FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol.3, N° 12, 2002, pp. 457 - 463

ON THE RESPONSE OF THE SECOND MODE OF A CANTILEVER BEAM

UDC 534.11:531.011

V. Marinca, B. Marinca, N. Herisanu

"Politehnica" University of Timisoara, Romania

Abstract. *In this paper it is considered that the displacement of a cantilever beam is the superposition of the contribution of the second mode and the prescribed harmonic base motion. The equation of motion is one weakly nonlinear and is solved by a combination of variational and averaging method. The results obtained here can be used in several experimental nonlinear identification techniques.*

1. INTRODUCTION

Several experimental nonlinear identification techniques have been presented in the literature. These techniques are based on a least-squares curve fit for time-domain data, which is linearly related to the unknown coefficients. They have the advantages of requiring little a priori knowledge of the system and requiring less time and effort for data acquisition than the sine well method used for the frequency-domain techniques. The frequency domain techniques avoid the problems associated with differentiation and observability of small terms but require considerably more theoretical effort.

In [1] is applied a time-domain technique and two frequency-domain techniques to the second mode of a cantilever beam. The frequency-domain techniques include a backbone curve-fitting technique based on the describing function method and on amplitude and frequency sweep technique. But on the other hand, the restoring-force-surface method cannot be used to determine the equation of motion. Several approaches require simultaneous measurement of position, velocity and acceleration or measuring the position using a fiber optic sensor and differentiating it to get the velocity and acceleration

Finally, we present the results of numerical integration using a Runge-Kutta routine and compare them with the analytical results: the agreement is very good.

Received July 04, 2002
From Program of Collaboration between Faculty of Mechanical Engineering Niš and University "Politechnica" Temisoara

2. PROBLEM FORMULATION

In [1] the equation of motion is determined by applying the assume modes method. We consider that the Lagrangian for a cantilever beam is described by:

$$L=\int_0^l\left[\frac{1}{2}m\dot{w}^2-\frac{1}{2}EI(w''^2+w''^2\,w'^2)+\frac{1}{8}m\left(\frac{\partial}{\partial t}\int_0^s w'^2\,ds\right)^2\right]ds \tag{1}$$

where L is Lagrangian of the system, m is mass per unit length of the beam, w is transverse displacement of the beam, E is Young's modulus, I is moment of inertia, s is arc length, t is time.

Assuming that the displacement $w(s,t)$ is the superposition of the contribution of the second mode and the prescribed motion, the displacement to be written as:

$$w(s,t)=\phi_2(s)u(t)+x_b(t) \tag{2}$$

where $\phi_2(s)$ is the shape of the second mode, $u(t)$ is amplitude of the motion and x_b is the prescribed motion. $\phi_2(s)$ is normalized, so that

$$\int_0^l \phi_2^2(s)ds=l \tag{3}$$

If ω is the frequency of this mode, then

$$EI\int_0^l {\phi_2''}^2 ds=ml\omega^2 \tag{4}$$

Substituting Eq.(2) into (1), we obtain:

$$L=\frac{1}{2}ml\dot{u}^2+m\xi\dot{u}\dot{x}_b+\frac{1}{2}ml\dot{x}_b^2-\frac{1}{2}ml\omega^2u^2-\frac{1}{2}\varepsilon ml\alpha u^4-\frac{1}{2}\varepsilon ml\beta\dot{u}^2u^2 \tag{5}$$

where ε is a small parameter and

$$\xi=\int_0^l\phi_2(s)ds;\qquad \varepsilon ml\alpha=2EI\int_0^l {\phi_2'}^2{\phi_2''}^2ds;\qquad \varepsilon\beta l=\int_0^l\left(\int_0^s {\phi_2'}^2ds\right)^2ds \tag{6}$$

The equation of motion can now be written as

$$ml\ddot{u}+m\xi\ddot{x}_b+\varepsilon ml\beta(u\dot{u}^2+u^2\ddot{u})+ml\omega^2u+\varepsilon ml\alpha u^3=Q \tag{7}$$

where the generalized force Q, accounting for viscous damping, can be written

$$Q=-c\dot{u} \tag{8}$$

In [1] is considered a harmonic base motion and is described it by

$$x_b=-\frac{1}{\Omega^2}a_b\cos\Omega t \tag{9}$$

where a_b is the input acceleration and Ω is the driving frequency. For the case of primary resonance, we define a detuning parameter such that:

$$\Omega^2=\omega^2+\varepsilon\sigma \tag{10}$$

Substituting Eq. (8-10) into (7) and dividing by ml gives

$$\ddot{u}+\Omega^2 u=\varepsilon\sigma u-\varepsilon\beta(u\dot{u}^2+u^2\ddot{u})-\varepsilon\alpha u^3-2\varepsilon\mu\dot{u}+2\eta\varepsilon a_b\cos\Omega t \tag{11}$$

where

$$\varepsilon\mu=\frac{c}{2ml};\quad \varepsilon\eta=\frac{\xi}{l} \tag{12}$$

3. An approximate theory application

Consider the following equation:

$$\ddot{u}+\Omega^2 u=\varepsilon f(\Omega t,u,\dot{u},\ddot{u}) \tag{13}$$

where f is a nonlinear function of u, $\dot{u}$ and $\ddot{u}$, periodic of Ωt with period 2π which may be expanded in a Fourier series; parameter ε is assumed to be small, i.e. $0<\varepsilon<<1$.

According to [2,3], we can construct the following iteration formula:

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda(\tau,t)[u_n''(\tau)+\Omega^2 u(\tau)-\varepsilon f(\Omega\tau,\tilde{u}_n,\tilde{u}_n',\tilde{u}_n'')]\,d\tau \tag{14}$$

where $\lambda(\tau,t)$ is called a general Lagrange multiplier, which can be identified optimally via variational theory, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n^{(k)}=0$, $k=0,1,2$, and $'=\partial/\partial\tau$. The function $\lambda(\tau,t)$ which extremizes the functional

$$\int_0^l \lambda(\tau,t)[u_n''(\tau)+\Omega^2 u_n(\tau)-\varepsilon f(\Omega\tau,\tilde{u}_n,\tilde{u}_n',\tilde{u}_n'')]\,d\tau \tag{15}$$

can be readily identified:

$$\lambda(\tau,t)=\frac{1}{\Omega}\sin\Omega(\tau-t) \tag{16}$$

As a result, we obtain the following iteration formula, named correction functional:

$$u_{n+1}(t)=u_n(t)+\frac{1}{\Omega}\int_0^t \sin\Omega(\tau-t)[u_n''(\tau)+\Omega u_n(\tau)-\varepsilon f(\Omega\tau,u_n,u_n',u_n'')]\,d\tau \tag{17}$$

In this method the problems are initially approximated with possible unknowns and so far do not depend on small parameters. The method reveals that the approximate solution obtained by the proposed method converge rapidly.

For $\varepsilon=0$, Eq.(13) has the solution $x(t)=a\cos\psi$, where $\dot{a}=0$, $\dot{\psi}=\Omega$. For $\varepsilon\neq 0$, we try a solution of the form:

$$u_0(t)=a\cos\psi+\varepsilon v(t)+0(\varepsilon^2) \tag{18}$$

which is called the approximation of the second order, v(t) is supposed to be periodic function of period 2π in both Ωt and ψ. However, the amplitude and the total phase of the first harmonic are supposed to satisfy the differential equations with separable variables ("slowly varying") [4]:

$$\begin{aligned}\dot{a} &= \varepsilon A(a,\theta)\\ \dot{\psi} &= \Omega + \varepsilon B(a,\theta)\\ \dot{\theta} &= \varepsilon B(a,\theta)\end{aligned} \tag{19}$$

Eq. (19_1) and (19_3) will be used calculating the functions $a(t)$ and $\theta(t)$ after the functions $A(a,\theta)$ and $B(a,\theta)$ have been found.

We first try to determine the functions $v(t)$, $A(a,\theta)$, $B(a,\theta)$ so that Eq. (13) be satisfied to within an error of the second order in ε will be systematically omitted in the following. By differentiating Eq. (18) twice with respect t, we obtain:

$$\dot{u}_0(t) = \dot{a}\cos\psi - a\dot{\psi}\sin\psi + \varepsilon\dot{v}(t) \tag{20}$$

$$\ddot{u}_0(t) = \ddot{a}\cos\psi - 2\dot{a}\dot{\psi}\sin\psi - a\dot{\psi}^2\cos\psi - a\ddot{\psi}\sin\psi + \varepsilon\ddot{v}(t) \tag{21}$$

Next, we deduce from (19) that

$$\ddot{a} = 0; \quad \ddot{\psi} = 0; \quad \dot{a}\dot{\psi} = \varepsilon\Omega A; \quad \dot{\psi}^2 = \Omega^2 + 2\varepsilon\Omega B \tag{22}$$

Introducing these expressions, as well as (19) into (20) and (21) yields:

$$\dot{u}_0 = -\Omega a\sin\psi + \varepsilon(A\cos\psi - aB\sin\psi + \dot{v}) \tag{23}$$

$$\ddot{u}_0 = -\Omega^2 a\cos\psi + \varepsilon(-2\Omega A\sin\psi - 2\Omega aB\cos\psi + \ddot{v}) \tag{24}$$

On the other hand, by taking into consideration (18), (23) and (24) it follows that:

$$\varepsilon f(\Omega\tau, u_0, \dot{u}_0, \ddot{u}_0) = \varepsilon f(\Omega\tau, a\cos\psi, -\Omega a\sin\psi, -\Omega^2 a\cos\psi) + 0(\varepsilon^2) \tag{25}$$

Finally, by substituting (18), (24) and (25) into (17), we obtain for n=0:

$$\begin{aligned}u_1(t) = u_0(t) + \frac{\varepsilon}{\Omega}\int_0^t \sin\Omega(\tau - t)[v''(\tau) + \Omega^2 v(\tau) - 2\Omega A\sin(\Omega\tau+\theta) - 2\Omega aB\cos(\Omega\tau+\theta) -\\ - f(\Omega\tau, a\cos(\Omega\tau+\theta), -\Omega a\sin(\Omega\tau+\theta), -\Omega^2 a\cos(\Omega\tau+\theta)]d\tau + 0(\varepsilon^2)\end{aligned} \tag{26}$$

in what follows, we should propose that $u_1(t) = u_0(t)$ in Eq. (26). In this case it is clear that the sequence $u_n(t)$ is constant and therefore, the solution of Eq. (13) can be obtained by only two iterations step due to the fact that the Lagrange multiplier can be exactly identified. The coefficient of ε/Ω in Eq.(26), would have to vanish and can be written as follows:

$$\begin{aligned}&\int_0^t \sin\Omega(\tau - t)[v''(\tau) + \Omega^2 v(\tau) - 2\Omega A\sin(\Omega\tau+\theta) - 2\Omega aB\cos(\Omega\tau+\theta) -\\ &- f(\Omega\tau, a\cos(\Omega\tau+\theta), -a\Omega\sin(\Omega\tau+\theta), -a\Omega^2\cos(\Omega\tau+\theta)]d\tau = v'(0)\sin\Omega t +\\ &+ \Omega v(0)\cos\Omega t - \Omega v(t) - \Omega A\left[t\cos(\Omega t+\theta) - \frac{\sin(\Omega t+\theta) + \sin(\Omega t - \theta)}{2\Omega}\right] +\\ &+ a\Omega B\left[\frac{\cos(\Omega t - \theta) - \cos(\Omega t+\theta)}{2\Omega} - t\sin(\Omega t+\theta)\right] -\\ &- \int_0^t \sin\Omega(\tau - t) f(\Omega\tau, a\cos(\Omega t+\theta), -a\Omega\sin(\Omega t+\theta), -a^2\Omega^2\cos(\Omega t+\theta))d\tau = 0\end{aligned} \tag{27}$$

In order to ensure no secular terms appear in (27), resonance must be avoided and thus, the coefficients of $t\sin(\Omega t+\theta)$ and $t\cos(\Omega t+\theta)$ must vanish. The functions A and B will be determined from these conditions and the function $v(t)$ will be determined from Eq.(27).

In our problem, we have:

$$f(\Omega t,u,\dot{u},\ddot{u})=\sigma u-\beta(u,\dot{u}^2+u^2\ddot{u})-\alpha u^3-2\mu\dot{u}+2\eta a_b\cos\Omega t \tag{28}$$

Substituting (18), (23) and (24) into (28), after several manipulations, we obtain:

$$\begin{aligned}&f(\Omega t,a\cos(\Omega t+\theta),-a\Omega\sin(\Omega t+\theta),-a\Omega^2\cos(\Omega t+\theta))=\\&=(\sigma a+\frac{1}{2}\beta a^3\Omega^2-\frac{3}{4}\alpha a^3)\cos(\Omega t+\theta)+2\mu a\Omega\sin(\Omega t+\theta)+\\&+(\frac{1}{2}\beta a^3\Omega^2-\frac{1}{4}\alpha a^3)\cos(3\Omega t+3\theta)+2\eta a_b\cos\Omega t\end{aligned} \tag{29}$$

Now, substituting (29) into (27), we obtain:

$$\begin{aligned}&v'(0)\sin\Omega t+\Omega v(0)\cos\Omega t-\Omega v(t)+\\&+t\sin\Omega t[\Omega(A+\mu a)\sin\theta+(\frac{1}{2}\sigma a-a\Omega B+\frac{1}{4}\beta a^3\Omega^2-\frac{3}{8}\alpha a^3)\cos\theta+\eta a_b]+\\&+t\cos\Omega t[(\frac{1}{2}\sigma a-a\Omega B+\frac{1}{4}\beta a^3\Omega^2-\frac{3}{8}\alpha a^3)\sin\theta-\Omega(A+\mu a)\cos\theta]+\\&+\frac{\sin(\Omega t+\theta)+\sin(\Omega t-\theta)}{2}(A+\mu a)+\\&+\frac{\cos(\Omega t-\theta)-\cos(\Omega t+\theta)}{2\Omega}(aB\Omega+\frac{1}{2}\sigma a+\frac{1}{4}\beta a^3\Omega^2-\frac{3}{8}\alpha a^3)-\\&-\frac{(2\beta\Omega^2-\alpha)a^3}{32\Omega}[\cos(3\Omega t+3\theta)+\cos(\Omega t-3\theta)-2\cos(\Omega t+3\theta)]=0\end{aligned} \tag{30}$$

The functions A and B are determined from the equations:

$$\Omega(A+\mu a)+\eta a_b\sin\theta=0 \tag{31}$$

$$\frac{1}{2}\sigma a-aB\Omega+\frac{1}{4}\beta a^3\Omega^2-\frac{3}{8}\alpha a^3+\eta a_b\cos\theta=0 \tag{32}$$

We obtain

$$A=-\mu a-\frac{\eta}{\Omega}a_b\sin\theta \tag{33}$$

$$aB=\frac{1}{2}\frac{\sigma a}{\Omega}+\frac{1}{4}\beta a^3\Omega-\frac{3}{8}\frac{\alpha a^3}{\Omega}+\frac{\eta a_b}{\Omega}\cos\theta \tag{34}$$

The parameters $a(t)$ and $\theta(t)$ are to be calculated by solving equations (9) and thus:

$$\dot{a}=-\varepsilon\mu a-\varepsilon\frac{\eta}{\Omega}a_b\sin\theta \tag{35}$$

$$a\dot{\theta} = \varepsilon\frac{\sigma a}{2\Omega} + \frac{1}{4}\varepsilon\beta a^3\Omega - \frac{3}{8}\frac{\varepsilon\alpha a^3}{\Omega} + \frac{\varepsilon\eta a_b}{\Omega}\cos\theta \tag{36}$$

Taking into account that ε is a small parameter and by applying averaging method, Eq. (35) and (36) can be written in the form:

$$\dot{a} = -\varepsilon\mu a \tag{37}$$

$$\dot{\theta} = \varepsilon\left(\frac{1}{2}\frac{\sigma}{\Omega} + \frac{1}{4}\beta a^2\Omega - \frac{3}{8}\frac{\alpha a^2}{\Omega}\right) \tag{38}$$

with the solutions:

$$a = a_0\exp(-\varepsilon\mu t) \tag{39}$$

$$\theta = \theta_0 + \frac{\sigma}{2\Omega}\varepsilon t + \frac{(2\beta\Omega^2 - 3\alpha)}{16\mu\Omega}a_0^2\left[-\exp(-2\varepsilon\mu t) + 1\right] \tag{40}$$

where a_0 and θ_0 are constants.

From (30), (39) and (40), we obtain:

$$v(t) = \frac{(\alpha - 2\beta\Omega^2)}{32\Omega^2}a_0^3\cos\left\{3\Omega t + 3\theta_0 + \frac{3\sigma}{2\Omega}\varepsilon t - \frac{3(3\alpha - 2\beta\Omega^2)}{16\mu\Omega}a_0^2\left[1 - \exp(-2\varepsilon\mu t)\right]\right\} \tag{41}$$

From (18) we obtain the approximation of the second order of Eq.(13):

$$u_0(t) = a_0\exp(-\varepsilon\mu t)\cos\left\{\Omega t + \theta_0 + \frac{\sigma}{2\Omega}\varepsilon t - \frac{3\alpha - 2\beta\Omega^2}{16\mu\Omega}a_0^2\left[1 - \exp(-2\varepsilon\mu t\right]\right\} + \varepsilon v(t) \tag{42}$$

On the other hand, fixed points of Eq. (35) and (36) are given by:

$$\eta a_b\sin\theta = -\mu a\Omega \tag{43}$$

$$\eta a_b\cos\theta = -\frac{1}{2}\sigma a - \frac{1}{4}\beta a^3\Omega^2 + \frac{3}{8}\alpha a^3 \tag{44}$$

Squaring and adding Eq. (43) and (44) gives:

$$(3\alpha - 2\beta\Omega^2)^2 a^6 - 8\sigma(3\alpha - 2\beta\Omega^2)a^4 + 16(\sigma^2 + 4\mu^2\Omega^2)a^2 - 64\eta^2 a_b^2 = 0 \tag{45}$$

Collecting data first for response amplitude a as a function of input amplitude a_b with driving frequency Ω held constant and then for a as a function of Ω with a_b held constant, allows a least-squares curve fit to be used to determine the coefficients μ, σ, α, β and η by comparing experimental force - and frequency - response curves with those predicted by Eq. (45).

The phase plane of this problem is plotted in Fig. 1 for the equation (11) and in Fig. 2 for the equation (42) for the following parameters: $\Omega = 26$, $\mu = 0.3$, $\eta = 0.5$, $\sigma = 2$, $\alpha = 0.4$, $\beta = 0.2$, $\varepsilon = 1/20$, $a_b = 1$, $a_0 = 0.1$, $\theta_0 = 0$.

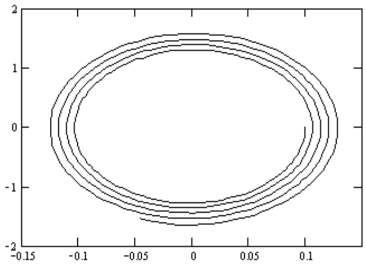

Fig. 1. Phase plane for eq. 11

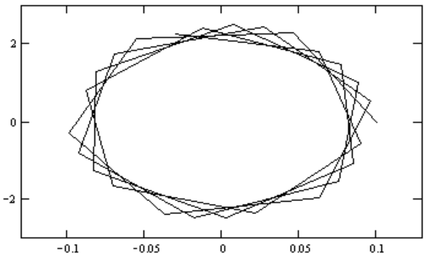

Fig. 2. Phase plane for eq. 42

REFERENCES

1. Krauss, W.R, Nayfeh, A.N, (1999) *Comparison of experimental identification techniques for a nonlinear SDOF system*, In Proceedings of the 17-th International Modal Analysis Conference, IMAC, pp.1182-1187
2. Marinca, V., (2002) *An Approximate Solution of the Equations Governed by One-dimensional Weakly Nonlinear Oscillations* (to appear)
3. He, J.H., (1999) *Variational iteration method - a kind of non-linear analytical technique: some examples*, Int. J. of Non-Linear Mechanics, 34, pp.699-70
4. Mitropolsky, Yu.A, (1965) *Problems of the Asymetric Theory of Non-stationary Vibration*, D.Davey N.Y.
5. Marinca V., (2002) *Application of Variational Iteration Method in Weakly Nonlinear Oscillation*, Bul. St. Univ. "Politehnica" Timişoara, Tom 47(61), pp.271-278

O ODZIVU DRUGOG MODA KONZOLNE GREDE

V. Marinca, B. Marinca, N. Herisanu

U ovom radu proučen je doprinos drugog moda pomeranju konzolne grede u odnosu na osnovnmo harmonijsko oscilovanje. Jednačina kretanja je slabo nelinearna i rešena prilogom varijacionoj i metodi usrednjenja. Rezultati ovde dobijeni bi mogli biti upotrebljeni za eksperimentalne nelinearne identifikacione tehnike.