_z}{dt} &= \frac{1}{2I_z}\rho Sbm_{z0}V^2 + \frac{1}{2I_z}\rho Sbm_z^\alpha V^2\alpha.\end{aligned} \qquad (12)$$

The computational complicity of Eqs. (12) is difficult because V is in the denominator of some in the right parts of Eqs. (12). Therefore we express all variables by a reference value plus a perturbation

$$V = V_0 + v \qquad (13a)$$

$$\alpha = \alpha \qquad (13b)$$

$$\omega_z = \omega_z \qquad (13c)$$

$$\theta = \theta \qquad (13d)$$

Moreover we use a Taylor series

$$\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - ... \qquad (14)$$

$$\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - ... \qquad (15)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ... \qquad (16)$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ... \qquad (17)$$

and a binomial series

$$\frac{1}{V_0 + v} = \frac{1}{V_0} - \frac{v}{V_0^2} + \frac{v^2}{V_0^3} - \frac{v^4}{V_0^4} \qquad (18)$$

Substituting expressions (13a), (13b), (13c), (13d), (14), (15), (16) and (18) into Eqs.(12) and retaining nonlinear terms up to third-order inclusive, we obtain quasilinear differential equations (19).

$$\begin{aligned}\frac{dV}{dt} &= \frac{P}{m} - \frac{1}{2m}\rho S(c_{x0}+kc_{y0}^2)V_0^2 - \frac{1}{m}\rho S(c_{x0}+kc_{y0}^2)V_0 v - \\ &-\left[\left(\frac{1}{m}Skc_{y0}c_y^\alpha + \frac{1}{2m}Sc_{y0}\right)V_0^2 - g\right]\alpha - g\theta - \\ &-\frac{2}{m}\rho S(c_{x0}+kc_{y0}^2)v^2 - \left(\frac{1}{2m}Skc_{y0}c_y^\alpha + \frac{1}{m}\rho Sc_{y0}\right)V_0 v\alpha - \\ &-\frac{P}{m}\alpha^2 - \left(\frac{1}{m}Skc_{y0}c_y^\alpha + \frac{1}{2m}\rho Sc_{y0}\right)v^2\alpha + \frac{g}{6}\theta^3 + \frac{g}{6}\alpha^2\theta - \frac{g}{6}\alpha^3 - \frac{g}{2}\alpha\theta^2,\end{aligned} \qquad (19)$$

$$\frac{d\alpha}{dt} = \omega_z + \frac{1}{m}\rho S c_{y0} V_0 + \frac{g}{V_0} - \left(\frac{1}{2m}\rho S c_{y0} V_0 + \frac{g}{V_0}\right)v -$$
$$- \left[\frac{1}{2m}\rho S c_y^\alpha - \frac{1}{2m}\rho S (c_{x0} + k c_{y0}^2) V_0 + \frac{P}{m V_0}\right]\alpha + \frac{g}{V_0^3} v^2 -$$
$$- \left[\frac{1}{2m}\rho S c_y^\alpha + \frac{1}{2m}\rho S (c_{x0} + k c_{y0}^2) - \frac{P}{m V_0^3}\right] v\alpha + \tag{19}$$
$$+ \left(\frac{1}{4m}\rho S c_{y0} + \frac{1}{m}\rho S k c_{y0} c_y^\alpha\right) V_0 \alpha^3 + \frac{g}{2}\alpha\theta - \frac{g}{V_0^4} v^3 - \frac{P}{m V_0^3} v\alpha^2 + \frac{P}{6 m V_0}\alpha^3 ,$$
$$\frac{d\omega_z}{dt} = \frac{1}{2I_z}\rho S b m_{z0} V_0^2 + \frac{1}{I_z}\rho S b m_{z0} V_0 v + \frac{1}{2I_z}\rho S b m_{z0} v^2 + \frac{1}{2I_z}\rho S b m_z^\alpha V_0^2 +$$
$$\frac{1}{2I_z}\rho S b m_z^\alpha V_0^2 \alpha + \frac{1}{2I_z}\rho S b m_z^\alpha V_0 v\alpha + \frac{1}{2I_z}\rho S b m_z^\alpha v^2 \alpha .$$

3. EXAMPLES

The differential equations of longitudinal motion are solved for 3 aircrafts.
1) Statically stable fighter - McDonnell Douglas F-4 Phantom II;
2) Statically unstable fighter - F-16;
3) Statically stable transport aircraft - Boeing 747.
The aerodynamic, geometric and inertia data are given in Table 1 [6] [2], [3], [4], [5], [7].

Table.1

	F-4	B-747	F-16
M	0,9	0,9	0,9
H, km	10,2	11,8	10,2
V_0, m/s	267	265	265
ρ, kg/m^3	0,4	0,316	0,4
m, kg	17690	288778	12040
I_z, kg m^2	2×10^5	$7{,}11\times10^7$	$1{,}42\times10^5$
S, m^2	49,2	511	27,9
l, m	11,6	59,64	9,45
b, m	4,88	8,32	3,20
P, kN	14,8	178	106,3
c_{x0}	0,0205	0,0439	0.0205
c_{y0}	0,1	0,29	0,1
c_y^α, rad^{-1}	3,75	5,5	3,75
m_{z0}	0.025	0	0,025
m_z^α, rad^{-1}	–0,4	–1,6	0,0808

The disturbance is an impulse, that arise the angle of attack.
The results are shown in fig. 1-20. With continuous line the solution for quasilinear

system differential equations (19) are given and with interrupted line - linear system differential equations, i.e., the nonlinear terms are ignored.

The statically stable aircrafts F-4 and Boeing 747 have the short-period and phugoid modes (see fig.1-4 and fig.9-12). In the short-period mode (see fig. 5-8 and fig.13-16) there are differences in amplitudes in the quasilinear system differential equations at α, θ, ω_z, while the frequency are retained.

For statically unstable fighter F-16 the parameters of motion (see fig.17-29) heavily increase and the differences of solutions of the systems differential equations increase in time.

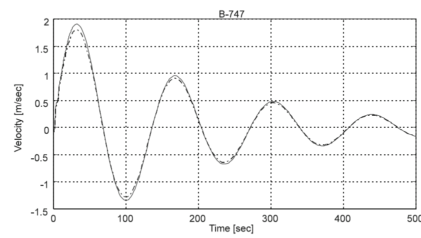

Fig.1. Phugoid mode of Boeing 747. Perturbation of velocity.

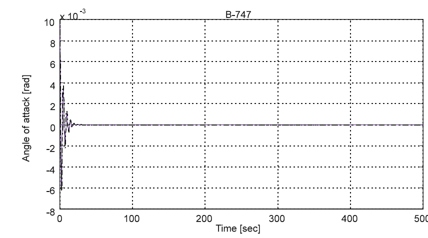

Fig.2. Short-period and phugoid modes of Boeing 747. Perturbation of angle of attack.

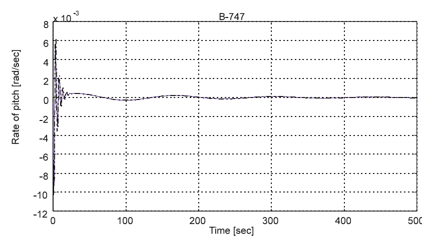

Fig.3. Short-period and phugoid modes of Boeing 747. Perturbation of angular velocity of pitch.

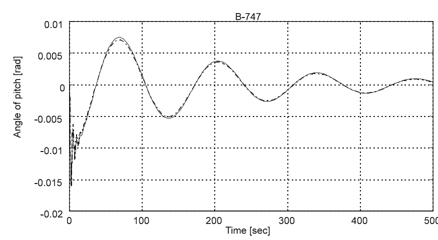

Fig.4. Short-period and phugoid modes of Boeing 747. Perturbation of flight-path angle.

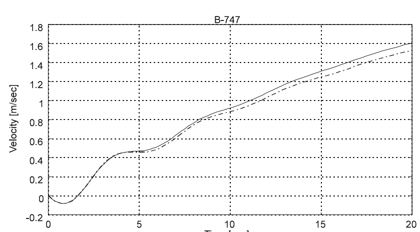

Fig.5. Short-period mode of Boeing 747. Perturbation of velocity.

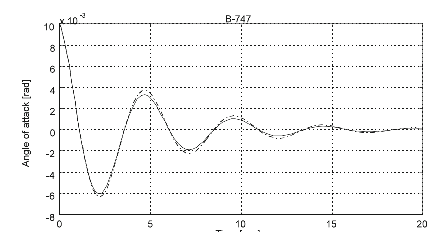

Fig.6. Short-period mode of Boeing 747. Perturbation of angle of attack.

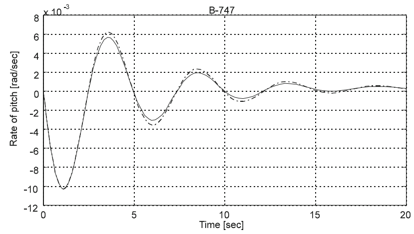

Fig.7. Short-period mode of Boeing 747. Perturbation of angular velocity of pitch.

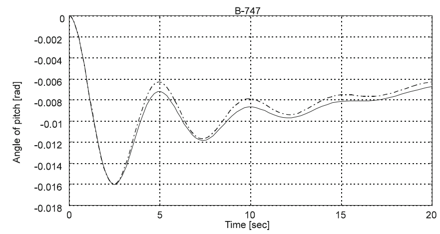

Fig.8. Short-period mode of Boeing 747. Perturbation of flight-path angle.

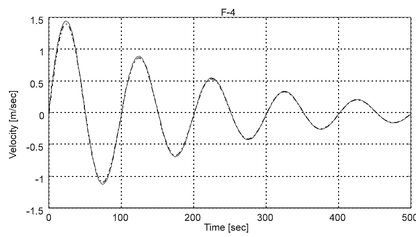

Fig.9. Phugoid mode of F-4. Perturbation of velocity.

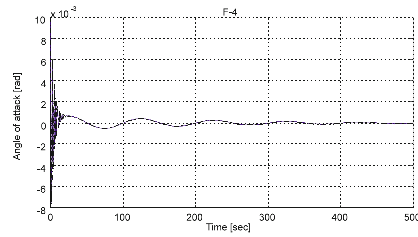

Fig.10. Short-period and phugoid modes of F-4. Perturbation of angle of attack.

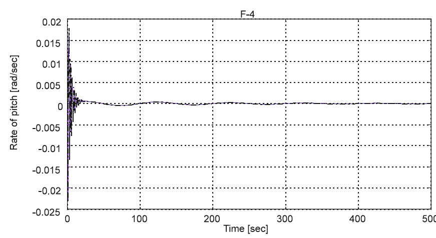

Fig.11. Short-period and phugoid modes of F-4. Perturbation of angular velocity of pitch.

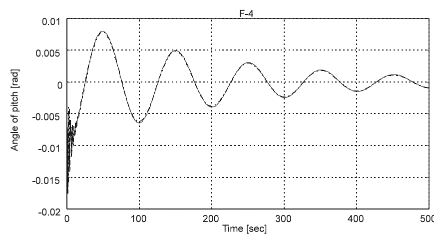

Fig.12. Short-period and phugoid modes of F-4. Perturbation of flight-path angle.

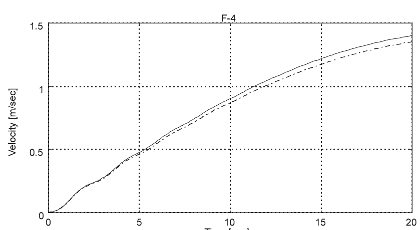

Fig.13. Short-period mode of F-4. Perturbation of velocity.

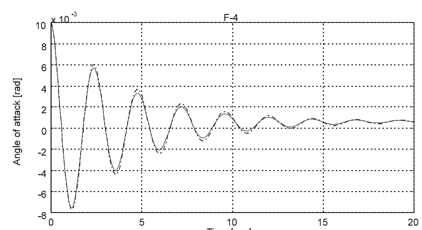

Fig.14. Short-period mode of F-4. Perturbation of angle of attack.

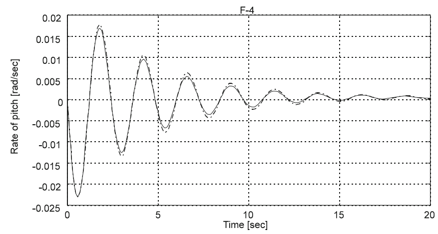

Fig.15.Short-period mode of F-4.
Perturbation of angular velocity of pitch.

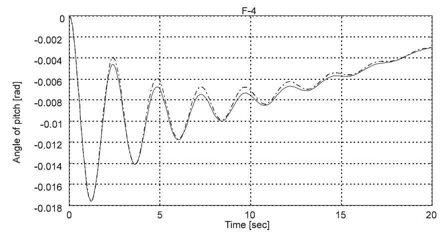

Fig.16.Short-period mode of F-4.
Perturbation of flight-path angle.

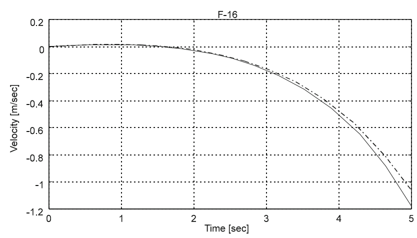

Fig.17. F-16 - Perturbation of velocity.

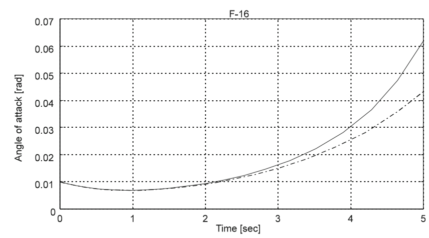

Fig.18.F-16 - Perturbation of angle of attack.

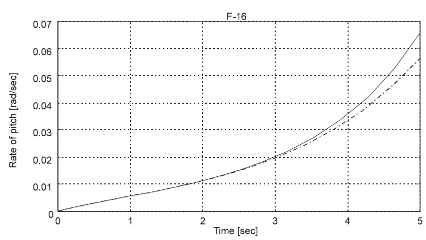

Fig.19.F-16 - Perturbation of angular velocity of pitch.

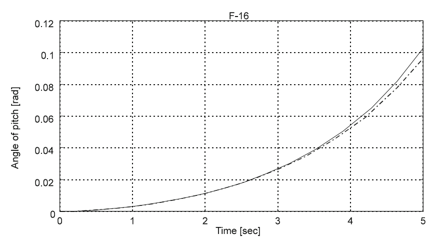

Fig.20.F-16 - Perturbation of flight-path angle.

4. CONCLUSIONS

For statically stable aircrafts the short periods in the short-period mode require pilot interrupted intervention in the flight control. Therefore at the preliminary calculation and design of the control system give an account of the nonlinear terms can provide to exactly results.

The statically unstable aircrafts give improved maneuverability and require active control to protection from departure at high angle of attack and/or sideslip. For that reason the consideration of nonlinear regimes will have even wider meaning for design of system of active control.

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KVAZI LINEARNA LONGITUDINALNA STABILNOST LETILICE

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Jednačine longitudinalnog kretanja uobičajeno prikazane u literaturi, su rešene metodom linearizacije oko srednjeg stanja ustaljenog leta, radi dobijanja frekvencije i prigušenja i kratko-periodičnih modova.

U ovom radu kvazilinearne diferencijalne jednačine su dobijene za konstantnu gustinu vazduha i konstantan potisak duže putanje leta.

Egzaktno uravnoteženje uzgona, otpora, težine i sila inercije je uzeto u račun, pod pretpostavkom da je let režimu mak-ovog broja.

Tri numerička primera su dobijena: statički stabilnih flater, statički nestabilni flater, kao i za transportnu letilicu.