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VALIDITY OF LIU'S SYSTEMATIC METHOD ON VARIATIONAL PRINCIPLES *

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Abstract. *Serious deflects are found in Liu's systematic method on variational principle (Acta Mech. 140, 2000,73-89), and the invalidity of Liu's method is illustrated by some examples. The necessary condition of the validity of Liu's method is first pointed out. The deflects can be completely eliminated by a reliable modification.*

Key words: *inverse problem of calculus of variations*

1. INTRODUCTION

In recent years the inverse problem of calculus of variations has brought about a renewed interest in continuum mechanics. It emanates from the powerful applications to finite element methods [1] and meshfree particle methods [2].

In 2000, Liu [3] proposed a systematic approach to derivation of variational principles from the partial differential equations. The basic idea of Liu's approach appeared in Liu's previous publications as early as 1990 [4,5]. The idea in Ref. [3] is not new or superior to the Refs. [4,5]. Liu's method can be applied not only to fluid mechanics, but also to solid mechanics [6]. As early as 1996, the present author found the invalidity of Liu's method to establishment of variational principles for one-dimensional unsteady compressible flow [7]. Using Liu's method, Liu himself [8] also obtained a wrong functional for transonic blade-to-blade flow (see Eq. (33) of Ref. [8], which was corrected by He in Ref. [9]). In Ref. [10] the present author first points out the contradictions in Liu's method. So Liu's method leaves much space to be further improved.

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* The editor calls the readers to discuss this paper

2. BASIC IDEAS OF LIU'S APPROACH

Liu's systematic method of derivation and transformation of variational principles consists of two major lines. There is a great vast of literature on the subject on variational principles, e.g. the classical monography by Wei-Zang Chien [11]. The first line of Liu's approach is also discussed by Chien [11] in details. The question of determining whether a set of field equations can be derived from a functional may be systematically elucidated by recourse to Veinberg's theorem, which also provides a formula for the computation of the corresponding functional. So the first line of Liu's approach sees nothing new. The key contribution of Liu's method lies on the second line, which provides a method of searching for a generalized variational principles directly from the field equations. Unfortunately, we found the second line contains serious deflects or contradictions, leading to very limited validity of this approach.

Consider a very simple system [3]

$$\begin{cases} \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} + f = 0, \\ \dfrac{\partial \Psi}{\partial x} = -v, \qquad \dfrac{\partial \Psi}{\partial y} = u. \end{cases} \tag{1}$$

According to Liu's second line, we can construct a trial-functional in the form [3]

$$J(\Psi,u,v,\mu_1,\mu_2,\mu_3) = \iint \left\{ F + \mu_1\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) + \mu_2\left(\frac{\partial \Psi}{\partial x} + v\right) + \mu_3\left(\frac{\partial \Psi}{\partial y} - u\right)\right\} dxdy, \tag{2}$$

where μ_1, μ_2, μ_3 are Lagrange multipliers, and F is an unknown function to be further determined.

Calculating variation of the functional (2), we obtain equations (1) and the following adjoint equations

$$\begin{cases} \dfrac{\partial \mu_2}{\partial x} + \dfrac{\partial \mu_3}{\partial y} - \dfrac{\partial F}{\partial \Psi} = 0, \\ \dfrac{\partial \mu_1}{\partial x} = \mu_2 + \dfrac{\partial F}{\partial v}, \qquad \dfrac{\partial \mu_1}{\partial y} = \mu_3 - \dfrac{\partial F}{\partial u}. \end{cases} \tag{3a,b,c}$$

The basic idea of Liu's second approach is to convert the adjoint system (3) into the original system (1). Comparing the original system (1) with the adjoint system (3), we can identify the unknown F and multipliers as $F = f\psi$, $\mu_1 = \psi$, $\mu_2 = -v$ and $\mu_3 = u$. Hence we obtained the required generalized variational principle [3]:

$$J(\Psi,u,v) = \iint \left\{ f\Psi + \Psi\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) - v\left(\frac{\partial \Psi}{\partial x} + v\right) + u\left(\frac{\partial \Psi}{\partial y} - u\right)\right\} dxdy. \tag{4}$$

The system discussed by Liu is too simple to consider its advantages. We will illustrate below that a slight modification of the system (1) lead to invalidity of Liu's approach.

3. CONTRADICTION IN LIU'S APPROACH

In [10], the author first points out the serious deflects or contradiction in Liu's method. The present author also illustrates the inherent deflects in the famous Lagrange multiplier method [12].

From (2) there might exist such a functional

$$\tilde{J} = \iint F dA, \tag{5}$$

subject to the three equations of system (1).

It is obvious that ***all*** the field equations (1) become constraints of an unknown functional (5), so there exists a contradiction in (2).The contradiction will leads to wrong results.

4. INVALIDITY OF LIU'S APPROACH

The incompatibility in Liu's basic assumption in (2) leads to very limited applications of this method. Now we slightly modify system (1) as follows

$$\begin{cases} \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} + f = 0, \\ \dfrac{\partial \Psi}{\partial x} = \dfrac{u}{\sqrt{u^2+v^2}}, \qquad \dfrac{\partial \Psi}{\partial y} = -\dfrac{v}{\sqrt{u^2+v^2}}. \end{cases} \tag{6a,b,c}$$

According to Liu's approach, the following functional can be constructed

$$J(\Psi,u,v,\mu_1,\mu_2,\mu_3) = \iint F dA + \\ + \iint \left\{ \mu_1\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) + \mu_2\left(\frac{\partial \Psi}{\partial y} - \frac{u}{\sqrt{u^2+v^2}}\right) + \mu_3\left(\frac{\partial \Psi}{\partial x} + \frac{v}{\sqrt{u^2+v^2}}\right)\right\} dA, \tag{7}$$

We, therefore, obtain the following adjoint system

$$\frac{\partial F}{\partial \Psi} - \frac{\partial \mu_2}{\partial y} - \frac{\partial \mu_3}{\partial x} = 0, \tag{8}$$

$$\frac{\partial F}{\partial u} + \frac{\partial \mu_1}{\partial y} - \frac{\mu_2}{\sqrt{u^2+v^2}} + \frac{\mu_2 u^2 - \mu_3 uv}{(u^2+v^2)\sqrt{u^2+v^2}} = 0, \tag{9}$$

$$\frac{\partial F}{\partial v} - \frac{\partial \mu_1}{\partial x} + \frac{\mu_3}{\sqrt{u^2+v^2}} + \frac{\mu_2 uv - \mu_3 v^2}{(u^2+v^2)\sqrt{u^2+v^2}} = 0. \tag{10}$$

By a parallel identification, if we set $F = f\psi$, $\mu_1 = \psi$, $\mu_2 = u$ and $\mu_3 = -v$, then Eq. (8) is reduced to Eq. (6a), however, Eqs. (9) and (10) violate Eqs. (6b) and (6c) respectively. There exits no simple way to convert the adjoint system (8)~(10) to the original one (6). So Liu's method is invalid for this simple case.

If we set $F = -2f\psi - u\psi_y + v\psi_x$, $\mu_1 = \psi$, $\mu_2 = -u$ and $\mu_3 = v$, then Eqs. (9) and (10) vanish completely. Such a special case has not been discussed by Liu, theoretically this should also lead to a correct solution, for the lost Eqs. (9) and (10) can be recovered from the stationary conditions with respect to μ_1 and μ_2. Accordingly we obtain the following functional:

$$J(\Psi, u, v) = \\ = \iint \left\{ -2f\Psi - u\Psi_y + v\Psi_x + \Psi\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - u\left(\frac{\partial \Psi}{\partial y} - \frac{u}{\sqrt{u^2+v^2}}\right) + v\left(\frac{\partial \Psi}{\partial x} + \frac{v}{\sqrt{u^2+v^2}}\right)\right\} dA. \tag{11}$$

It is eaisy to prove that the obtained functional is wrong.

To illustrate our approach is reasonable according to Liu's theory, we re-consider Liu's model example, e.g., system (1). From (3), we can set $F = f\psi + (u^2 + v^2)/2$, $\mu_1 = C$ (a constant), $\mu_2 = -v$ and $\mu_3 = u$, then Eqs. (3b) and (3c) vanish completely. So we obtain the following functional:

$$J(\Psi, u, v) = \iint \left\{ f\Psi + (u^2+v^2)/2 + C\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) - v\left(\frac{\partial \Psi}{\partial x} + v\right) + u\left(\frac{\partial \Psi}{\partial y} - u\right)\right\} dxdy. \tag{12}$$

It is obvious that all Euler equations of functional (12) satisfy Eqs. (3). The reliable modification of Liu's method is also illustrated in the last section of this paper (see Eq. 42).

Now a question arises why the same approach leads to different results. The contradiction in Liu's method can be partly eliminated by trial-and-error method of identification of the multipliers and the unknown F, so Liu's method might lead to a correct functional for simple problems. However, Liu's approach fails for relatively complicated cases where the trial-and-error method can not compensate for the contradiction.

Now consider another simple example. Consider the system

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{13}$$

By Liu's approach, Liu obtained the following functional[3]

$$J(u, v) = \iint \left\{ v\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right\} dxdy. \tag{14}$$

As pointed out by Liu [3], we can not obtain any Euler equations from the above functional.

5. HE'S APPROACH

To eliminate the contradiction in Liu's basic assumption in (2), we can assume that the unknown functional

$$J(\Psi) = \iint F dxdy \tag{15}$$

is under the constraints of two equations in (1), let's say the last two equations. By Lagrange multiplier method, we have

$$J_1(\Psi,u,v,\mu_2,\mu_3)=\iint\left\{F+\mu_2\left(\frac{\partial\Psi}{\partial x}+v\right)+\mu_3\left(\frac{\partial\Psi}{\partial y}-u\right)\right\}dxdy, \tag{16}$$

From $\delta J_1 = 0$ we obtain the following Euler's equations

$$\Psi_x=-v\ ,\ \Psi_y=u\ ,\ \delta_\Psi F-\mu_{2x}-\mu_{3y}=0\ \ ,\ \delta_u F-\mu_3=0\ ,\ \delta_v F-\mu_2=0\ , \tag{17}$$

where the subscripts denote partial differentials and $\delta_\Psi F$ is variational differential, which is defined as

$$\delta_\Psi F=\frac{\partial F}{\partial\Psi}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial\Psi_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial\Psi_y}\right)$$

By a similar way as above, we have

$$\delta_\Psi F=f,\ \delta_u F=-u\ ,\ \delta_v F=-v\ . \tag{18}$$

We can, therefore, identify the unknown F as follows

$$F=f\,\Psi-\frac{1}{2}(u^2+v^2)\,. \tag{19}$$

So we identify the assumed functional (15), which reads

$$J(\Psi)=\iint\left\{f\,\Psi-\frac{1}{2}(u^2+v^2)\right\}dxdy\,, \tag{20}$$

and obtain the following generalized variational principle

$$J_1(\Psi,u,v,\mu_2,\mu_3)=\iint\left\{f\,\Psi-\frac{1}{2}(u^2+v^2)+v\left(\frac{\partial\Psi}{\partial x}+v\right)-u\left(\frac{\partial\Psi}{\partial y}-u\right)\right\}dxdy, \tag{21}$$

which is the same as Liu's result.
For the system (6), we can construct the following functional:

$$J(\Psi,u,v,\mu_2,\mu_3)=\iint\left\{F+\mu_2\left(\frac{\partial\Psi}{\partial y}-\frac{u}{\sqrt{u^2+v^2}}\right)+\mu_3\left(\frac{\partial\Psi}{\partial x}+\frac{v}{\sqrt{u^2+v^2}}\right)\right\}dA\,. \tag{22}$$

We, therefore, obtain the following adjoint system:

$$\frac{\delta F}{\delta\Psi}-\frac{\partial\mu_2}{\partial y}-\frac{\partial\mu_3}{\partial x}=0\,, \tag{23}$$

$$\frac{\delta F}{\delta u}-\frac{\mu_2}{\sqrt{u^2+v^2}}+\frac{\mu_2 u^2-\mu_3 uv}{(u^2+v^2)\sqrt{u^2+v^2}}=0\,, \tag{24}$$

$$\frac{\delta F}{\delta v}+\frac{\mu_3}{\sqrt{u^2+v^2}}+\frac{\mu_2 uv-\mu_3 v^2}{(u^2+v^2)\sqrt{u^2+v^2}}=0\,. \tag{25}$$

We can identify the multipliers and F as follows:

$$\mu_2 = u\ ,\ \mu_3 = -v\ , F = f\ \Psi\ . \tag{26}$$

Hence, we obtain the following generalized variational principle:

$$J(\Psi, u, v) = \iint \left\{ f\ \Psi + u\left(\frac{\partial \Psi}{\partial y} - \frac{u}{\sqrt{u^2+v^2}} \right) - v\left(\frac{\partial \Psi}{\partial x} + \frac{v}{\sqrt{u^2+v^2}} \right) \right\} dA. \tag{27}$$

It is easy to prove that all Euler equations satisfy all field equations.

For the system (13), we can write the following trial-functional

$$J(u, v, \Psi) = \iint \left\{ F + \Psi\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right\} dxdy\ , \tag{28}$$

where ψ is a Lagrange multiplier.

The stationary conditions of the above functional read:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0\ , \tag{29}$$

$$\frac{\delta F}{\delta u} + \frac{\partial \Psi}{\partial y} = 0\ , \tag{30}$$

$$\frac{\delta F}{\delta v} - \frac{\partial \Psi}{\partial x} = 0\ . \tag{31}$$

From the equations (30) and (31), we have

$$\frac{\partial}{\partial x}\left(\frac{\delta F}{\delta u} \right) + \frac{\partial}{\partial y}\left(\frac{\delta F}{\delta v} \right) = 0\ , \tag{32}$$

which should be the field equation $u_x + v_y = 0$, so we set:

$$\frac{\delta F}{\delta u} = u\ , \text{ and } \frac{\delta F}{\delta v} = v\ , \tag{33}$$

from which we identify the unknown F as follows:

$$F = \frac{1}{2}(u^2 + v^2)\ . \tag{34}$$

So we obtain the following variational principle:

$$J = \iint \frac{1}{2}(u^2 + v^2) dxdy\ , \tag{35}$$

and the following generalized variational principle:

$$J(u, v, \Psi) = \iint \left\{ \frac{1}{2}(u^2 + v^2) + \Psi\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right\} dxdy\ . \tag{36}$$

In this approach, the unknown F has a special meaning, and the multipliers can be introduced in a more meaningful way.

6. FURTHER REMARKS

Why does Liu's method, sometimes, lead to a correct result? Under a wrong assumption, and by a trial-and-error method, Liu's method might lead to a correct solution. Now consider a general system:

$$g_i(u_i, u_i') = 0, \ (i=1\sim n). \tag{37}$$

In view of Liu's method, a functional can be constructed as follows:

$$J(u_i, \mu_i) = \int [F + \sum_{i=1}^{n} \mu_i g_i] dx . \tag{38}$$

Its adjoint system reads:

$$G_i = \frac{\partial F}{\partial u_i} + \sum_{i=1}^{n} \mu_i \frac{\partial g_i}{\partial u_i} - \frac{d}{dx}\left(\sum_{i=1}^{n} \mu_i \frac{\partial g_i}{\partial u_i'} \right) + \cdots = 0 . \tag{39}$$

By a careful choice of the multipliers and the unknown F, we can convert the above adjoint system to the original system (37). Generally, G_i can be written in the forms

$$G_i = \sum_{j=1}^{n} a_{ij} g_j , \tag{40}$$

where a_{ij} are constants, which should meet the condition that the determinant of a_{ij} is different from zero, i.e. $\det(a_{ij}) \neq 0$. After identification of F and the multipliers, we have

$$J(u_i) = \int [F(u_i) + \sum_{i=1}^{n} \mu_i(u_i) g_i(u_i, u_i')] dx . \tag{41}$$

Supposing that F and the multipliers can be expressed the functions of u_i, we have the following Euler equations. Its Euler equations are:

$$\frac{\partial F}{\partial u_i} + \sum_{i=1}^{n} \frac{\partial \mu_i}{\partial u_i} g_i(u_i, u_i') + \sum_{j=1}^{n} \mu_j \frac{\partial g_j}{\partial u_i} - \frac{d}{dx}\left(\sum_{i=1}^{n} \mu_i \frac{\partial g_i}{\partial u_i'} \right) = 0 . \tag{41}$$

Recalling (39) and (40) , we have:

$$\sum_{j=1}^{n} \left(a_{ij} + \frac{\partial \mu_j}{\partial u_i} \right) g_j(u_i, u_i') = 0 . \tag{42}$$

So the necessary condition of validity of Liu's method is:

$$\det\left(a_{ij} + \frac{\partial \mu_j}{\partial u_i} \right) \neq 0 . \tag{43}$$

It is difficult to choose such multipliers to satisfy the above condition. It is also reasonable that some equations in an adjoint system vanish (for example $a_{ij} = 0$ (j =3,8)).

Note: If F and the multipliers are functions of u_i and their derivatives, a responding relation for the validity of Liu's method can be obtained.

The semi-inverse method [11] is proved to be a powerful tool of searching for various variational principles for physical problems. For example, for the system (6), we can construct the following trial-functional:

$$J(\Psi,u,v)=\iint\left\{\Psi\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+F\right\}dxdy\,, \tag{44}$$

where no multiplier is introduced, and F is an unknown function of u and v.

The Euler equation with respect to ψ satisfies one of its field equation. The Euler equations with respect to u and v read:

$$\frac{\partial\Psi}{\partial y}+\frac{\delta F}{\delta u}=0\,, \tag{45}$$

$$-\frac{\partial\Psi}{\partial x}+\frac{\delta F}{\delta v}=0\,. \tag{46}$$

If we set

$$\frac{\delta F}{\delta u}=\frac{v}{\sqrt{u^2+v^2}}\,, \tag{47}$$

and

$$\frac{\delta F}{\delta v}=\frac{u}{\sqrt{u^2+v^2}}\,, \tag{48}$$

then equations (45) and (46) reduce to the field equations. From (47) and (48), we can immediately identify the unknown F, which reads:

$$F=\sqrt{u^2+v^2}\;. \tag{49}$$

We, therefore, obtain the needed functional:

$$J(\Psi,u,v)=\iint\left\{\Psi\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\sqrt{u^2+v^2}\right\}dxdy\,. \tag{50}$$

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VALJANOST LIU-OVOG SISTEMATSKOG METODA VARIJACIONOG PRINCIPA

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Ozbiljni nedostaci su nadjeni u Liu-ovoj sistematskoj metodi varijacionog principa, a to da Liu-ova metoda ne valja ilustrovano je nekim primerima. Neophodni uslovi valjanosti Liu-ove metode su istaknuti. Nedostaci mogu da budu eliminasni pouzdanim modifikacijama.