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LOW MACH NUMBER, HIGH ORDER RAREFIED GAS FLOW IN MICRO-CHANNELS

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Abstract. *Isothermal rarefied gas flow in micro-channels of slowly varying cross section is treated in this paper. It is assumed that the ratio of the reference Mach number square and the reference Reynolds number is a small quantity, so that inertia can be neglected and the effect of viscosity is spread over the whole cross-section of the channel. Higher order slip boundary condition on the wall is used for the solution of governing equations. Gas rarefaction leads to increase of mass flow rate for the same inlet and outlet pressure ratio.*

Key words: *micro-channels, rarefied gas flow, higher order slip boundary condition.*

1. INTRODUCTION

More than three decades had passed since famous physicist Richard Feynman, in his inspiring lecture presented on the Annual Meeting of the American Physical Society, held at Caltech, in 1959, under the title "There is plenty of room at the bottom", raised the question of making miniature electro-mechanical devices, before his visionary ideas became reality. In recent years several manufacturing processes have been developed which can create extremely small machines, so that the term Microelectromechanical systems (MEMS) technology is now widely used to refer to devices with characteristic dimensions measured in microns. It is usually thought that MEMS technology "is a giant step, and it cannot be excluded that microelectromechanical systems will have in the near future the same impact on society and economy as the IC has had since the early 1960s", [2].

Problems of fluid mechanics, and in particular of gas dynamics, arise in several microdevices intended for important industrial and medical applications. They can be successfully applied for measurements in turbulence (even on Kolmogorov microscales!), for active flow control, as micropumps and microturbines, for integrated cooling of electronic circuits and superconducting magnets, in cryo-coolers for infra-red detectors and diode lasers, etc.

Low pressure gas flows in several microdevices, say in micro-channels, can seldom be treated as a continuum flow with no-slip boundary conditions, because the values of the

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Knudsen number attained in such a flow are usually not extremely small. Rather, slip boundary conditions should be used for the solution of Navier-Stokes equations, or at extremely low pressures gas should be considered as a collection of molecules, and not as a continuum. In this paper we treat a continuum gas flow in a micro-channel of variable cross section. We assume the flow is compressible, isothermal and of low Mach number, so that viscosity prevails over inertia in the entire cross section of the channel. At that we employ the higher order slip boundary conditions at the channel walls, defined recently in [1].

2. Problem statement and governing equations

We study the problem depicted in Fig. 1 in which the upper half of a symmetric channel is presented. The flow in the channel is supposed to be a steady, two-dimensional, isothermal, compressible flow of a rarefied, perfect gas. System of equations governing such a flow consists of equation of continuity, the momentum equations in *x* and *y* direction (Fig. 1), and the equation of state. They will be written in nondimensional form by using the following scales (Fig. 1): δ_0 for all lengths, and average velocity in *x* direction, pressure and density at the entrance ($x = 0$) cross section of the channel, u_0, p_0 and ρ_0 for all velocities, pressure and density, respectively.

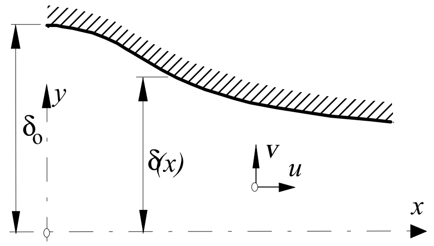


Fig.1 Rarefied gas flow in a micro-channel.

Also, they will be simplified by assuming that $\gamma M_0^2/\mathrm{Re} = \varepsilon << 1$ is a small parameter, where γ is the ratio of specific heats, M_0 is the reference Mach number, and Re is the reference Reynolds number, and the channel varies its cross-section slowly, on the scale of ε, so that a slow coordinate $\xi = \varepsilon x$ can be introduced in order to make these slow variations explicit. The equations read:

- continuity equation

$$\partial(pu)/\partial\xi + \partial(pV)/\partial y = 0 \tag{1}$$

- momentum equation in *x*-direction,

$$\gamma M_0^2 p\left(u\frac{\partial u}{\partial \xi} + V\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial \xi} + \frac{\partial^2 u}{\partial y^2} + O(\varepsilon^2) \tag{2}$$

- momentum equation in *y*-direction,

$$\partial p/\partial y = O(\varepsilon^2)\ . \tag{3}$$

where the nondimensional form of the equation of state: $p = \rho$ has been incorporated and $V(\xi,y)$ is defined as: $v(x,y) = \varepsilon V(\xi,y)$, where $v(x,y)$ is the small transverse velocity component. This system should be solved by using the following symmetry and boundary conditions:

$$y = 0: \qquad \partial u/\partial y = \partial^3 u/\partial y^3 = \ldots = 0, \; V = 0,$$

$$y = \delta(\xi): \quad u = u_w = \frac{2-\sigma_v}{\sigma_v}\left[-\frac{\mathrm{Kn}}{p}\frac{\partial u}{\partial y} + \frac{\mathrm{Kn}^2}{2p^2}\frac{\partial^2 u}{\partial y^2} + O(\mathrm{Kn}^3)\right], \; V = V_w = u_w \frac{d\delta}{d\xi},$$

where σ_v is the accommodation coefficient of the solid wall surface, and Kn is the reference Knudsen number defined as the ratio between the free molecular path at the entrance of the channel and δ_0. Since for an isothermal flow free molecular path is inversely proportional to pressure, Kn/p appearing in the slip boundary conditions obviously represents a local value of the Knudsen number. As mentioned in the Introduction, the slip conditions stated in this form were first proposed by Beskok at al. [1] with the aim to extend their validity to as high values of the Knudsen number as possible – the highest predicted value being about 0.6, if it is defined via the width of the channel. Since in this problem the Knudsen number is defined via the half width of the channel (δ_0), we may expect that the theory presented will be applicable to even higher values of Kn. In what follows we will apply our results for Kn up to 0.9.

It follows from (2) that for high subsonic and supersonic flow inertia terms on the left is of the same order of magnitude as the dominant viscous term, and the problem is one of boundary layer type. However, for low subsonic Mach numbers, inertia term can be neglected, and the flow is viscosity dominated. This case is particularly simple because equation (2), taking into account (3), can be easily integrated yielding:

$$u = -\frac{1}{2}\frac{dp}{d\xi}\left[\delta^2 - y^2 + \frac{2-\sigma_v}{\sigma_v}\frac{\mathrm{Kn}}{p}\left(2\delta - \frac{\mathrm{Kn}}{p}\right)\right]. \tag{4}$$

Since we are primarily interested in the derivation of an equation for the pressure distribution inside the channel, we will now circumvent the determination of V from (1). We will simply integrate (1) in y from 0 to $\delta(\xi)$, apply the boundary conditions: for $y = 0$, $V = 0$, and for $y = \delta(\xi)$, $V = V_w = u_w d\delta/d\xi$, and Leibnitz's formula to get: $\int_0^\delta p u dy = 1.$ Finally, utilizing (4), the following equation governing pressure is obtained:

$$\frac{dp}{d\xi} = -\frac{3}{p\delta\left[\delta^2 + \frac{2-\sigma_v}{\sigma_v}\frac{3\mathrm{Kn}}{2p}\left(2\delta - \frac{\mathrm{Kn}}{p}\right)\right]}, \tag{5}$$

which should be solved with the boundary condition: for $\xi = 0$, $p = 1$. Obviously, equation possesses two singular points – cross sections of the channel: one in which $p = 0$, and one in which the term in the bracket is zero for some $p > 0$. The integration of (5) makes sense only up to the cross section in which $dp/d\xi \to \infty$ for some finite non-zero value of p. In what follows we will call this cross section the critical one (ξ_k) and the pressure in it - p_k, the critical pressure.

3. RESULTS AND DISCUSSION

For a channel consisting of plane walls ($\delta = 1$) the solution of (5) can be readily obtained by quadrature in the form: $\xi = \xi(p,\mathrm{Kn},\sigma_v)$. For $\sigma_v = 1$ it is:

$$\xi = \frac{1}{6}(1-p^2) + \mathrm{Kn}(1-p) + \frac{\mathrm{Kn}^2}{2}\ln p\ ,$$

and is plotted in Fig.2, for $p_k \le p \le 1$. Critical pressure increases linearly with Kn as: $p_k = 0.436\mathrm{Kn}$, while the critical length of the channel increases whit Kn up to the maximum value of $\xi_k = 0.36$ for and decreases for higher values.

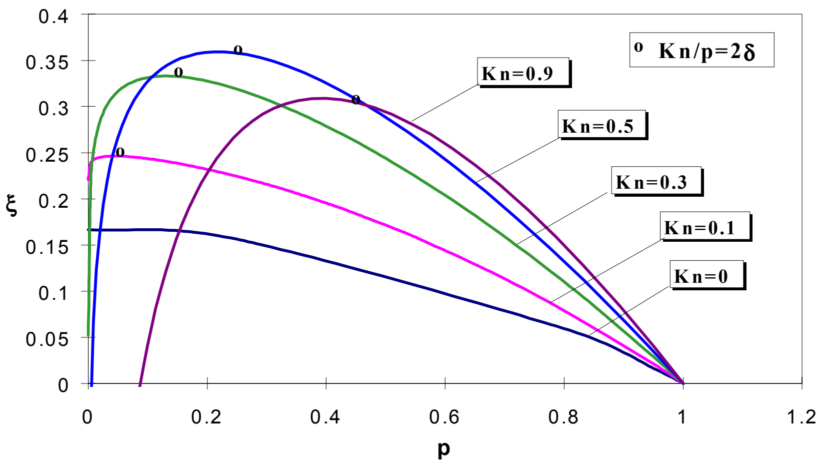
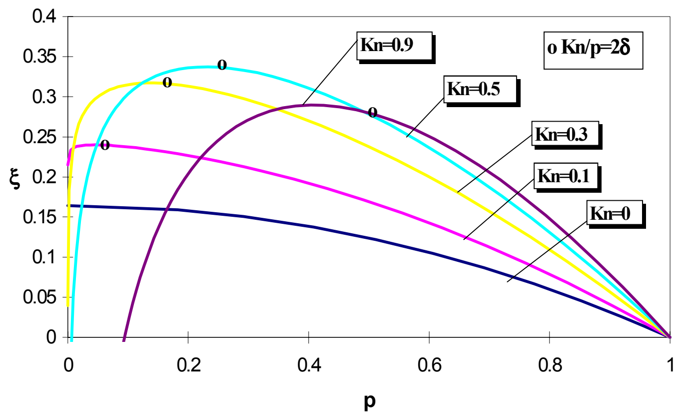


Fig. 2. Pressure distribution between parallel walls for different Kn-number.

The drop of the pressure to a non-zero critical value at a finite distance from the channel entrance bears some resemblance to the phenomenon of so-called "mathematical" choking, described for the first time by Schwartz [3]. While in Schwarz's problem the flow near the critical cross section was not realistic due to an infinite increase of the Mach number, whereby the basic assumption of low Mach number flow was violated, here, in addition to the same reason for the break of the theory (note that $u \to \infty$ as $dp/d\xi \to \infty$ (4)), there is another reason for which the above stated result should be accepted with caution. Namely, since Kn/p represents the local value of the Knudsen number, it increases downstream and reaches the value of 2δ in front of the critical length, which means that the mean free path in the cross section in which $\mathrm{Kn}/p = 2\delta$ becomes equal to the width of the channel! In spite of using higher order slip boundary conditions on the wall, aimed at increasing values of the Knudsen number for which the theory is applicable, we do not believe that such an extension of the theory can be acceptable in this case. Anyhow, cross sections in which $\mathrm{Kn}/p = 2\delta$ are designated in Fig. 2, and figures to follow.

a)



b)

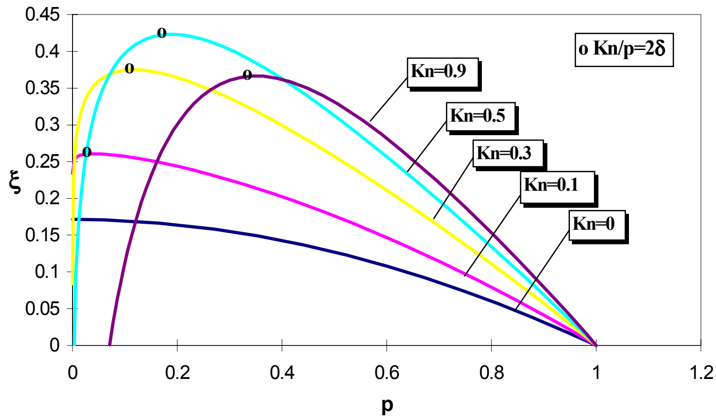


Fig. 3 Pressure distribution in a convergent channel (a), and in a divergent channel (b) for different Knudsen number.

In order to get some insight into the effect of the varying cross section of the channel, we performed the numerical integration of equation (5) for: $\delta(\xi) = (1-a)/(1+\xi^2) + a$, for two values of a: $a = 0.5$ – convergent channel, and $a = 2$ – divergent channel. Results presented in Fig. 3 are qualitatively very similar to those obtained for parallel walls. Critical pressure increases with the opening of the exit cross section, while at the same time critical length of the channel decreases.

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STRUJANJE GASOVA U MIKRO-KANALIMA PRI MALIM VREDNOSTIMA MAHOVOG BROJA, KORIŠĆENJEM GRANIČNIH USLOVA KLIZANJA VIŠEG REDA

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U radu se tretira izotermsko strujanje razređenog gasa u mikro-kanalima promenljivog poprečnog preseka. Pretpostavlja se da je odnos kvadrata Mahovog broja i Rejnoldsovog broja mali, tako da se inercioni članovi u osnovnim jednačinama mogu zanemariti, a uticaj viskoznosti je dominantan u celom preseku kanala. Koriste se granični uslovi klizanja gasa na zidu višeg reda i pokazuje da razređenost gasa dovodi do povećanja masenog protoka pri istom odnosu ulaznog i izlaznog pritiska.