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CONSERVATION LAWS IN THE DYNAMICS OF COSSERA CURVES - APPLICATION TO THE FRACTURE MECHANICS

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Abstract. *The recent theory of elastic directed curves is used to derive explicit expression for the set of the conservation law. Using Euclidean group of transformation, the equivalence between conservation law and Euclidean invariance is demonstrated.*

Key words: *Conservation laws, dynamics, Euclidean group of transformation*

1. INTRODUCTION

The definition of a rod as a curve with a triad of director leads to a complete description of the strain in a rod, as shown by Ericksen and Truesdell [1]. They presented a modern generalised version of the work of E. and F. Cosserat [2] and developed non-linear theories of strain for directed curves and surfaces, i.e. curves and surfaces at each point of which a triad of non-coplanar director vectors is defined.

The general theory of [3] can be considered as a generalisation of the statical non-linear theory of elastic directed curves presented by Cohen [4] who obtained the equilibrium equations, boundary conditions and constitutive relations by postulating principles of virtual work and of material frame indifference.

A non-linear dynamical theory of elastic directed curves was developed by Whitman and DeSilva [3], which can be regarded as an extension of Cohen's results. He postulated a Hamilton's principle, conservations of mass and invariance of the action density function under rigid body variations. These three postulates yield a complete dynamical theory of directed curves, including equations of motion, boundary conditions, and constutive equations. We then consider alternate forms of the general equations, namely the tensor and so-called anhonolomic forms. Finally, the above theory is reduced to the case of a Cosserat curve, i.e. a curve with three rigid directors.

In this paper we formulate Noether's theorem for the dynamical, non-linear theory of elastic directed curves and the appropriate general conservation law. Then, using Euclidean group of transformations, the equivalence between conservation law and

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Euclidean invariance is demonstrated. Finally, it was shown that one of obtained integrals (analogon Eshelby's energy-momentum tensor) has a physical meaning of the energy release rate.

2. NOETHER'S THEOREM

Let $\xi = \xi(\xi^{\alpha}) \subset R_{\alpha}$, $\alpha = 0,1,...,n$, be the independent and $F = F(F_i) \subset R_i$, $i = 1,2,...,m$, dependent vector variables, which describe the behaviour of material system under consideration, and L is the action density, defined and twice continuously differentiable in R_i.

To develop a dynamical theory of directed curves, we modify the action principle given by Toupin [6] for three-dimensional elastic directed media. We define the action A to be the functional

$$A = \int_t \int_c \rho L(s,t) ds\, dt = \int_R L(F) d\xi \,, \quad F = (x,d) \tag{2.1}$$

We can define a special form of Noether's theorem, which is used here to derive the conservation laws [5].

Theorem. Suppose that F satisfies Euler-Lagrange equations, then L is infinitesimally invariant of F with respect to the family of transformations if, and only if,

$$\frac{\partial}{\partial \xi^{\alpha}} (\{L_{F,\alpha}, m\} + L\alpha^{\alpha}) + \{m, Q\} = 0 \tag{2.2}$$

where the abbreviated notation suggested by Ericksen:

$$\{X, Y\} = \{(a_1, a_2, a_3), (b_1, b_2, b_3)\} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

is used.

The proof of this theorem can be found in [5].

3. A DYNAMICAL THEORY OF ELASTIC DIRECTED CURVES

The starting point of the study is the theory of elastic directed curves by A. B. Whitman and C. N. DeSilva [3].

A directed curve is defined as a curve imbedded in a Euclidean three-space with a non-coplanar triad of deformable vectors called directors associated with each point of the curve. Any deformed configuration c of a directed curve at some time t is described by the functions:

$$\mathbf{r} = \mathbf{r}(s,t)\,; \quad \mathbf{d}_{\alpha} = \mathbf{d}_{\alpha}(s,t)\,; \quad \alpha = 1,2,3 \tag{3.1}$$

where $\mathbf{r}$ locates a point on c with respect to a fixed curvilinear coordinate system x^i with metric $\mathbf{g}$, $\mathbf{d}_{\alpha}$ are the deformed directors, and s is the arc length along c.

We assume the existence of an undeformed reference configuration C at $t = 0$ such that

$$\mathbf{R}(S) = \mathbf{r}\big|_{t=0}\,;\ \mathbf{D}_{\alpha}(S) = \mathbf{d}_{\alpha}\big|_{t=0} \tag{3.2}$$

where **R** locates a point on C, $\mathbf{D}_\alpha$ are the undeformed directors, and S is the arc length along C. In order to complete the description of a continuous deformation of C into c, we must specify, in addition to (2.1) and (2.2), the mapping:

$$s = s(S,t) \text{ and its inverse } S = S(s,t). \tag{3.3}$$

We define the velocity $\boldsymbol{v}$ of points on the curve and the director velocities $\boldsymbol{w}_\alpha$ by

$$v = \dot{r} = \dot{x}^i g_i\,; \quad w_\alpha = \dot{d}_\alpha = \dot{d}^i g_\alpha$$

To indicate differentiation of a tensor quantity $\boldsymbol{F}$ with respect to arc lengths s and S, we adopt the notations

$$\hat{F}(s,t) \equiv \frac{\partial F}{\partial S}; \quad \hat{F}(S,t) \equiv \frac{\partial F}{\partial S}$$

Various component forms of these arc derivations can be defined. For example of vector and a second order tensor with components F_i and F_{ij} respectively.

The basic field equations for directed curves are [3]

$$\tau_i' + \rho f_i = \rho \dot{v}_i, \quad \mu_i'^{\alpha} - \Phi_i^\alpha + \rho h_i^\alpha = \rho \dot{\omega}_i^\alpha \tag{3.4}$$

where ρ - mass density, τ_i' - stress vector ; $\dot{\mu}_i{}^{\alpha}$ - corresponding to directed deformations; v - velocity of points on the curve - director velocities.

We now assume that the action density can be written as the difference between strain and kinetic energy parts

$$L = \varepsilon - K = L(\xi, F, \nabla F) \tag{3.5}$$

where ε is the strain energy density, and K is the kinetic energy density which we take in the form

$$K = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} g_{ij} A^{\alpha\beta} \dot{d}_\alpha^i \dot{d}_\beta^j \tag{3.6}$$

such that

$$p\varepsilon = W, \quad \tau_i = \rho\lambda \frac{\partial W}{\partial x} i, \quad \mu_i^\alpha = \rho\lambda \frac{\partial W_i}{\partial d_\alpha}, \quad \Phi_i^\alpha = \frac{\partial W_i}{\partial d_\alpha} \tag{3.7}$$

The function ρ_o is the mass density of the reference configuration C, and the "stretch" $\lambda = ds/dS$ is analogous to the Jacobian in 3-D continuum mechanics.

It may by verified that

$$\begin{aligned} &\tau = L, x'; \quad L, \dot{x}_i = -\rho v_i = -P_i \\ &\mu_i^\alpha = \rho\lambda W,_{d_\alpha'^i}, \quad \Phi_i^\alpha = \rho W,_{d_\alpha^i}, \quad L,_{\dot{d}_\alpha^i} = -\rho A^{\alpha\beta} \omega\beta_i = P_i^\alpha \end{aligned} \tag{3.8}$$

and the laws (3.2) are equivalent to the Euler-Lagrange equations.

Then, from relations (2.2), together with (3.8) and analogous divergence theorem, it follows that

$$\frac{d}{dt} \int (\{P^0, m\} + L\alpha_0) ds + \int (\{T, m\} + L\alpha),_s ds + \int \{Q, m\} ds = 0 \tag{3.9}$$

where

$$P = L,_{\dot{x}}, \quad P^{\alpha} = L,^{\dot{d}}, \quad P = [P^0, P^{\alpha}], \quad T = [\tau, \mu_{\alpha}], \quad Q = \rho[f, h^{\alpha}] \tag{3.91}$$
$$\mathbf{m} = \mathbf{p} - \nabla F \, \alpha$$

4. INVARIANCE AND CONSERVATION

The equivalence of Euclidean invariance of the action density and certain conservative laws is demonstrated in this section. Following Toupin [6], we postulate the invariance of the action density function as

$$L(\xi, F, \nabla F) = L(\xi^*, F^*, \nabla F^*) \tag{4.1}$$

under the class of Euclid displacements transformation:

$$\begin{aligned} s^* &= s + C\eta, \quad x^* = x + (\Omega x + D)\eta \\ t^* &= t + C\eta, \quad d^*_{\alpha} = d_{\alpha} + \Omega d_{\alpha}\eta \end{aligned} \tag{4.2}$$

where Ω is an arbitrary constant, orthogonal, antisymmetric matrix and C, D and C_0 are arbitrary constants.

Conservation laws. We postulate that the numbers of independent constants in transformation (4.2) determine the number of independent conservation laws.

The case when,

$D \neq 0$

$$\int_C \rho \cdot f ds + \tau_i \Big|_{s1}^{s2} = \frac{d}{dt} \int_C P ds \tag{4.3}$$

$\Omega \neq 0$

$$\int_C \rho \Omega (xf + d_{\alpha} h^{\alpha}) ds + \Omega (x\tau + d_{\alpha} \mu^{\alpha}) \Big|_{s1}^{s2} = \frac{d}{dt} \int_C S ds \tag{4.4}$$

$C_0 \neq 0$, $S = \Omega(P^0 \dot{x} + P^{\alpha} \dot{d}_{\alpha})$

$$\frac{d}{dt} \int_C E ds = \int_C \rho (f\dot{x} + h^{\alpha} \dot{d}_{\alpha}) \, ds + (\tau \dot{x} + \mu^{\alpha} \dot{d}_{\alpha}) \Big|_{s1}^{s2} \Big| \tag{4.5}$$

where we have defined an energy function E by

$$E = P\dot{x} + P^{\alpha} d_{\alpha} - \rho \cdot L \,. \tag{4.6}$$

Equations (4.3)-(4.5) express the balance laws of linear momentum, director momentum and energy. Equation (4.5) demonstrates that the rate of change of energy of any deforming segment c of the directed curve is balanced by the rate of working of the body forces f, h^{α} along c and the rate of working of the τ, μ^{α} at the ends of c.

The reduced forms of equations (4.3) and (4.4) for the static theory of directed curves were recorded by Cohen [4].

Now we consider the case when $C \neq 0$. Under these conditions we obtain the J-integral type of relation

$$\frac{d}{dt}\int_C \{P, \nabla F\} ds - \int_C (L - \{T, \nabla F\}), s \cdot ds + \int_C \{Q, \nabla F\} ds = 0 \tag{4.7}$$

Discusion

Clasical elastostatics. In this case the **x** and d_α are independent of time, and action density L is not dependent of the **x** and d_α ; thus **P** and d_α $\mathbf{P}_\alpha$ vanish. The reduced forms of equations (4.3) and (4.4) for the statics theory of directed curves were recorded by Cohen [4].

The expression (4.7) reduces to

$$B = (W - \tau\nabla x - \mu_\alpha \nabla d^\alpha)\Big|_{s1}^{s2} = (W - \{T, \nabla F\})\Big|_{s1}^{s2} \tag{4.8}$$

which is analogous to the well known to Eshelby's energy-momentum tensor [7].

5. The relation between J integral and G

Denoting J to be conservation integral Eq. (4.7), takes the form

$$J = B = (W - \tau\nabla x - \mu_\alpha \nabla d^\alpha)\Big|_{s1}^{s2} + \frac{d}{dt}\int_C \{P^0, \nabla F\}\, ds \tag{5.1}$$

This result can also be obtained by differentiating the relation for the total energy with respect to *s*.

In this case the rate of change of energy at any deforming segment *C* at the director curve is:

$$G = \frac{dE^*}{ds} \tag{5.2}$$

On the other hand, the total energy is

$$\begin{aligned} E^*(a) &= \int_S W ds - (\tau\dot{x} + \mu_\alpha d^\alpha)\Big|_{s1}^{s2} + \frac{d}{dt}\int_C (P^0, \nabla F)\, ds = \\ &= \int_C W ds - (T, \nabla F)\Big|_{s1}^{s2} - \int_C [\dot{P}^0, \nabla F]\, ds \end{aligned} \tag{5.3}$$

Differiating (5.3) with respect to s, and applied of the analogue divergence theorem becomes

$$\frac{-dE^*}{ds} = \int_C \frac{\partial W}{\partial s} ds - \{T, \nabla F\}\Big|_{s1}^{s2} + \frac{d}{dt}\int_C \{P^0, \nabla F\}\, ds$$

respectively,

$$\frac{-dE^*}{ds} = W - \{T, \nabla F\}\Big|_{s1}^{s2} + \frac{d}{dt}\int_C \{P^0, \nabla F\}\, ds \tag{5.4}$$

From (5.2) and (5.5) we can conclude, that the value of J integral is identical equal to the rate of change at energy of any deformating segment c of the directed curve, i.e.

$$J(s) = G(a)$$

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ZAKONI ODRŽANJA U DINAMICI COSSERA KRIVIH - PRIMENA KOD MEHANIKE LOMA

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U ovom radu je formulisana teorema E. Noether za teoriju polja i dat opšti oblik zakona konzervacije za isto – elastične direktorske krive. Potom koristeći Eduklidovu grupu transformacija pokazuje se ekvivalentnost između zakona održanja i istih. Na kraju se jednom od dobijenih zakona održanja (Eshelby – Energy moment tensor) može fizički interpretirati kao brzina oslobođene energije duž lučne koordinate s.

Ključne reči: *zakoni konzervacije, dinamika, Euklidova grupa transformacija.*