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NUMERICAL COMPARISON OF THE SCALAR, PSEUDOINVARIANT AND INVARIANT APPROACH IN THE DERIVATION OF FINITE ELEMENT EQUATIONS OF MOTION IN CURVILINEAR COORDINATES

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Abstract. *Finite element equations of motion - based on a consistent use of invariant approximations in arbitrary curvilinear coordinates - are numerically compared with the usual ones. The superiority of the proposed invariant, vector-valued approach - although rejected a relatively long time ago as "less accurate" than the usual, scalar one - is demonstrated in the case of determining the nodal displacements in a typical membrane problem in polar coordinates.*

Key words: *finite elements, curvilinear coordinates, scalar approximations, pseudoinvariant approximations, invariant approximations.*

INTRODUCTION

The usual equations of motion of a typical finite element[1] of a continuum in curvilinear coordinates x^i, when a *scalar* approximation[2]

$$w^i = w^{Ni}\ \Psi_N \tag{1}$$

of the displacement field $\mathbf{u} = w^i\ \mathbf{g}_i$ (w^i are the contravariant components of displacement, $\mathbf{g}_i$ are the covariant base vectors of the curvilinear coordinates in question) is used[3] in their derivation, read (s. p. 190 in [3])

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[1] The classical, "displacement" type finite element analysis is in question.

[2] Einstein's summation convention for diagonally repeated indices will be used. Lowercase Latin indices have the range {1,2,3}. Uppercase Latin indices pertain to nodes and have the range from 1 to the total number of nodes of the element.

[3] I.e. each of its coordinates is approximated separately - as a scalar function.

$$
\begin{aligned}
m_{NM}\ddot{w}^{Mi} &+ \int_{v_o} t^{qj}(\delta^i_j \Psi_{N,q} - \Psi_N \Gamma^i_{jq})dv_o + \\
&+ \int_{v_o} t^{qj}(\Psi_{M,j}\Psi_{N,q}\delta^i_m + \Psi_M \Psi_{N,q}\Gamma^i_{mj} - \\
&- \Psi_{M,j}\Psi_N \Gamma^i_{mq} - \Psi_M \Psi_N \Gamma^r_{mj}\Gamma^i_{rq})dv_o w^{Mm} = p^i_N \quad ,
\end{aligned}
\tag{2}
$$

where m_{NM} is the consistent mass matrix (s. (13.37) in [3])

$$
m_{NM} = \int_{v_o} \rho_o \Psi_N \Psi_M dv_o \ , \tag{3}
$$

p^i_N is the total generalized force at node N (s. (13.54) in [3])

$$
p^i_N = \int_{v_o} \hat{F}^i \Psi_N dv_o + \int_{A_o} \hat{S}^i \Psi_N dA_o \quad , \tag{4}
$$

and $\hat{\mathbf{F}}$ and $\hat{\mathbf{S}}$ are the body forces and surface forces, respectively; further, Ψ_N are the interpolation functions and the comma in $\Psi_{N,m}$ denotes partial differentiation with respect to the corresponding curvilinear coordinates; Γ^i_{jk} are the Christoffel symbols of the second kind. The motion is referred to the reference configuration of the element; hence, t^{mj} is the stress tensor measured per unit area of the undeformed element of mass density, volume and surface area ρ_o, v_o and A_o, respectively.

On the other hand, "*a less accurate but considerably simpler form of the equations of motion in general coordinates*"

$$
m_{NM}\ddot{w}^{Mi} + \int_{v_o} t^{qj}\Psi_{N,q}(\delta^i_j + \Psi_{M,j}w^{Mi})dv_o = p^i_N \tag{5}
$$

is quoted (but not derived!) in [3] (p. 191), supposing - instead of the scalar approximations (1) of the components - a *"vector-valued"* approximation

$$
\mathbf{u} = \Psi_N \mathbf{u}^N \ , \tag{6}
$$

where $\mathbf{u}^N$ is the value of the vector field $\mathbf{u}$ at node N. However, the fact that the shifting operators (the Euclidean shifters; [2], p. 806) do not appear in (5) — although their presence, in accordance with Ericksen's concept of integration of tensor fields in curvilinear coordinates, is expected (s. [2], p. 808) — was a good reason to treat this approach as a *pseudoinvariant* one (s. [8]).

Finally, by a consistent use of the *invariant* approximation (6) of the displacement field[4]

$$
\mathbf{u} = \Psi_N \mathbf{u}^N = \Psi_N w^{Ni}\mathbf{g}_{(N)i} = \Psi_N w^N_i \mathbf{g}^{(N)i} \tag{7}
$$

[4] The placement of an index in parentheses means that the summation convention is not applied to the corresponding member — for example in summation over N in (7) this member is simply associated to the other members with this index.

(w^{Ni} and w_i^N are the contravariant and covariant components of displacement at node N; $\mathbf{g}_{(N)i}$ and $\mathbf{g}^{(N)i}$ are the base vectors at N), we have obtained the following equations of motion of a finite element in arbitrary curvilinear coordinates (s. [8])

$$m_{NM}\ddot{w}^{Ml} g^{(N)k}_{(M)l} + \int_{v_o} t^{ij}\Psi_{N,j} g_l^{(N)k}(\delta_i^l + \Psi_{M,i} g^l_{(M)m} w^{Mm}) dv_o = \tilde{p}_N^k \quad , \tag{8}$$

where $g_i^{(N)k} = \mathbf{g}^{(N)k} \bullet \mathbf{g}_i$ and $g^{(N)k}_{(M)l} = \mathbf{g}^{(N)k} \bullet \mathbf{g}_{(M)l}$ are the Euclidean shifters and

$$\tilde{p}_N^k = \int_{v_o} \hat{F}^i \Psi_N g_i^{(N)k} dv_o + \int_{A_o} \hat{S}^i \Psi_N g_i^{(N)k} dA_o \tag{9}$$

is the componental form in curvilinear coordinates of the total generalized force at N.

The aim of this communication is to make a numerical comparison of equations (2), (5) and (8).

FINITE ELEMENT EQUATIONS OF EQUILIBRIUM OF HOOKEAN MATERIALS

In order to perform this comparison, we shall consider the static behaviour of bodies - to obtain the corresponding equations of equilibrium, we simply drop the inertia terms in (2), (5) and (8). Then, for the sake of this comparison, we introduce (in the subintegral expressions in these equations) the local material curvilinear coordinates[5] ξ^α; hence

$$t^{ij} = \frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial x^j}{\partial \xi^\beta}\sigma^{\alpha\beta} \, . \tag{10}$$

Further, supposing linearly elastic materials, we have

$$\sigma^{\alpha\beta} = E^{\alpha\beta\gamma\delta} e_{\gamma\delta} \tag{11}$$

($E^{\alpha\beta\gamma\delta}$ are the first-order elasticities), where

$$e_{\gamma\delta} = \frac{1}{2}(w_{\gamma,\delta} + w_{\delta,\gamma}) \, . \tag{12}$$

Thus, assuming the displacement gradients to be infinitesimals and neglecting their products, we arrive from (2), (5) and (8) at the following three groups of finite element equations of equilibrium of Hookean materials

$$k^{ij}_{MN} w_j^M = p_N^i \qquad \textit{(scalar approach)} \quad , \tag{13}$$

where

$$k^{ij}_{MN} \equiv \int_{v_o} (x^i_\alpha \Psi_{N,\beta} - \Psi_N \Gamma^i_{mn} x^m_\alpha x^n_\beta) E^{\alpha\beta\gamma\delta} \\ (x^j_\gamma \Psi_{M,\delta} - \Psi_M \Gamma^j_{pq} x^p_\gamma x^q_\delta) dv_o \quad , \tag{14}$$

then

$$\bar{k}^{ij}_{MN} w_j^M = p_N^i \qquad \textit{(pseudoinvariant approach)} \quad , \tag{15}$$

[5] Lowercase Greek indices have the range {1,2,3}.

where

$$\bar{k}^{ij}_{MN} \equiv \int_{v_o} x^i_\alpha \Psi_{N,\beta} E^{\alpha\beta\gamma\delta} x^j_\gamma \Psi_{M,\delta} dv_o \,, \tag{16}$$

and finally

$$\tilde{k}^{ij}_{MN} w^M_j = \tilde{p}^i_N \qquad \textit{(invariant approach)} \,, \tag{17}$$

where[6]

$$\tilde{k}^{ij}_{MN} \equiv \int_{v_o} g^{(N)i}_\alpha \Psi_{N,\beta} E^{\alpha\beta\gamma\delta} g^{(M)j}_\gamma \Psi_{M,\delta} dv_o \,, \tag{18}$$

The expressions (14), (16) and (18) are stiffness matrices for linearly elastic materials in the scalar, pseudoinvariant and invariant approach, respectively. Obviously, in the case of rectangular Cartesian coordinates $z^i \equiv x^i$, when the Christoffel symbols are zero and the Euclidean shifters are the Kronecker delta, these expressions reduce to

$$k^{ij}_{MN} \equiv \int_{v_o} z^i_\alpha \Psi_{N,\beta} E^{\alpha\beta\gamma\delta} z^j_\gamma \Psi_{M,\delta} dv_o \,, \tag{19}$$

i.e. to the well-known stiffness matrix in classical infinitesimal elasticity (cf. e.g. with (16.13) in [3]).

The above mentioned numerical comparison of three approaches (the scalar, the pseudoinvariant and the invariant one) will be based on an in-house STATA (STATic Analysis) finite element code (described in [6]) and its modification in the part where these approaches are implemented. We shall consider the determination of nodal displacements, in polar coordinates, in a typical membrane problem with quadrilateral finite element meshes. It should be noted that the interpolation functions for the quadrilateral isoparametric finite element under consideration (based on [4] and [5]) are given as

$$\begin{aligned}
\Psi_1 &= \tfrac{1}{4}(1-\xi^1)(1-\xi^2) \\
\Psi_2 &= \tfrac{1}{4}(1+\xi^1)(1-\xi^2) \\
\Psi_3 &= \tfrac{1}{4}(1+\xi^1)(1+\xi^2) \\
\Psi_4 &= \tfrac{1}{4}(1-\xi^1)(1+\xi^2) \quad .
\end{aligned} \tag{20}$$

BENDING OF A CIRCULAR ARC

A cantilever curved beam (inner radius $r_i = 5$, outer radius $r_o = 20$, arc = 90º, thickness $t = 1$, $E = 1000$, $\nu = 0.3$) is analysed under two load conditions: a transverse end load (with a resultant force 10) and pure bending (with a bending moment of 150). The results of the three approaches are presented in Tables 1 and 2, while the theoretical solutions are obtained according to [1]. These tables show the convergence of the corresponding tip displacement with increasing mesh refinement.

As for the rate of convergence, this invariant approach is obviously superior to the scalar one (although the latter was proclaimed *"a better approximation"*; s. p. 48 in [3]), while the famous *"less accurate"* approach (rejected a relatively long time ago; p. 191 in [3]) is, in essence, what we refer to as the pseudoinvariant approach.

[6] Cf. with (3.74) in [7].

Table 1.

	Circular arc under transverse end load		
mesh	(inv. approx.)	(scal. approx.)	(pseudoinv. approx.)
1×1	-0.02039	-0.01730	-0.07186
2×2	-0.04200	-0.03370	-0.08661
3×3	-0.04908	-0.04312	-0.09099
4×4	-0.05233	-0.04848	-0.09350
5×5	-0.05425	-0.05170	-0.09520
6×6	-0.05550	-0.05376	…
7×7	-0.05636	-0.05513	…
8×8	-0.05699	-0.05609	…
9×9	-0.05745	-0.05679	…
10×10	-0.05780	-0.05730	…
	theoretical solution: $u_r^A = -0.06234$		

Table 2.

	Circular arc under pure bending		
mesh	(inv. approx.)	(scal. approx.)	(pseudoinv. approx.)
2×2	0.05399	0.03940	0.02122
4×4	0.05627	0.04953	0.01852
6×6	0.05697	0.05366	…
8×8	0.05720	0.05529	…
10×10	0.05737	0.05614	0.01829
	theoretical solution: $u^C_{<\varphi>} = 0.06354$		

CONCLUDING REMARKS

Finite element equations of equilibrium - based on a *consistent* use of invariant FE approximations in arbitrary curvilinear coordinates - are numerically compared with the usual ones. The efficiency of this approach is demonstrated by calculating the nodal displacements in a typical membrane problem in polar coordinates.

Finally, without hurrying to immediately proclaim this numerical example as crucial evidence to the superiority of the proposed *invariant* (*covariant*) approach, we only wish to emphasize something that is undisputable - the least that this approach deserves is to be fully reconsidered once again (especially bearing in mind that the invariant approach can be successfully applied in local and global stress smoothing procedures, too; s. [9] and [10]).

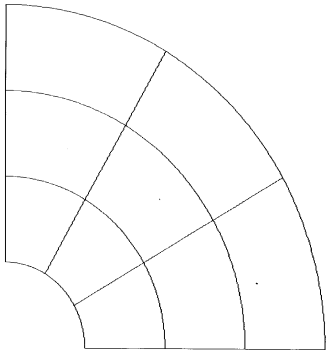

Fig. 1

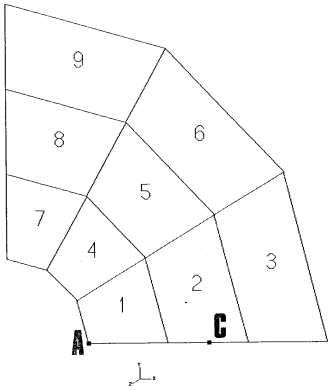

Fig. 2

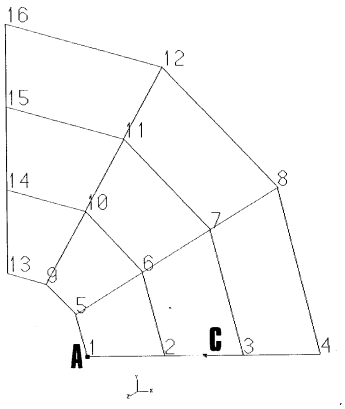

Fig. 3

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REFERENCES

1. S. Timoshenko and J.N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York - Toronto - London, 1951.
2. J.L. Ericksen, *Tensor Fields*, Handbuch der Physik, Bd. III/1, Springer-Verlag, Berlin - Göttingen - Heidelberg, l960.
3. J.T. Oden, *Finite Elements of Nonlinear Continua*, McGraw-Hill, New York, 1972.
4. M. Berković, *Thin shell isoparametric elements*, In Proc. II World Congress on Finite Element Methods, Bournemouth, (1978).
5. M. Berković, *Thin shell analysis*, Advanced Topics and New Developments in Finite Element Analysis, MARC, Rijswijk, The Netherlands, (1979), 1-43.
6. M. Berković and Z. Drašković, *Structural analysis software for microcomputers*, In Proc. Conference of Engineering Software for Microcomputers, ed. B.A. Schrefler, R.W. Lewis and S.A. Odorizzi, Pineridge Press, Swansea, (1984), 487-498.
7. S. Vujić, M. Berković, D. Kuzmanović, P. Milanović, A. Sedmak and M. Mićić, *The Application of the Finite Element Method to Geostatic Analysis in Mining*, Faculty of Mining and Geology, Beograd, 1991. *(in Serbian)*
8. Z. Drašković, *Contribution to the invariant derivation of finite element equations of motion in curvilinear coordinates*, Facta Universitatis, Series "Mechanics, Automatic Control and Robotics", 2 (6), (1995), 25-32.
9. Z. Drašković, *Contribution to a more accurate nodal stresses determination in the classical finite element method*, Naučno-tehnički PREGLED, XLV (9), (1995), 3-8. *(in Serbian)*
10. D. Mijuca, Z. Drašković and M. Berković, *Stress recovery procedure based on the known displacements*, Facta Universitatis, Series "Mechanics, Automatic Control and Robotics", 2 (7/2), (1997), 513-523.

NUMERIČKO POREĐENJE SKALARNOG, PSEUDOINVARIJANTNOG I INVARIJANTNOG PRISTUPA U IZVOĐENJU JEDNAČINA KRETANJA KONAČNIH ELEMENATA U KRIVOLINIJSKIM KOORDINATAMA

Zoran Drašković

Jednačine kretanja konačnih elemenata - zasnovane na doslednom korišćenju invarijantnih aproksimacija u proizvoljnim krivolinijskim koordinatama - numerički su upoređene sa uobičajenim jednačinama. Superiornost predloženog invarijantnog, vektorskog pristupa - premda dosta davno odbačenog kao "manje tačnog" od uobičajenog, skalarnog pristupa - pokazana je u slučaju određivanja čvornih pomeranja u jednom tipičnom primeru u polarnim koordinatama.