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LONGITUDINAL CREEP VIBRATIONS OF A FRACTIONAL DERIVATIVE ORDER RHEOLOGICAL ROD WITH VARIABLE CROSS SECTION *

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Abstract. *Longitudinal creep vibrations of a fractional derivative rheological rod with variable cross section are examined. Partial differential equation and particular solutions for the case of natural creep longitudinal vibrations of the rod of creep material of a fractional derivative order is accomplished. For the case of natural creep vibrations, eigenfunction and time-function, for different examples of boundary conditions are determined. Different boundary conditions are analyzed and series of eigenvalues and natural circular frequencies of longitudinal creep vibrations, as well as tables of these values are completed. By using MathCad a graphical presentation of the time-function is presented.*

Key words: *Longitudinal creep vibrations, fractional derivative order, variable cross section, boundary conditions, series of eigenvalues, MathCad.*

I INTRODUCTION

Mechanics of hereditary medium (material) is presented in scientific literature by the array of fundamental monographs and papers [8], [9], [28], [30], [31], [32] and [34] and is widely used in engineering analyses of strength and deformability of constructions made of new construction materials. This field of mechanics is being intensively developed and filled up with new research monographs [28] and [9]. Actuality of that direction of development of mechanics is conditioned by engineering

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practice with utilizing the new construction materials on synthetic base, the mechanical properties of which often have pointed creep rheological character [32].

Nowadays proportion of utilization of these materials can be compared with size of using the metals. New construction materials possess both high strength and different useful physical characteristics as: dielectric's properties, radio conductivity, transparentness, high deformability and low (small) weight are, that make them irreplaceable in many cases. Successes of chemistry are enabling production of new synthetic materials with ordered properties [32].

The university books D.P. Rašković [29] and V.A. Vujičić[36] contain the classical theory of longitudinal oscillations of homogeneous rods and beams, and in [24] we can find mathematical theory of corresponding partial differential equations. R.E.D Bishop's paper [5] contain some results on longitudinal waves in beams and the paper [6] by Coehen H.and Whitman A.B. present research concerning waves in elastic rods. The effect of an arbitrarily mass on the longitudinal vibrations of a bar is investigated by M.A. Cutchins [10].

A series of papers [15, 16, 17], by K.S. Hedrih and A. Filipovski, presents results of original research on nonlinear oscillations of longitudinal vibrations of an elastic and rheological rod with variable cross section, which has application in engineering systems such as ultrasonic transducers, and ultrasonic concentrator (see Ref. [1]). Paper by L. G. Merkulov [26] contain method for numerical processing of the vibration state of the ultrasonic concentrator in the form of a rod with variable cross section.

Two paper [20, 21] by K.S.Hedrih present results on transversal vibrations of prismatic beam of hereditary material. Papers [23] and [19] contain some models of discrete continuum with hereditary light standard element as the constrants and with light standard creep element as constraints of the fractional derivatives in the behavior of materials. Standard hereditary element is constraint in the systems which are investigated and described in the papers [18] and [22]. P.O. Agrawal presented paper [2] about a new Largangian and a new Lagrange equation of motion for fractionally damped systems.

II FUNDAMENTALS OF MECHANICS OF CREEP AND HEREDITARY SYSTEMS

In present literature notion ***hereditary elasticity*** and ***viscose-elasticity*** are equivalent. J.M. Rabotnov (Юю М. Работнов) [28], being conducted via papers of V. Volter (*V. Volter*), believes that notion ***hereditary elasticity*** is more exact and a better description of the essence of the phenomenon. This term expresses the ability of rheological body to specifically "***remember***" history of loading (stretching). Viscose-elastic body possesses the particularity of deforming [9], [34], [25]:

For the *short-time-loading*, *fast form (shape) reconstruction of the body form* after unload occurs;

for the *long-time-loading*, establishing of the form (shape) needs necessary *long-time-period after unloading*, i.e. ***viscous-elastic*** bodies "***remember***" ("***memorize***"), which reflexes in term "***hereditary elasticity***".

More or less, all solid bodies practically hold hereditary properties [28], [30]; and [9]. For example, forced by long-time-loading (period of many years), steel spring changes the length (wearing, fatigue), and after unloading it regains the former length in the time period that is calculable with loading period. These advents are known for practitioners which put heavy tracks for durable storage (conservation) on stiff supports to unload springs. In this specified example, recording hereditary properties requires many hours of measurement for investigation. For viscous-elastic synthetic materials, as rubber or polymers-threads, manifestation time for hereditary properties is measured by second and minutes.

Thus, not less then other theories, as it was shown in ref. [28], [30], hereditary theory is relevant for describing internal friction, even in metals with small stress-amplitudes.

Material laws and constitutive theories are the fundamental bases for describing the mehanical behavior of materials under multiaxual states of stress involving creep and creep rupture (see J. Betten's Ref. [4]). In creep mechanics one can differentiate between three stages: the primary, secondary and tertiary creep stage [4]. These terms correspond to a decreasing, constant and increasing creep strain-rate, respectively. In order to describe the creep behavior of metals in the primary stage, tensorial nonlinear constitutive equations involving the strain-hardering hypothesis are proposed.

III MODEL OF CREEP RHELOGICAL BODY

For modeling processes of solidification and relaxation, models of Kelvin's viscous-elastic material and Maxwell's ideal-elastic-viscous fluid, [31], [9], [12] and [13], are being used. In their paper, [8], A. O. Goroshko and N. P. Puchko, have used model of standard hereditary body to modeling dynamics of mechanical systems with rheological links. Studying elements of mechanics of hereditary systems in their monograph [31], G. N. Savin. and Yu.Ya. Ruschisky, gave survey of both structure and analysis of the rheological models of simple and complex laws for linear deformable hereditary-elastic media, as well as theory of growing old of hereditary-elastics systems. Rheological models can be found in R. Stojanovic's monograph [33], as in the university's publications [12] and [13] from K. (Stevanović) Hedrih, and in the monograph [9] by A. O. Goroshko and K.S. Herdrih.

Recently, there is a noticable interest in using fractional derivatives to describe creep behavior of material. In solid mechanics particularly for describing problems related to material creep behavior including viscoelastic and viscoplastic effects, fractional derivatives have a longer history (see Ref. [35], [3], [11], [23]). Mathematical basis of the fractional derivative and short complete of fractional calculus are presented in the monograph papar [7] by R. Gorenflo and F. Mainardi.

Paper [11] by Dli Gen-guo, Zhu Zheng-you and Cheng Chanh-jun contain the consideration of dynamical stability of viscoelastic column with fractional derivative constitutive relation. Paper [3] by B.S. Bašlić and T. M. Atanacković considered stability and creep of a fractional derivative order viscoelastic rod.

By using stress-strain relation from cited refernces, a single-axis stress state of the creep hereditary type material is described by fractional order time derivative differential relation in the form of three parameter model:

$$\sigma(t) = -\{E_0 \varepsilon(t) + E_\alpha \mathfrak{D}_t^\alpha[\varepsilon(t)]\} \tag{1}$$

where $\mathfrak{D}_t^\alpha[\bullet]$ is operator of fractional derivative - the α^{th} derivative of strain $\varepsilon(t)$ with respect to time t in the following form:

$$\mathfrak{D}_t^{\alpha}[\varepsilon(t)] = \frac{d^{\alpha}\varepsilon(t)}{dt^{\alpha}} = \varepsilon^{[\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\varepsilon(\tau)}{(t-\tau)^{\alpha}}d\tau \tag{2}$$

where E_0 and E_α are instant and prolonged elasticity modulus, respectively, while α is relaxation parameter, ratio number from interval $0 < \alpha < 1$, and $\Gamma(1-\alpha)$ is Euler gama function. We shall use relation (2) only for $t \geq 0$.

IV LONGITUDINAL CREEP VIBRATIONS EQUATION OF A FRACTIONAL DERIVATIVE ORDER RHEOLOGICAL ROD WITH VARIABLE CROSS SECTION

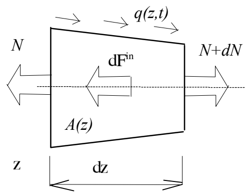

Fig. 1. An element of the rod of variable cross section

Consider a deformable rod of a fractional derivative order with variable cross section, whose axis is straight.

Figure 1. shows an element of the rod of variable cross section $A(z)$, where z is axis's length coordinate of the rod. Normal force acting on the cross section at the distance z measured from left side of the rod is:

$$N(z,t) = A(z)\sigma_z(z,t) \tag{3}$$

while it's value in cross section on distance $z + dz$ is:

$$N(z+dz,t) = A(z)\sigma_z(z,t) + \frac{\partial}{\partial z}[A(z)\sigma_z(z,t)]\,dz \tag{4}$$

where t is time, and $\sigma_z(z,t)$ is normal stress in the points of cross section that is, according to introduced assumption, invariable on the cross-section. Moreover deplaning of cross section are neglected considering that all points have the same axial displacement determined by coordinate $w(z,t)$.

According to the D'Alambert's principle following equation could be written for dynamical equilibrium of forces acting on rod's element:

$$\rho A(z)dz\frac{\partial^2 w(z,t)}{\partial t^2} = -N(z,t) + N(z+dz,t) + q(z,t)A(z)dz = \frac{\partial N(z,t)}{\partial z}dz + q(z,t)A(z)dz \tag{5}$$

where ρ is rod material's density, and $q(z,t)$ is distributed volume force. Substituting expression (1) into equation (3) leads to:

$$\frac{\partial^2 w(z,t)}{\partial t^2} - \frac{1}{\rho A(z)}\frac{\partial}{\partial z}[A(z)\sigma_z(z,t)] = \frac{1}{\rho}q(z,t) \tag{6}$$

We assume that rod is made of creep rheological material and therefore the stress-strain-state equation written in the form (1).

Taking that strain in axis's direction of rod is:

$$\varepsilon_z(z,t) = \frac{\partial w(z,t)}{\partial z}\,, \tag{7}$$

previous stress-strain-state relation (1) can be written in following form as:

$$\sigma_z(z,t) = E_0 \frac{\partial w(z,t)}{\partial z} + E_\alpha \mathfrak{D}_t^\alpha \left[\frac{\partial w(z,t)}{\partial z} \right] \tag{8}$$

Introducing previous fractional derivative stress-strain relation into equilibrium's equation (6), following fractional derivative-partial-differential equation can be gotten:

$$\frac{\partial^2 w(z,t)}{\partial t^2} - \frac{1}{\rho A(z)} \frac{\partial}{\partial z} \left[E_0 A(z) \frac{\partial w(z,t)}{\partial z} \right] = \frac{1}{\rho A(z)} \frac{\partial}{\partial z} \left\{ E_\alpha A(z) \mathfrak{D}_t^\alpha \left[\frac{\partial w(z,\tau)}{\partial z} \right] \right\} + \frac{1}{\rho} q(z,t) \tag{9}$$

If we mark $c_0^2 = \frac{E_0}{\rho}$ and $c_\alpha^2 = \frac{E_\alpha}{\rho}$ than previous equation gets the following form:

$$\frac{1}{c_0^2} \frac{\partial^2 w(z,t)}{\partial t^2} - \frac{1}{A(z)} \frac{\partial}{\partial z} \left[A(z) \frac{\partial w(z,t)}{\partial z} \right] = \frac{c_\varepsilon^2}{c_0^2} \frac{1}{A(z)} \frac{\partial}{\partial z} \left[A(z) \mathfrak{D}_t^\alpha \left[\frac{\partial w(z,t)}{\partial z} \right] \right] + \frac{1}{E_0} q(z,t). \tag{10}$$

V NATURAL LONGITUDINAL CREEP VIBRATIONS OF A FRACTIONAL DERIVATIVE ORDER RHEOLOGICAL ROD WITH VARIABLE CROSS SECTION

Solution of the following fractional derivative-partial-differential equation:

$$\frac{1}{c_0^2} \frac{\partial^2 w(z,t)}{\partial t^2} - \frac{1}{A(z)} \frac{\partial}{\partial z} \left[A(z) \frac{\partial w(z,t)}{\partial z} \right] = \frac{c_\varepsilon^2}{c_0^2} \frac{1}{A(z)} \frac{\partial}{\partial z} \left[A(z) \mathfrak{D}_t^\alpha \left[\frac{\partial w(z,t)}{\partial z} \right] \right] \tag{11}$$

can be looked for Bernoulli's method of particular integrals in the form of multiplication of two functions, from which the first $Z(z)$ depends only on space coordinate z, and the second is time function $T(t)$:

$$w(z,t) = Z(z)T(t) \tag{12}$$

Assumed solution is introduced in previous equation bringing to the following expression:

$$\frac{1}{c_0^2} Z(z)\ddot{T}(t) - \frac{1}{A(z)} \frac{\partial}{\partial z} [A(z)Z'(z)T(t)] = \frac{c_\alpha^2}{c_0^2} \frac{1}{A(z)} \frac{\partial}{\partial z} [A(z)Z'(z) \mathfrak{D}_t^\alpha [T(t)]] \tag{13}$$

Introducing the constant $\omega_0^2 = k^2 c_0^2$ it is easy to share previous equation on following two:

* first, a second order differential equation on unknown eigenfunction $Z(z)$ of space coordinate z, with variable coefficients :

$$Z''(z) + \frac{A'(z)}{A(z)} Z'(z) + k^2 Z(z) = 0 \tag{14}$$

and * second, fractional-differential equation on unknown time-function $T(t)$:

$$\ddot{T}(t) + \omega_0^2 T(t) = -\omega_\alpha^2 \mathfrak{D}_t^\alpha [T(t)] \tag{15}$$

or in the form:

$$\ddot{T}(t)+\omega_\alpha^2 \mathfrak{D}_t^\alpha[T(t)]+\omega_0^2 T(t)=0 \tag{15.a}$$

Both equations can be solved independently. These are connected only with characteristic coupled constants $\omega_0^2 = k^2 c_0^2$. The first differential equation (14), can be, in some cases, solved for characteristically specified function of variation of cross section of the rod. As it was solved in ref. [10] for different cases of functions of variation of cross section, in following, we will recall the outcomes from that paper.

VI THE TIME-FUNCTION SOLUTION OF A FRACTIONAL DIFERENTIAL EQUATION

The second, fractional-differential equation on unknown time-function $T(t)$ we can rewrite in the following form:

$$\ddot{T}(t)+\omega_\alpha^2 T^{(\alpha)}(t)+\omega_0^2 T(t)=0 \tag{15.b}$$

This fractional-differential equation (15*) on unknown time-function $T(t)$, can be solved applying Laplace transforms (see Ref. [29] and [33]). Upon that fact Laplace transform of solution is in form:

$$\mathfrak{T}(p)=\mathfrak{L}[T(t)]=\frac{pT(0)+\dot{T}(0)}{p^2+\omega_0^2\left[1+\frac{\omega_\alpha^2}{\omega_0^2}\mathbf{R}(p)\right]} \tag{16}$$

where $\mathfrak{L}[\mathfrak{D}_t^\alpha[T(t)]]=\mathbf{R}(p)\mathfrak{L}[T(t)]$ is Laplace transform of a fractional derivative $\frac{d^\alpha T(t)}{dt^\alpha}$ for $0 \le \alpha \le 1$. For creep rheological material those Laplace transforms are in the form:

$$\mathfrak{L}[\mathfrak{D}_t^\alpha[T(t)]]=\mathbf{R}(p)\mathfrak{L}[T(t)]-\frac{d^{\alpha-1}}{dt^{\alpha-1}}T(0)=p^\alpha \mathfrak{L}[T(t)]-\frac{d^{\alpha-1}}{dt^{\alpha-1}}T(0) \tag{17}$$

where the initial initial value are:

$$\left.\frac{d^{\alpha-1}T(t)}{dt^{\alpha-1}}\right|_{t=0}=0 \tag{17a}$$

So, in that case Laplace transform of time-function is given by following expression:

$$\mathfrak{L}\{T(t)\}=\frac{pT_0+\dot{T}_0}{[p^2+\omega_\alpha^2 p^\alpha+\omega_0^2]} \tag{18}$$

For boundary cases, when material parameters α take following values: $\alpha = 0$ and $\alpha = 1$ we have the two special simple cases, whose corresponding fractional-differential equations and solutions are known. In these cases fractional-differential equations are:

$$1^*\ \ddot{T}(t)+\tilde{\omega}_0^2 T^{(0)}(t)+\omega_0^2 T(t)=0, \text{ for } \alpha=0, \tag{19}$$

where $T^{(0)}(t)=T(t)$, and

$$2^*\ \ddot{T}(t)+\omega_1^2 T^{(1)}(t)+\omega_0^2 T(t)=0\text{, for } \alpha=1 \tag{20}$$

where $T^{(1)}(t)=\dot{T}(t)$.

The solutions of equations (19) and (20) are:

$$1^*\ T(t)=T_0\cos t\sqrt{\omega_0^2+\tilde{\omega}_0^2}+\frac{\dot{T}_0}{\sqrt{\omega_0^2+\tilde{\omega}_0^2}}\sin t\sqrt{\omega_0^2+\tilde{\omega}_0^2} \tag{21}$$

for $\alpha = 0$.

$$2^*\text{ a. } T(t)=e^{-\frac{\omega_1^2}{2}t}\left\{T_0\cos t\sqrt{\omega_0^2-\frac{\omega_1^4}{4}}+\frac{\dot{T}_0}{\sqrt{\omega_0^2-\frac{\omega_1^4}{4}}}\sin t\sqrt{\omega_0^2-\frac{\omega_1^4}{4}}\right\} \tag{22}$$

for $\alpha = 1$, and for $\omega_0>\frac{1}{2}\omega_1^2$ (for soft creep) or for strong creep:

$$2^*\text{ b. } T(t)=e^{-\frac{\omega_1^2}{2}t}\left\{T_0 Ch\, t\sqrt{\frac{\omega_1^4}{4}-\omega_0^2}+\frac{\dot{T}_0}{\sqrt{\frac{\omega_1^4}{4}-\omega_0^2}}Sh\, t\sqrt{\frac{\omega_1^4}{4}-\omega_0^2}\right\} \tag{23}$$

for $\alpha = 1$, and for $\omega_0<\frac{1}{2}\omega_1^2$.

For critical case:

$$2^*\text{ c. } T(t)=e^{-\frac{\omega_1^2}{2}t}\left\{T_0+\frac{2\dot{T}_0}{\omega_1^2}t\right\}\text{ for } \alpha=1 \text{ and for } \omega_0=\frac{1}{2}\omega_1^2. \tag{24}$$

Fractional-differential equation (15. b*) for the general case, when α is real number from interval $0 < \alpha < 1$ can be solved by using Laplace's transformation. By using that is:

$$\mathfrak{L}\left\{\frac{d^\alpha T(t)}{dt^\alpha}\right\}=p^\alpha\mathfrak{L}\{T(t)\}-\left.\frac{d^{\alpha-1}T(t)}{dt^{\alpha-1}}\right|_{t=0}=p^\alpha\mathfrak{L}\{T(t)\} \tag{25}$$

and by introducing initial conditions of fractional derivatives in the form (17.a), and after taking Laplace's transform of the equation (15. b) we obtain the following.

By analysing previous Laplace transform (18) of solution we can conclude that we can consider two cases.

For the case when $\omega_0^2\neq 0$, the Laplace transform solution can be developed into series by following way:

$$\mathfrak{L}\{T(t)\}=\frac{pT_0+\dot{T}_0}{p^2\left[1+\frac{\omega_\alpha^2}{p^2}\left(p^\alpha+\frac{\omega_0^2}{\omega_\alpha^2}\right)\right]}=\left(T_0+\frac{\dot{T}_0}{p}\right)\frac{1}{p}\frac{1}{1+\frac{\omega_\alpha^2}{p^2}\left(p^\alpha+\frac{\omega_0^2}{\omega_\alpha^2}\right)} \tag{26}$$

$$\mathfrak{L}\{T(t)\} = \left(T_0 + \frac{\dot{T}_0}{p}\right)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k \omega_\alpha^{2k}}{p^{2k}}\left(p^\alpha + \frac{\omega_0^2}{\omega_\alpha^2}\right)^k \tag{27}$$

$$\mathfrak{L}\{T(t)\} = \left(T_0 + \frac{\dot{T}_0}{p}\right)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k \omega_\alpha^{2k}}{p^{2k}}\sum_{j=0}^{k}\binom{k}{j}\frac{p^{\alpha j}\omega_\alpha^{2(j-k)}}{\omega_o^{2j}}. \tag{28}$$

In writing (28) it is assumed that expanssion leads to convergent series [7, 4]. The inverse Laplace transform of previous Laplace transform of solution (26) in term-by-term steps is based on known theorem, and yield to following solution of differential equation (15. b) of time function in the following form of time series:

$$\begin{aligned} T(t) = \mathfrak{L}^{-1}\{T(t)\} = T_0\sum_{k=0}^{\infty}(-1)^k \omega_\alpha^{2k} t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{2j} t^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+1-\alpha j)} + \\ + \dot{T}_0\sum_{k=0}^{\infty}(-1)^k \omega_\alpha^{2k} t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{2j} t^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+2-\alpha j)} \end{aligned} \tag{29}$$

or

$$T(t) = \mathfrak{L}^{-1}\{T(t)\} = \sum_{k=0}^{\infty}(-1)^k \omega_\alpha^{2k} t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{2j} t^{-\alpha j}}{\omega_o^{2j}}\left[\frac{T_0}{\Gamma(2k+1-\alpha j)} + \frac{\dot{T}_0 t}{\Gamma(2k+2-\alpha j)}\right] \tag{30}$$

Two special cases of the solution for $\omega_0^2 = 0$ are:

$$T(t) = T_0\cos\tilde{\omega}_o t + \frac{\dot{T}_0}{\tilde{\omega}_0}\sin\tilde{\omega}_o t \quad \text{for } \alpha = 0 \text{ and } \omega_0^2 = 0. \tag{31}$$

$$T(t) = T_0 + \frac{\dot{T}_0}{\omega_1^2}(1 - e^{-\omega_1^2 t}) \quad \text{for } \alpha = 1 \text{ and } \omega_0^2 = 0. \tag{32}$$

For the case $\omega_0^2 = 0$ and when α is real number from interval $0 < \alpha < 1$ we can write following Laplace transform of solution:

$$\mathfrak{L}\{T(t)\} = \frac{pT_0 + \dot{T}_0}{[p^2 + \omega_\alpha^2 p^\alpha]} \tag{33}$$

and corresponding expasion into convergente series

$$\mathfrak{L}\{T(t)\} = \left(T_0 + \frac{\dot{T}_0}{p}\right)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k \omega_\alpha^{2k}}{p^{(2-\alpha)k+1}}. \tag{34}$$

Taking the inverse Laplace transform of (34) we obtain the general solution of time function corresponding to fractional differential equation (15.b) for the case $\omega_0^2 = 0$ and $0 < \alpha < 1$ in following form:

$$T(t) = \mathfrak{L}^{-1}\{T(t)\} = \sum_{k=0}^{\infty}(-1)^k \omega_\alpha^{2k} t^{(2-\alpha)k}\left[\frac{T_0}{\Gamma(2k+1-\alpha k)} + \frac{\dot{T}_0 t}{\Gamma(2k+2-\alpha k)}\right] \tag{35}$$

In Figure 2. numerical simulations and graphical presentation of the solution (30) of the fractional-differential equation (15.b) of the system are presented. Time functions $T(t,\alpha)$ surfaces for different rod (beam) longitudinal vibrations kinetic and creep material parameters in the space $(T(t,\alpha),t,\alpha)$ for interval $0 \le \alpha \le 1$ are visible:

in **a*** for $\left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=1$; in **b*** for $\left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=\frac{1}{4}$; in **c*** for $\left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=\frac{1}{3}$; in **d*** for $\left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=3$.

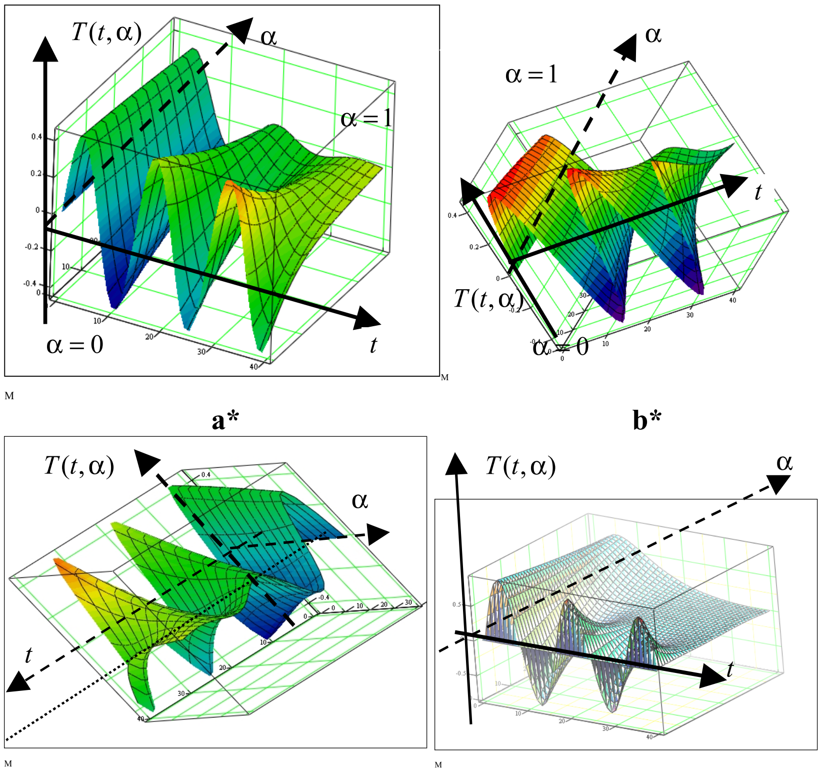

Fig. 2. Numerical simulations and graphical presentation of the results. Time functions $T(t,\alpha)$ surface for the different beam transversal vibrations kinetic and creep material parameters:

$$\mathbf{a^*}\ \left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=1;\ \mathbf{b^*}\ \left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=\frac{1}{4};\ \mathbf{c^*}\ \left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=\frac{1}{3};\ \mathbf{d^*}\ \left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=3$$

In Figure 3. the time functions $T(t,\alpha)$ surfaces and curves families for the different rod (beam) longitudinal vibrations kinetic and discrete values of the creep material parameters $0 \le \alpha \le 1$ are presented. In Figures **a*** and **c*** for $\left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=1$; in Figures b* and **d*** for $\left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=\frac{1}{4}$; in Figure **e*** for $\left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=\frac{1}{3}$; and in Figure **f*** for $\left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right)=3$.

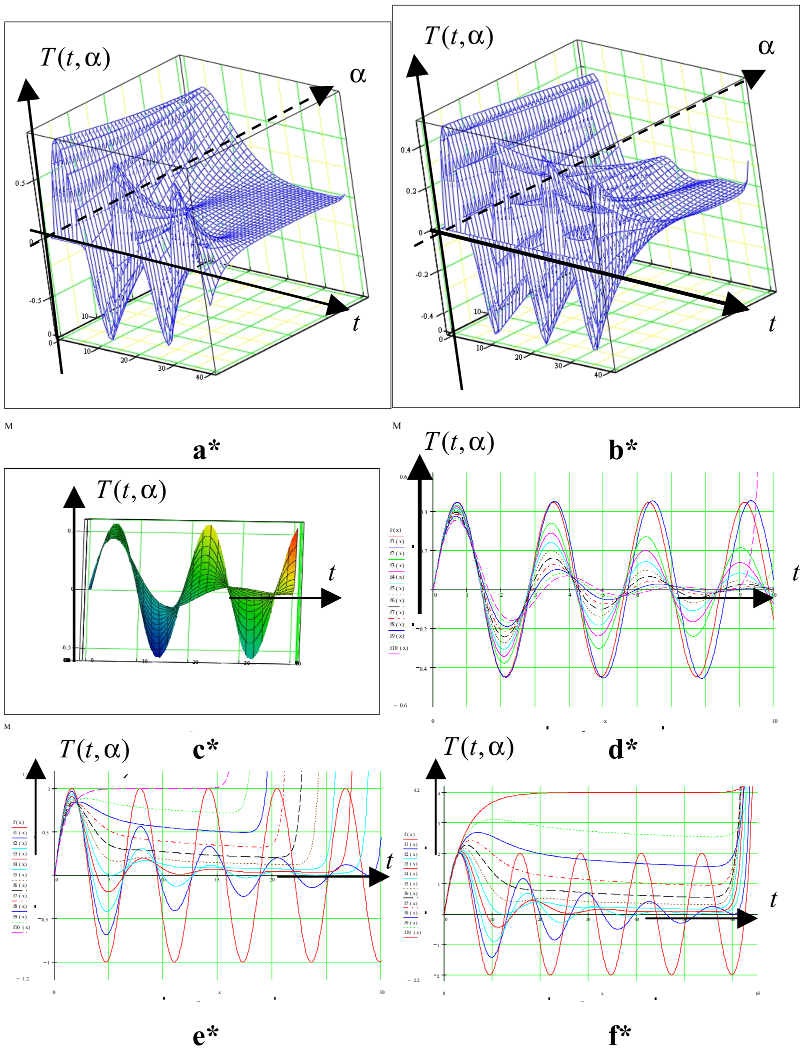

Fig. 3. Numerical simulations and graphical presentation of the results. Time functions $T(t,\alpha)$ surface and curves families for the different beam transversal vibrations kinetic and discrete values of the creep material parameters $0 \le \alpha \le 1$:

$$\mathbf{a^*} \text{ and } \mathbf{c^*} \left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right) = 1\,;\ \text{b* and } \mathbf{d^*} \left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right) = \frac{1}{4}\,;\ \mathbf{e^*} \left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right) = \frac{1}{3}\,;\ \mathbf{f^*} \left(\frac{\omega_{\alpha x}}{\omega_{0x}}\right) = 3\,.$$

VII BOUNDARY CONDITIONS FOR DIFFERENT CASES ROD FORMS

In order to determine characteristic numbers $\omega_0^2 = k^2 c_0^2$ and $\omega_\alpha^2 = k^2 c_\alpha^2$ for different cases of boundary conditions of the rod of variable cross section that vibrate longitudinally, stresses and displacements of boundary cross sections would be expressed in dependence on eigenfunction and time-function.

Axial displacements of the ends of the rod are:

$$w(0,t) = Z(0)T(t)\,,\; w(\ell,t) = Z(\ell)T(t)\,. \tag{36}$$

Normal stress in interior cross sections is:

$$\sigma_z(z,t) = Z'(z)[E_0 T(t) + E_\alpha \mathfrak{D}_t^\alpha[T(t)]] = -\frac{1}{\omega_0^2} E_0 Z'(z)\ddot{T}(t)\,. \tag{37}$$

Therefore in left and right base-cross-section it will be:

$$\sigma_z(0,t) = Z'(0)[E_0 T(t) + E_\alpha \mathfrak{D}_t^\alpha[T(t)]] = -\frac{1}{\omega_0^2} E_0 Z'(0)\ddot{T}(t) \tag{38}$$

$$\sigma_z(\ell,t) = Z'(\ell)[E_0 T(t) + E_\alpha \mathfrak{D}_t^\alpha[T(t)]] = -\frac{1}{\omega_0^2} E_0 Z'(\ell)\ddot{T}(t) \tag{39}$$

Example I: Eigenvalue equation, eigenfunctions of conical-shape rod.

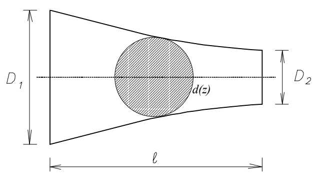

Fig. 4. The rod of variable cross section $A(z)$.

Following Figure 4. shows the rod of variable cross section $A(z)$, of length ℓ, with geometrical axis z , and with diameters D_1 and D_2 at the left and right bound respectively.

Let, diameter of cross section $d(z)$ changes according to expression:

$$d(z) = D_1(1 - \alpha z) \tag{40}$$

where α is parameter:

$$\alpha = \frac{1}{\ell}\left[1 - \left(\frac{D_2}{D_1}\right)\right] = \frac{1}{\ell}[1 - N] \tag{41}$$

For this case, taking in account differential equation (14), eigenfunction is:

$$Z(z) = \frac{1}{1 - az}(C_1 \cos kz + C_2 \sin kz) \tag{42}$$

If we consider rod with free ends, stresses on these free boundary bases must be equal to zero and therefore boundary conditions can be written in the form:

$$\begin{aligned} \sigma_z(0,t) &= 0 \quad Z'(0) = 0 \\ \sigma_z(\ell,t) &= 0 \quad Z'(\ell) = 0 \end{aligned} \tag{43}$$

For boundary conditions defined with above relations, eigenvalue equation can be written in the form of determinant:

$$\Delta(k\ell) = \begin{vmatrix} \alpha & k \\ \alpha - k(1-\alpha\ell)tgk\ell & \alpha tgk\ell + k(1-\alpha\ell) \end{vmatrix} = 0 \tag{44}$$

When relation between diameters of end of the rod $N = D_2/D_1$ is taken into account, eigenvalue equation, introducing non-dimensional number $\xi = k\ell$, can be written in the form of transcendental equation:

$$tg\xi = \frac{\xi}{\dfrac{\xi^2 N}{(1-N)^2} + 1} \tag{45}$$

whose roots (eigenvalues) were given in Table 1.

Table 1. Eigenvalues of eigenvalue equation for the rod with free ends.

N	0	0.01	0.02	0.06	0.08	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
ξ_1	4.493	4.448	4.403	4.230	4.148	4.070	3.749	3.529	3.383	3.286	3.222	3.181	3.157	3.145	π
ξ_2	7.725	7.648	7.572	7.290	7.169	7.062	6.702	6.520	6.420	6.360	6.325	6.303	6.291	6.285	2π
ξ_3	10.90	10.79	10.69	10.32	10.18	10.06	9.732	9.591	9.518	9.477	9.453	9.438	9.430	9.426	3π
ξ_4	14.07	13.93	13.79	13.36	13.21	13.10	11.81	11.69	11.64	11.61	11.59	11.58	11.57	11.57	4π

Table 2. Eigenvalues for the cantilever rod with exponential shape

N	0.01	0.02	0.04	0.06	0.08	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
ξ_1	1.624	1.562	1.484	1.429	1.385	1.347	1.202	1.093	1.000	1.918	1.841	1.770	1.701	1.635	$k\pi/2$
ξ_2	5.417	5.344	5.261	5.208	5.167	5.134	5.022	4.951	4.897	4.854	4.818	4.787	4.759	4.735	
ξ_3	8.358	8.295	8.227	8.185	8.154	8.130	8.051	8.003	7.968	7.941	7.918	7.899	7.882	7.867	
ξ_4	11.380	11.328	11.274	11.241	11.217	11.198	11.139	11.104	11.078	11.058	11.042	11.028	11.016	11.005	

Eigenfunctions take following form:

$$Z_n(z) = \frac{C_{on}}{1-\alpha z}\left(\cos k_n z - \frac{\alpha}{k_n}\sin k_n z\right); \; n = 1,2,... \tag{46}$$

Amplitude magnification factor on n-th eigenvalue can be defined with relation between values of eigenfunctions at the ends of the rod:

$$\left|\frac{Z_n(\ell)}{Z_n(0)}\right| = \left|\frac{1}{1-\alpha\ell}\left(\cos k_n\ell - \frac{\alpha}{k_n}\sin k_n\ell\right)\right|; \; n = 1,2,... \tag{47}$$

Example II: Eigenvalue equation, eigenfunctions of exponential-shape rod

We consider the rod of exponential shape whose length is ℓ, geometrical axis z and with diameters D_1 and D_2 at the left and right bound. Cross section diameter $d(z)$ changes according to expression:

Table 3. Eigenvalues for the exponential shape cantilever rod with weight on the free end

N	$\mu_2 = 0.4$			$\mu_2 = 1.0$			$\mu_2 = 8.0$		
	ξ_1	ξ_2	ξ_2	ξ_1	ξ_2	ξ_2	ξ_1	ξ_2	ξ_2
0.01	3.521	6.564	9.655	3.261	6.639	9.513	3.152	6.294	9.433
0.02	3.595	6.607	9.664	3.289	6.405	9.518	3.154	6.295	9.434
0.03	3.638	6.619	9.669	3.308	6.411	9.520	3.155	6.295	9.434
0.04	0.318	3.668	6.627	3.322	6.415	9.522	3.157	6.296	9.434
0.05	0.738	3.689	6.663	3.334	6.418	9.522	3.159	6.296	9.434
0.06	0.927	3.707	6.637	3.344	6.421	9.524	3.160	6.297	9.435
0.07	1.045	3.719	6.640	3.353	6.423	9.525	3.160	6.297	9.435
0.08	1.128	3.730	6.643	3.360	6.425	9.525	3.161	6.297	9.435
0.09	1.189	3.734	6.645	3.366	6.426	9.526	3.162	6.297	9.435
0.10	1.236	3.745	6.646	3.372	6.427	9.526	3.163	6.297	9.435
0.20	1.413	3.776	6.654	1.198	3.406	6.434	3.167	6.298	9.435
0.30	1.433	3.780	6.655	0.749	3.420	6.436	3.169	6.299	9.435
0.40	1.416	3.777	6.653	0.892	3.427	6.438	3.171	6.299	9.435
0.50	1.382	3.770	6.652	0.946	3.430	6.438	3.172	6.299	9.435
0.60	1.341	3.763	6.650	0.960	3.431	6.438	3.172	6.299	9.435
0.70	1.296	3.755	6.649	0.952	3.431	6.438	0.904	3.173	6.308
0.80	1.247	3.747	6.647	0.931	3.430	6.438	0.265	3.173	6.300
0.90	1.196	3.740	6.645	0.899	3.428	6.438	0.310	3.173	6.300
1.00	1.142	3.732	6.643	0.860	3.425	6.437	0.311	3.173	6.300

$$d(z) = D_1 e^{-\delta z} \tag{48}$$

where

$$\delta = \frac{1}{\ell} ln\left(\frac{D_1}{D_2}\right) = -\frac{1}{\ell} lnN \tag{49}$$

and eigenfunction is:

$$Z(z) = e^{\delta z}(C_1 \cos k'z + C_2 \sin k'z), \tag{50}$$

where $k' = \sqrt{k^2 - \delta^2}$.

For the case of cantilever rod, boundary conditions are:

$$\begin{aligned} &w(0,t) = 0 \quad Z(0) = 0 \\ &\sigma_z(\ell,t) = 0 \quad Z'(\ell) = 0 \end{aligned} \tag{51}$$

and with non-dimensional number: $\xi' = k'\ell$, eigenvalue equation takes form:

$$tg\xi' = -\frac{\xi'}{\delta\ell} = \frac{\xi'}{\ln N} \tag{52}$$

and eigenfunctions take form:

$$Z_n(z) = C_n e^{\delta z} sink'_n z \tag{53}$$

Table 3a. Eigenvalues for the catenary shape cantilever rod with weights on the free ends

	$\mu_1 = 1.0$									$\mu_1 = 10.0$								
	$\mu_2 = 0.0$			$\mu_2 = 1.0$			$\mu_2 = 10.0$			$\mu_2 = 0.0$			$\mu_2 = 1.0$			$\mu_2 = 10.0$		
N	ξ_1	ξ_2	ξ_3	ξ_1	ξ_2	ξ_3	ξ_1	ξ_2	ξ_3	ξ_1	ξ_2	ξ_3	ξ_1	ξ_2	ξ_3	ξ_1	ξ_2	ξ_3
0.02	1.628	4.811	7.943	3.334	6.481	9.591	3.241	6.389	9.515	4.723	7.863	11.003	3.254	6.396	9.518	3.162	6.304	9.442
0.04	1.645	4.825	7.950	3.377	6.501	9.600	3.263	6.401	9.520	4.725	7.864	11.004	3.279	6.408	9.523	3.166	6.306	9.443
0.06	1.660	4.835	7.954	3.407	6.513	9.605	3.279	6.407	9.523	1.581	4.726	7.865	3.297	6.415	9.526	3.170	6.307	9.443
0.08	1.672	4.842	7.957	3.430	6.522	9.609	3.291	6.412	9.525	1.583	4.727	7.865	3.311	6.419	9.528	3.172	6.308	9.444
0.10	1.684	4.849	7.959	3.449	6.528	9.611	3.302	6.416	9.527	1.584	4.727	7.865	3.323	6.423	9.530	3.175	6.309	9.444
0.20	1.734	4.868	7.965	3.514	6.548	9.619	3.339	6.428	9.531	1.591	4.729	7.866	3.362	6.434	9.534	3.183	6.311	9.445
0.40	1.818	4.886	7.972	3.583	6.565	9.625	3.383	6.439	9.535	1.602	4.731	7.866	3.403	6.443	9.537	3.192	6.313	9.445
0.60	1.893	4.899	7.975	0.654	3.623	6.574	3.412	6.445	9.537	1.612	4.732	7.866	3.427	6.448	9.538	3.197	6.314	9.446
0.80	1.964	4.907	7.978	1.066	3.651	6.580	0.460	3.434	6.449	1.622	4.733	7.867	0.694	3.442	6.451	3.201	6.314	9.446
0.90	1.996	4.910	7.978	1.199	3.663	6.583	0.747	3.443	6.451	1.627	4.733	7.867	0.830	3.448	6.452	3.203	6.315	9.446
1.00	1.029	4.913	7.979	1.307	3.673	6.585	0.929	3.452	6.452	1.632	4.734	7.867	0.929	3.452	6.452	0.444	3.204	6.315

Boundary conditions for cantilever rod with weight (mass $\mathbf{M_2}$) on the free end can be written:

$$w(0,t)=0\ ,\ Z(0)=0$$

$$\left[\mathbf{E}A(\ell)\frac{\partial w}{\partial z}\right]\Bigg|_{z=\ell}=-\left[\mathbf{M}_2\frac{\partial^2 w}{\partial t^2}\right]\Bigg|_{z=\ell}\ ,\ Z'(\ell)=\mu_2 k^2\ell Z(\ell) \tag{54}$$

where we wrote: $\mu_2=\mathbf{M}_2/\rho A_2\ell$, and got eigenvalue equation:

$$ctg\xi'=\mu_2\left(\xi'+\frac{\delta^2\ell^2}{\xi'}\right)-\frac{\delta\ell}{\xi'}\ , \tag{55}$$

Eigenfunctions take the form:

$$Z_n(z)=C_n e^{\delta z}\sin k'_n z\ \ ;\ \mathrm{n}=1,2,...\alpha. \tag{56}$$

Eigenvalues for the exponential shape cantilever rod with weight on the free end where is given in Table 3.

Example III: Eigenvalue equation, eigenfunctions of catenary shape rod

As in previous cases we consider oscillatory characteristics of rod of length ℓ, with geometrical axis z, with diameters D_1 and D_2 of the cross section but with catenary law of change the diameter of cross section:

$$d(z)=D_2 Ch\gamma(\ell-z) \tag{57}$$

where γ is denoted as:

$$\gamma=\frac{1}{\ell}ArCh\left(\frac{D_1}{D_2}\right)=\frac{1}{\ell}ArCh\left(\frac{1}{N}\right) \tag{58}$$

Eigenfunction has the form :

$$Z(z)=\frac{1}{Ch\gamma(\ell-z)}(C_1\cos k'z+C_2\sin k'z), \tag{59}$$

where: $k'=\sqrt{k^2-\gamma^2}$.

Boundary conditions for the rod with weights at the free ends are given in the form:

$$\begin{aligned}&\left[\mathbf{E}A(0)\frac{\partial w}{\partial z}\right]\Bigg|_{z=0}=\left[\mathbf{M}_1\frac{\partial^2 w}{\partial t^2}\right]\Bigg|_{z=0}\quad ;\ Z'(0)=-\mu_1 k^2\ell Z(0)\\&\left[\mathbf{E}A(\ell)\frac{\partial w}{\partial z}\right]\Bigg|_{z=\ell}=-\left[\mathbf{M}_2\frac{\partial^2 w}{\partial t^2}\right]\Bigg|_{z=\ell}\quad ;\ Z'(\ell)=\mu_2 k^2\ell Z(\ell)\end{aligned} \tag{60}$$

Denoting as non-dimensional mass factor $\mu_i=\mathbf{M}_i/\rho A_i\ell$, $(i=1,2)$ eigenvalue equation can be written in the form:

$$tgk'\ell = \frac{\{(\mu_1+\mu_2)[(k'\ell)^2+(\gamma\ell)^2]+\gamma\ell th\gamma\ell\}k'\ell}{(k'\ell)^2(\mu_1\mu_2 k'^2\ell^2-1)+\{\mu_1\mu_2\gamma\ell[(\gamma\ell)^2+2(k'\ell)^2]+\mu_2[(k'\ell)^2+(\gamma\ell)^2]th\gamma\ell\}\gamma\ell} \quad (61)$$

Eigenfunctions are in the following form:

$$Z_n(z) = \frac{C_n}{Ch\gamma(\ell - z)}\left\{cosk'_n z - \left[\frac{\gamma\ell th\gamma\ell}{k'_n\ell} + \mu_1\left(\frac{k'^2_n\ell^2+\gamma^2\ell^2}{k'_n\ell}\right)\right]sink'_n z\right\};\ n = 1,2,...\infty \quad (62)$$

Amplitude magnification factor on the n-th eigenvalue is:

$$\left|\frac{Z_n(\ell)}{Z_n(0)}\right| = \left|Ch\gamma\ell\left\{cosk'_n\ell - \left[\frac{\gamma\ell th\gamma\ell}{k'_n\ell} + \mu_1\left(\frac{k'^2_n\ell^2+\gamma^2\ell^2}{k'_n\ell}\right)\right]sink'_n\ell\right\}\right|; \quad (63)$$

VIII THE FINAL EXPRESSION OF THE SOLUTIONS

From transcendent eigenvalue (characteristic) equation we can find roots set ξ_n, $n = 1,2,3,4,...$ (see Table 1,2,3) and eigenvalues $k_n = \xi_n/\ell$, $n = 1,2,3,4,...$ of the longitudinal vibrations of the rod with changeable cross section for chosen boundary conditions. By using these eigenvalues we obtain: a* eigen frequency values or chracteristic kinetic parameters of the free creep longitudinal vibrations $\omega^2_{on} = k^2_n c^2_0 = \frac{\xi^2_n}{\ell^2}c^2_0$, $n = 1,2,3,4,...$, and $\omega^2_{\alpha n} = k^2_n c^2_\alpha = \frac{\xi^2_n}{\ell^2}c^2_\alpha,\ \ 0<\alpha<1$, $n = 1,2,3,4,...$; b* set of eigen orthogonal functions $Z_n(z)$, $n = 1,2,3,4,...$ and c* set of the time functions $T_n(t)$, $n = 1,2,3,4,...$. Than we can write set of particular solutions in the form:

$$w_n(z,t) = Z_n(z)T_n(t)\ ,\ n = 1,2,3,4.... \quad (A)$$

each one of which satisfies the boundary conditions. Generalized family solutions which satisfies the boundary conditions is:

$$w(z,t) = \sum_{n=1}^{n=\infty} Z_n(z)T_n(t) \quad (A^*)$$

For different cases of parameter $0 < \alpha < 1$, time functions are in the following forms:

1* for $\alpha = 0$

$$T_n(t) = T_{0n}\cos t\sqrt{\omega^2_{0n}+\tilde{\omega}^2_{0n}} + \frac{\dot{T}_{0n}}{\sqrt{\omega^2_{0n}+\tilde{\omega}^2_{0n}}}\sin t\sqrt{\omega^2_{0n}+\tilde{\omega}^2_{0n}}\ ,\ n = 1,2,3,4.... \quad (64)$$

2* a. for $\alpha = 1$ and for $\omega_{0n} > \frac{1}{2}\omega^2_{1n}$, $n = 1,2,3,4.....$ (for soft creep)

$$T_n(t) = e^{-\frac{\omega^2_{1n}}{2}t}\left\{T_{0n}\cos t\sqrt{\omega^2_{0n}-\frac{\omega^4_{1n}}{4}} + \frac{\dot{T}_{0n}}{\sqrt{\omega^2_{0n}-\frac{\omega^4_{1n}}{4}}}\sin t\sqrt{\omega^2_{0n}-\frac{\omega^4_{1n}}{4}}\right\}\ ,\ n = 1,2,3,4.... \quad (65)$$

or **2* b.** for $\alpha = 1$ and for $\omega_{0n} < \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4....$ (for strong creep)

$$T_n(t) = e^{-\frac{\omega_{1n}^2}{2}t}\left\{T_{0n} Ch\, t\sqrt{\frac{\omega_{1n}^4}{4} - \omega_{0n}^2} + \frac{\dot{T}_{0n}}{\sqrt{\frac{\omega_{1n}^4}{4} - \omega_{0n}^2}} Sh\, t\sqrt{\frac{\omega_{1n}^4}{4} - \omega_{0n}^2}\right\},\ n = 1,2,3,4.... \tag{66}$$

2* c. For $\omega_{0n} = \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4.....$ (for critical case):

$$T_n(t) = e^{-\frac{\omega_{1n}^2}{2}t}\left\{T_{0n} + \frac{2\dot{T}_{0n}}{\omega_{1n}^2}t\right\} \text{ for } \alpha = 1. \tag{67}$$

3* For $0 < \alpha < 1$

$$T_n(t) = \mathfrak{L}^{-1}\{T_n(t)\} = \sum_{k=0}^{\infty}(-1)^k \omega_{\alpha n}^{2k} t^{2k} \sum_{j=0}^{k}\binom{k}{j}\frac{\omega_{\alpha n}^{2j} t^{-\alpha j}}{\omega_{on}^{2j}}\left[\frac{T_{0n}}{\Gamma(2k+1-\alpha j)} + \frac{\dot{T}_{0n} t}{\Gamma(2k+2-\alpha j)}\right], \tag{68}$$
$$n = 1,2,3,4....$$

Two special cases of the solution for $\omega_{0n}^2 = 0$ are:

$$T_n(t) = T_{0n}\cos\tilde{\omega}_{on} t + \frac{\dot{T}_{0n}}{\tilde{\omega}_0}\sin\tilde{\omega}_{on} t \quad \text{for } \alpha = 0 \text{ and } \omega_{0n}^2 = 0. \tag{69}$$

$$T_n(t) = T_{0n} + \frac{\dot{T}_{0n}}{\omega_{1n}^2}(1 - e^{-\omega_{1n}^2 t}) \quad \text{for } \alpha = 1 \text{ and } \omega_{0n}^2 = 0. \tag{70}$$

Sets of eigen orthogonal functions $Z_n(z)$, $n = 1,2,3,4....$ are in following forms:

a* Eigenfunctions of conical rod with free ends take following form:

$$Z_n(z) = \frac{C_{on}}{1 - \alpha z}\left(\cos k_n z - \frac{\alpha}{k_n}\sin k_n z\right);\ n = 1,2,... \tag{71}$$

b* Eigenfunctions of exponential-shape rod, for the case of cantilever rod, take form:

$$Z_n(z) = C_n e^{\delta z} sin k'_n z$$

c* Eigenfunctions of exponential-shape rod, for the case of cantilever rod with weight on the free end takes the form:

$$Z_n(z) = C_n e^{\delta z} sin k'_n z \ ;\ n = 1,2,...\alpha. \tag{72}$$

Eigenvalues for the exponential shape cantilever rod with weight on the free end are given in Table 3.

d* Eigenfunctions of catenary shape rod, and boundary conditions for the rod with weights at the free ends

$$Z_n(z) = \frac{C_n}{Ch\gamma(\ell - z)}\left\{cos k'_n z - \left[\frac{\gamma\ell th\gamma\ell}{k'_n \ell} + \mu_1\left(\frac{{k'_n}^2\ell^2 + \gamma^2\ell^2}{k'_n \ell}\right)\right] sin k'_n z\right\}. \tag{73}$$

VIII 1* Some examples of solutions of longitudinal creep vibrations of conical rod with free ends

1* Solution of longitudinal vibrations of conical rod with free ends, for $\alpha = 0$ take following form:

$$w(z,t) = \sum_{n=1}^{n=\infty}\left\{\frac{C_{0n}}{1-\beta z}\left(\cos k_n z - \frac{\beta}{k_n}\sin k_n z\right)\right\} \bullet \left\{T_{0n}\cos t\sqrt{\omega_{0n}^2+\tilde{\omega}_{0n}^2} + \frac{\dot{T}_{0n}}{\sqrt{\omega_{0n}^2+\tilde{\omega}_{0n}^2}}\sin t\sqrt{\omega_{0n}^2+\tilde{\omega}_{0n}^2}\right\} \tag{74}$$

2. a*. Solution of longitudinal vibrations of conical rod with free ends, for $\alpha = 1$ and for $\omega_{0n} > \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4....$. (for soft creep) take following form:

$$w(z,t) = \sum_{n=1}^{n=\infty}\left\{\frac{C_{0n}}{1-\beta z}\left(\cos k_n z - \frac{\beta}{k_n}\sin k_n z\right)\right\} e^{-\frac{\omega_{1n}^2}{2}t} \bullet \left\{T_{0n}\cos t\sqrt{\omega_{0n}^2-\frac{\omega_{1n}^4}{4}} + \frac{\dot{T}_{0n}}{\sqrt{\omega_{0n}^2-\frac{\omega_{1n}^4}{4}}}\sin t\sqrt{\omega_{0n}^2-\frac{\omega_{1n}^4}{4}}\right\} \tag{75}$$

or

2. b*. for $\alpha = 1$ and for $\omega_{0n} < \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4....$. (for strong creep)

$$w(z,t) = \sum_{n=1}^{n=\infty}\left\{\frac{C_{0n}}{1-\beta z}\left(\cos k_n z - \frac{\beta}{k_n}\sin k_n z\right)\right\} e^{-\frac{\omega_{1n}^2}{2}t} \bullet \left\{T_{0n} Ch\, t\sqrt{\frac{\omega_{1n}^4}{4}-\omega_{0n}^2} + \frac{\dot{T}_{0n}}{\sqrt{\frac{\omega_{1n}^4}{4}-\omega_{0n}^2}} Sh\, t\sqrt{\frac{\omega_{1n}^4}{4}-\omega_{0n}^2}\right\}, \tag{76}$$

2. c*. For critical case: $\alpha = 1$ and for $\omega_{0n} = \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4....$

$$w(z,t) = \sum_{n=1}^{n=\infty}\left\{\frac{C_{0n}}{1-\beta z}\left(\cos k_n z - \frac{\beta}{k_n}\sin k_n z\right)\right\} e^{-\frac{\omega_{1n}^2}{2}t}\left\{T_{0n} + \frac{2\dot{T}_{0n}}{\omega_{1n}^2}t\right\} \tag{77}$$

3* For $0 < \alpha < 1$

$$w(z,t) = \sum_{n=1}^{n=\infty}\left\{\frac{C_{0n}}{1-\beta z}\left(\cos k_n z - \frac{\beta}{k_n}\sin k_n z\right)\right\} \bullet \left\{\sum_{k=0}^{\infty}(-1)^k \omega_{\alpha n}^{2k} t^{2k} \sum_{j=0}^{k}\binom{k}{j}\frac{\omega_{\alpha n}^{2j} t^{-\alpha j}}{\omega_{on}^{2j}}\left[\frac{T_{0n}}{\Gamma(2k+1-\alpha j)} + \frac{\dot{T}_{0n} t}{\Gamma(2k+2-\alpha j)}\right]\right\}. \tag{78}$$

Two special cases of the solution for $\omega_{0n}^2 = 0$ are:

$$w(z,t) = \sum_{n=1}^{n=\infty} \left\{ \frac{C_{0n}}{1-\beta z} \left(\cos k_n z - \frac{\beta}{k_n} \sin k_n z \right) \right\} \left\{ T_{0n} \cos \tilde{\omega}_{on} t + \frac{\dot{T}_{0n}}{\tilde{\omega}_0} \sin \tilde{\omega}_{on} t \right\} \text{ for } \alpha = 0 \text{ and } \omega_{0n}^2 = 0 . \quad (79)$$

$$w(z,t) = \sum_{n=1}^{n=\infty} \left\{ \frac{C_{0n}}{1-\beta z} \left(\cos k_n z - \frac{\beta}{k_n} \sin k_n z \right) \right\} \left\{ T_{0n} + \frac{\dot{T}_{0n}}{\omega_{1n}^2} (1 - e^{-\omega_{1n}^2 t}) \right\} \text{ for } \alpha = 1 \text{ and } \omega_{0n}^2 = 0 . \quad (80)$$

VIII. 2* Some Examples of Solutions of Longitudinal Creep Vibrations of Exponential-Shape Rod

Solution of longitudinal vibrations of exponential-shape rod, for the case of cantilever rod (or of exponential-shape rod, for the case of cantilever rod with weight on the free end) take form:

1* For $\alpha = 0$

$$w(z,t) = \sum_{n=1}^{n=\infty} \{ C_n e^{\delta z} sin k'_n z \} \left\{ T_{0n} \cos t \sqrt{\omega_{0n}^2 + \tilde{\omega}_{0n}^2} + \frac{\dot{T}_{0n}}{\sqrt{\omega_{0n}^2 + \tilde{\omega}_{0n}^2}} \sin t \sqrt{\omega_{0n}^2 + \tilde{\omega}_{0n}^2} \right\} \quad (81)$$

2* a. For $\alpha = 1$ and for $\omega_{0n} > \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4,...$ (soft creep)

$$w(z,t) = \sum_{n=1}^{n=\infty} \{ C_n e^{\delta z} sin k'_n z \} e^{-\frac{\omega_{1n}^2}{2} t} \left\{ T_{0n} \cos t \sqrt{\omega_{0n}^2 - \frac{\omega_{1n}^4}{4}} + \frac{\dot{T}_{0n}}{\sqrt{\omega_{0n}^2 - \frac{\omega_{1n}^4}{4}}} \sin t \sqrt{\omega_{0n}^2 - \frac{\omega_{1n}^4}{4}} \right\} (82)$$

or

2* b. For $\alpha = 1$ and for $\omega_{0n} < \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4....$ (strong creep):

$$w(z,t) = \sum_{n=1}^{n=\infty} \{ C_n e^{\delta z} sin k'_n z \} e^{-\frac{\omega_{1n}^2}{2} t} \left\{ T_{0n} Ch\, t \sqrt{\frac{\omega_{1n}^4}{4} - \omega_{0n}^2} + \frac{\dot{T}_{0n}}{\sqrt{\frac{\omega_{1n}^4}{4} - \omega_{0n}^2}} Sh\, t \sqrt{\frac{\omega_{1n}^4}{4} - \omega_{0n}^2} \right\} ,(83)$$

2. c*. For $\alpha = 1$ and for $\omega_{0n} = \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4....$ critical case:

$$w(z,t) = \sum_{n=1}^{n=\infty} \{ C_n e^{\delta z} sin k'_n z \} e^{-\frac{\omega_{1n}^2}{2} t} \left\{ T_{0n} + \frac{2\dot{T}_{0n}}{\omega_{1n}^2} t \right\} \quad (84)$$

3* For general case $0 < \alpha < 1$

$$w(z,t) = \sum_{n=1}^{n=\infty} \left\{ \frac{C_{0n}}{1-\beta z} \left(\cos k_n z - \frac{\beta}{k_n} \sin k_n z \right) \right\}$$
$$\bullet \left\{ \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha n}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{\omega_{\alpha n}^{2j} t^{-\alpha j}}{\omega_{on}^{2j}} \left[\frac{T_{0n}}{\Gamma(2k+1-\alpha j)} + \frac{\dot{T}_{0n} t}{\Gamma(2k+2-\alpha j)} \right] \right\}. \tag{85}$$

Two special cases of the solution for $\omega_{0n}^2 = 0$ are:

$$w(z,t) = \sum_{n=1}^{n=\infty} \{C_n e^{\delta z} sink'_n z\} \left\{ T_{0n} \cos \tilde{\omega}_{on} t + \frac{\dot{T}_{0n}}{\tilde{\omega}_0} \sin \tilde{\omega}_{on} t \right\} \text{ for } \alpha = 0 \text{ and } \omega_{0n}^2 = 0 . \tag{86}$$

$$w(z,t) = \sum_{n=1}^{n=\infty} \{C_n e^{\delta z} sink'_n z\} \left\{ T_{0n} + \frac{\dot{T}_{0n}}{\omega_{1n}^2} \left(1 - e^{-\omega_{1n}^2 t} \right) \right\} \text{ for } \alpha = 1 \text{ and } \omega_{0n}^2 = 0 . \tag{87}$$

VIII 3* Some examples of solutions of longitudinal creep vibrations of catenary-shape rod with weights at the free ends.

Solution of longitudinal vibrations of *of catenary shape rod, and boundary conditions for the rod with weights at the free ends*, *takes form*:

1* for $\alpha = 0$

$$w(z,t) = \sum_{n=1}^{n=\infty} \left\{ \frac{C_n}{Ch\gamma(\ell - z)} \left\{ cosk'_n z - \left[\frac{\gamma\ell th\gamma\ell}{k'_n \ell} + \mu_1 \left(\frac{k'^2_n \ell^2 + \gamma^2 \ell^2}{k'_n \ell} \right) \right] sink'_n z \right\} \right\}$$
$$\bullet \left\{ T_{0n} \cos t\sqrt{\omega_{0n}^2 + \tilde{\omega}_{0n}^2} + \frac{\dot{T}_{0n}}{\sqrt{\omega_{0n}^2 + \tilde{\omega}_{0n}^2}} \sin t\sqrt{\omega_{0n}^2 + \tilde{\omega}_{0n}^2} \right\} \tag{88}$$

2* a. For $\alpha = 1$ and for $\omega_{0n} > \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4....$. (soft creep):

$$w(z,t) = \sum_{n=1}^{n=\infty} \left\{ \frac{C_n}{Ch\gamma(\ell - z)} \left\{ cosk'_n z - \left[\frac{\gamma\ell th\gamma\ell}{k'_n \ell} + \mu_1 \left(\frac{k'^2_n \ell^2 + \gamma^2 \ell^2}{k'_n \ell} \right) \right] sink'_n z \right\} \right\} e^{-\frac{\omega_{1n}^2}{2} t}$$
$$\bullet \left\{ T_{0n} \cos t\sqrt{\omega_{0n}^2 - \frac{\omega_{1n}^4}{4}} + \frac{\dot{T}_{0n}}{\sqrt{\omega_{0n}^2 - \frac{\omega_{1n}^4}{4}}} \sin t\sqrt{\omega_{0n}^2 - \frac{\omega_{1n}^4}{4}} \right\} \tag{89}$$

or

2. b*. For $\alpha = 1$ and for $\omega_{0n} < \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4.....$ (strong creep):

$$w(z,t) = \sum_{n=1}^{n=\infty}\left\{\frac{C_n}{Ch\gamma(\ell - z)}\left\{cos k'_n z - \left[\frac{\gamma\ell th\gamma\ell}{k'_n\ell} + \mu_1\left(\frac{k'^2_n\ell^2 + \gamma^2\ell^2}{k'_n\ell}\right)\right]sin k'_n z\right\}\right\}e^{-\frac{\omega_{1n}^2}{2}t}$$
$$\bullet\left\{T_{0n}Ch\, t\sqrt{\frac{\omega_{1n}^4}{4} - \omega_{0n}^2} + \frac{\dot{T}_{0n}}{\sqrt{\frac{\omega_{1n}^4}{4} - \omega_{0n}^2}}Sh\, t\sqrt{\frac{\omega_{1n}^4}{4} - \omega_{0n}^2}\right\}, \quad (90)$$

2* c. For $\alpha = 1$ and for $\omega_{0n} = \frac{1}{2}\omega_{1n}^2$, $n = 1,2,3,4.....$ (For critical case):

$$w(z,t) = \sum_{n=1}^{n=\infty}\left\{\frac{C_n}{Ch\gamma(\ell - z)}\left\{cos k'_n z - \left[\frac{\gamma\ell th\gamma\ell}{k'_n\ell} + \mu_1\left(\frac{k'^2_n\ell^2 + \gamma^2\ell^2}{k'_n\ell}\right)\right]sin k'_n z\right\}\right\}$$
$$\bullet e^{-\frac{\omega_{1n}^2}{2}t}\left\{T_{0n} + \frac{2\dot{T}_{0n}}{\omega_{1n}^2}t\right\} \quad (91)$$

3* For general case $0 < \alpha < 1$:

$$w(z,t) = \sum_{n=1}^{n=\infty}\left\{\frac{C_n}{Ch\gamma(\ell - z)}\left\{cos k'_n z - \left[\frac{\gamma\ell th\gamma\ell}{k'_n\ell} + \mu_1\left(\frac{k'^2_n\ell^2 + \gamma^2\ell^2}{k'_n\ell}\right)\right]sin k'_n z\right\}\right\}$$
$$\bullet\left\{\sum_{k=0}^{\infty}(-1)^k\omega_{\alpha n}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_{\alpha n}^{2j}t^{-\alpha j}}{\omega_{on}^{2j}}\left[\frac{T_{0n}}{\Gamma(2k+1-\alpha j)} + \frac{\dot{T}_{0n}t}{\Gamma(2k+2-\alpha j)}\right]\right\}. \quad (92)$$

Two special cases of the solution for $\omega_{0n}^2 = 0$ are:

$$w(z,t) = \sum_{n=1}^{n=\infty}\left\{\frac{C_n}{Ch\gamma(\ell - z)}\left\{cos k'_n z - \left[\frac{\gamma\ell th\gamma\ell}{k'_n\ell} + \mu_1\left(\frac{k'^2_n\ell^2 + \gamma^2\ell^2}{k'_n\ell}\right)\right]sin k'_n z\right\}\right\}$$
$$\bullet\left\{T_{0n}\cos\tilde{\omega}_{on}t + \frac{\dot{T}_{0n}}{\tilde{\omega}_0}\sin\tilde{\omega}_{on}t\right\} \quad (93)$$

for $\alpha = 0$ and $\omega_{0n}^2 = 0$.

$$w(z,t) = \sum_{n=1}^{n=\infty}\left\{\frac{C_n}{Ch\gamma(\ell - z)}\left\{cos k'_n z - \left[\frac{\gamma\ell th\gamma\ell}{k'_n\ell} + \mu_1\left(\frac{k'^2_n\ell^2 + \gamma^2\ell^2}{k'_n\ell}\right)\right]sin k'_n z\right\}\right\}$$
$$\bullet\left\{T_{0n} + \frac{\dot{T}_{0n}}{\omega_{1n}^2}(1 - e^{-\omega_{1n}^2 t})\right\} \quad (94)$$

for $\alpha = 1$ and $\omega_{0n}^2 = 0$.

VIII CONCLUDING REMARKS

From the obtained analytical and numerical results for natural longitudinal creep vibrations of a fractional derivative order hereditary rod with variable cross section, it can be seen that a fractional derivative order hereditary properties are convenient for changing time function depending on material creep parameters, and that fundamental eigen-function depending on space coordinate is dependent only on boundary conditions and geometrical properties for considered models.

The first four eigen values for natural longitudinal vibrations of rheological conical rod (with variable cross section) with free ends are monotonously decreasing when ratio between ends diameters is increasing in interval: [0,1].

Changes of the first four eigen values for natural longitudinal vibrations of a fractional derivative order hereditary rod with variable cross section for different boundary conditions, as well as for different forms of rod can be seen from tables as a result of numerical experiment shown in paper.

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LONGITUDINALNE OSCILACIJE REOLOŠKE GREDE, PROMENLJIVOG POPREČNOG PRESEKA, OD MATERIJALA KONSTUTUTIVNE RELACIJE IZRAŽENE IZVODIMA NECELOG REDA

Predstavljeni su rezultati izučavanja longitudinalnih oscilacija reološke grede promenljivog poprečnog preseka, a od materijala sa svojstvima puzanja za koje je konstitutivna relacija izražena izvodima necelog reda. Izvedena je parcijalna diferencijalna jednačina i određena rešenja za slučaj sopstvenih longitudinalnih oscilacija grede, čiji materijal ima svojstva puzanja, a koja se opisuju izvodima ne celog reda. Za slučaj sopstvenih oscilacija grede promenljivog poprečnog preseka određeni su sopstveni brojevi, sopstvene funkcije i vremenske funkcije za različite granične uslove na krajevima grede, koji se javljaju u inženjerskim primenama. Sastavljene su tablice sopstvenih vrednosti za različite granične uslove. Pomoću MathCad programa sastavljene su grafičke ilustracije svojstava vremenske funkcije pri promeni parametra puzanja materijala. Pokazano je da se za usvojeni model grede promenljivog poprečnog preseka sopstvena funkcija ne zavisi od parametra izvoda necelog reda, već samo funkcija vremena.