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NEW CLASS OF SOLUTIONS OF THE REDUCED WAVE EQUATION APPLICABLE TO CRACK PROBLEMS RELATED TO GRADIENT ELASTICITY

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Abstract. *Using complex variables, the two-dimensional reduced wave equation is transformed into an equation composed of two Hankel operators. Due to the symmetry of these operators, solutions of the transformed equation appear as products of Bessel functions, where each individual function's argument is associated with one of the newly defined independent variables. When a particular class of these solutions is identified with linear elastic displacement for finite length cracks, stresses are generated at the tips possessing the characteristic inverse square root singularity. As an application of this special class of solutions, an internal crack problem subject to the constitutive assumptions of the Aifantis strain gradient elastic theory is posed and solved. In this particular gradient elasticity theory, the analogous linear elastic displacement and stress is incorporated into the solution of the corresponding gradient elasticity problem.*

1. INTRODUCTION

In general, solutions of the reduced wave equation are very important for analyzing vibration problems in elastic media. In particular, they appear in dynamic studies related to the propagation of cracks in linear elastic material subject to harmonic excitation [1-3]. Besides dynamic crack problems, the reduced wave equation is also found as a governing equation in equilibrium crack problems [4-13] that are modeled using a particular strain gradient elasticity theory proposed and developed by Aifantis and co-workers [14-20]. Unlike its linear elastic counterpart, this gradient elasticity theory includes second order spatial derivatives of displacement in the constitutive equations relating stress to strain. This theory holds promise for modeling both non-local material behavior and material damage. A description of the role that gradient elasticity can play in the context of damage mechanics is provided in [21].

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Historically solutions of the reduced wave equation have presented great mathematical difficulties for crack problems. This is especially true for finite length cracks as the geometry of the crack itself introduces an inherent gage length [1], which further complicates solution over the semi-infinite crack. Typically elliptical coordinates are introduced for finite length crack problems, which allow separation of variables and a solution of a particular boundary value problem in terms of an infinite series of Mathieu functions [1-3]. However, textbooks on fracture mechanics rarely broach this subject because of the complexity of Mathieu functions [22-24] and lack of familiarity with them among engineers. In contrast, the solutions obtained here are represented in the form of Bessel functions, which are more familiar to engineers than Mathieu functions, and are more readily available in commercial software packages for evaluation.

The solution of the reduced wave equation, as derived here, takes the form of an infinite series of products of Bessel functions. To the best of the authors' knowledge, this particular class of solutions has not been explored previously, although it is likely to be recoverable from other solutions, as Mathieu functions themselves are expressible in terms of an infinite series of Bessel functions [22].

Attention is confined in this paper to the tearing mode of fracture, which is also known variously as the antiplane shear crack problem or the mode III crack problem. Mode III is chosen for this particular analysis because it is the simplest to describe mathematically of the three principal modes of fracture. However, the class of solutions derived here for the mode III problem can be readily extended to the other modes of fracture as well.

For the mode III crack problem, the reduced wave equation for the out-of-plane displacement $w(x,y)$ assumes the form

$$w + k\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0\,, \tag{1}$$

where x, y represent Cartesian coordinates, and the constant $k \in \mathsf{R}$, $k > 0$. The mathematical properties of equation (1) change dramatically with the sign of k. In a mechanical sense, $k < 0$ corresponds to a system with a negative stiffness. For $k > 0$, solutions assume forms that are typical of eigenvalue problems, including multiplicity of solutions for homogeneous boundary conditions. A detailed discussion of the effect of the sign of k on solutions and boundary value problems related to (1) is given in [25].

For forced vibration problems [3], (1) is generated by substituting a displacement of the form $w^*(x,y,t) = w(x,y)\exp i\omega t$ into the wave equation $\partial w^*_{tt} = (\rho/G)\nabla^2 w^*$, where t is time, ρ is the density of the material, G is the shear modulus, and ω is the circular frequency of the forcing function. In this particular case, (1) represents a combined statement of Hooke's law and Newton's second law for a continuum, where $k = \rho / G\omega$.

Alternatively, for the Aifantis gradient elasticity theory, an inhomogeneous form of (1) is determined from an integration of the fourth-order partial differential equation $\nabla^2 w + k\nabla^4 w = 0$, which results from combining the non-trivial equilibrium equation for the antiplane crack problem with constitutive equations that relate shear strain and its second-order spatial derivatives to the shear stress. The constant k in this case may be interpreted as a material parameter, the gradient elastic shear modulus.

The inhomogeneous form of (1), in the context of gradient elasticity theory, has the left hand side of (1) equated to a harmonic function w_0 . Moreover, this function w_0 is the particular solution of this inhomogeneous partial differential equation, and it corresponds physically to the displacement of the analogous traction boundary value problem as formulated for linear elasticity [5]. The complete solution for gradient elasticity problem is then obtained by superposing homogeneous solutions of the reduced wave equation to the particular solution w_0 , subject to certain additional boundary conditions beyond those required by linear elasticity.

In this regard it should be mentioned that there are several different types of displacement solutions involving crack problems for this particular gradient elasticity theory. Some solutions [4,5,10-13] were obtained under the assumption that the second derivative of displacement normal to the crack surface is zero, i.e., $\partial w_{nn} = 0$. For the case $k < 0$, this additional boundary condition ensures that the solution is unique and that the strain remains finite at the crack tip, unlike the corresponding strains of linear elasticity theory [20]. Mathematically these solutions are quite complicated and closed form solutions have been obtained only along the crack line. For $k > 0$, uniqueness of solution is lost; however, simple solutions which neglect the additional boundary condition along the crack surface are derivable in the neighborhood of the crack tip, while retaining the desirable feature that the strain remains finite at the tip [6,8]. These displacements are oscillatory rather than monotonic as in the previous case reflecting a significant change in response. However, when analogous asymptotic solutions, which neglect the additional boundary condition, are analyzed for $k < 0$, the crack surfaces open in a monotonic fashion, but in a direction opposite to the applied load [6,8]. Consequently, these solutions must be rejected on a physical basis despite the fact the strain remains bounded at the crack tip. This and more recent studies indicate that gradient elasticity problems where $k < 0$ and the additional boundary condition $\partial w_{nn} = 0$ is satisfied are generally more suitable for material modeling under static conditions.

However, gradient elasticity problems where $k > 0$ should not be dismissed altogether, as there are exceptions to the rule. For example, an equation analogous to (1) with $k > 0$ is derivable from a discrete model for application to granular materials with random packing structure [26]. Models of this type with $k > 0$ also provide insight into the behavior of dynamic gradient elasticity problems prior to the onset of instability using the dispersion of waves as a criterion [27-29]. Thus the interests of modern materials scientists / engineers coupled with innate importance of dynamic solutions of linear elastic crack problems provide more than sufficient justification for studying this new class of solutions in more detail.

2. Analysis

Defining the complex variables z and its complex conjugate $\bar{z}$ in terms of the real variables (x,y), i.e., $z = x + iy$, $\bar{z} = x - iy$, one obtains the following alternative form of (1) under this transformation of independent variables

$$w + 4k \frac{\partial^2 w}{\partial z \partial \bar{z}} = 0 . \tag{2}$$

Now let two complex functions $\alpha(z,\bar{z})$ and $\beta(z,\bar{z})$ be defined as follows

$$\alpha = \frac{1}{2}\sqrt{\frac{\bar{z}}{k}}[\sqrt{z+a} - \sqrt{z-a}] \tag{3}$$

$$\beta = \frac{1}{2}\sqrt{\frac{\bar{z}}{k}}[\sqrt{z+a} + \sqrt{z-a}]\,, \tag{4}$$

where a represents one half the length of an internal crack lying along the x-axis ($-a \le x \le a$) in an infinite plane, where the origin of the Cartesian coordinates is located at the crack center. Now, using a chain rule, the partial derivative of w with respect to z is described in terms of the complex functions (α,β) as follows

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial \alpha}\frac{\partial \alpha}{\partial z} + \frac{\partial w}{\partial \beta}\frac{\partial \beta}{\partial z}\,. \tag{5}$$

Differentiating α and β with respect to z and substituting the results into (5), produces the expression

$$\frac{\partial w}{\partial z} = \frac{\alpha\beta}{a(\beta^2 - \alpha^2)}\left[\beta\frac{\partial w}{\partial \beta} - \alpha\frac{\partial w}{\partial \alpha}\right]. \tag{6}$$

Similarly, another chain rule involving $\bar{z}$ generates the relationship

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = \frac{\partial \alpha}{\partial \bar{z}}\frac{\partial}{\partial \alpha}\left(\frac{\partial w}{\partial z}\right) + \frac{\partial \beta}{\partial \bar{z}}\frac{\partial}{\partial \beta}\left(\frac{\partial w}{\partial z}\right), \tag{7}$$

which upon substitution of $\partial w/\partial z$ from (6) yields

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = \frac{1}{4k(\beta^2 - \alpha^2)}\left[\beta^2\frac{\partial^2 w}{\partial \beta^2} + \beta\frac{\partial w}{\partial \beta} - \alpha^2\frac{\partial^2 w}{\partial \alpha^2}\alpha\frac{\partial w}{\partial \alpha}\right]. \tag{8}$$

Using (8) to rewrite (2) in terms of (α,β), a partial differential equation composed of two Hankel operators is generated, i.e.,

$$\alpha^2\frac{\partial^2 w}{\partial \alpha^2} + \alpha\frac{\partial w}{\partial \alpha} + \alpha^2 w = \beta^2\frac{\partial^2 w}{\partial \beta^2} + \beta\frac{\partial w}{\partial \beta} + \beta^2 w\,. \tag{9}$$

Assuming solutions of (9) to be of the form

$$w = A(\alpha)B(\beta), \tag{10}$$

separating variables, and setting both sides of the equal sign equal to a constant n^2 yields

$$\frac{1}{A}\left[\alpha^2\frac{\partial^2 A}{\partial \alpha^2} + \alpha\frac{\partial A}{\partial \alpha} + \alpha^2 A\right] = \frac{1}{B}\left[\beta^2\frac{\partial^2 B}{\partial \beta^2} + \beta\frac{\partial B}{\partial \beta} + \beta^2 B\right] = n^2\,. \tag{11}$$

Rewriting (11), one obtains the following system of ordinary differential equations

$$\alpha^2 \frac{\partial^2 A}{\partial \alpha^2} + \alpha \frac{\partial A}{\partial \alpha} + (\alpha^2 - n^2) A = 0 , \tag{12}$$

$$\beta^2 \frac{\partial^2 B}{\partial \beta^2} + \beta \frac{\partial B}{\partial \beta} + (\beta^2 - n^2) B = 0 . \tag{13}$$

Both equations (12) and (13) have linear independent solutions comprised of Bessel functions of the first $J_n(\ )$ and second $Y_n(\ )$ kinds respectively, i.e.,

$$A = A_n(\alpha) = D_n J_n(\alpha) + E_n Y_n(\alpha) \tag{14}$$

$$B = B_n(\beta) = F_n J_n(\beta) + G_n Y_n(\beta), \tag{15}$$

where the subscript n indicates the order of the Bessel Function, and D_n , E_n , F_n , and G_n represent arbitrary constants.

One particular class of these solutions is particularly useful for crack problems, i.e.,

$$w_n = C_n J_n(\alpha) Y_n(\beta) , \tag{16}$$

where C_n is a constant. By differentiating (16) with respect to z one obtains

$$\frac{\partial w_n}{\partial z} = \frac{C_n}{2\sqrt{z^2 - a^2}} [\beta J_n(\alpha) Y_{n-1}(\beta) - \alpha J_{n-1}(\alpha) Y_n(\beta)] . \tag{17}$$

Now, as one approaches the two crack tips along the upper crack surface, i.e., as $x \to \pm a, y \to 0^+$, the independent variables (α,β) approach the following limits

$$\alpha\Big|_{x\to\pm a, y\to 0^+} = \frac{a}{\sqrt{2k}}, \quad \beta\Big|_{x\to\pm a, y\to 0^+} = \pm\frac{a}{\sqrt{2k}} . \tag{18}$$

Further, by employing the following identities [20, 21] for Bessel functions together with (17) and (18), i.e.,

$$J_n(z) Y_{n-1}(z) - J_{n-1}(z) Y_n(z) = \frac{2}{\pi z} , \ n \text{ integer} \tag{19}$$

$$J_n(e^{m\pi i} z) = e^{mn\pi i} J_n(z) , \ m \text{ integer} \tag{20}$$

$$Y_n(e^{m\pi i} z) = e^{-mn\pi i} Y_n(z) + 2i \sin(mn\pi) \cot(n\pi) J_n(z) , \tag{21}$$

one can determine that the right hand side of (17) approaches asymptotically the following relationship at the crack tips

$$\frac{\partial w_n}{\partial z}\Big|_{z\to\pm a} \sim \begin{cases} \dfrac{C_n}{\pi\sqrt{z^2 - a^2}}, & n \text{ even integer} \\ \dfrac{\pm C_n}{\pi\sqrt{z^2 - a^2}}, & n \text{ odd integer,} \end{cases} \tag{22}$$

where n is the integer order of the Bessel functions in (16). Equation (22) is similar in form to the denominator of the classic Westergaard stress function $Z_{III}(z)$ [30], i.e.,

$$Z_{III}(z) = \frac{\tau_\infty z}{\sqrt{z^2 - a^2}}\,, \tag{23}$$

where τ_∞ is the magnitude of the traction applied at infinity. The real and imaginary parts of Z_{III} are the antiplane shear stresses τ_y and τ_x respectively.

The denominator of (23) is the term responsible for the characteristic inverse square root singularities found at the crack tips, i.e., $z = \pm a$. As in the case of the Westergaard stress function, the function w_n of (16) is a complex function. This is due to the asymmetry of the variables $(z, \bar{z})$ in the definitions of (α,β). Correspondingly, the strains determined with the aid of (22) for gradient elasticity theory have the familiar inverse square root singularities at the crack tips.

It is readily verified that the complex conjugate $\overline{w}_n$ of the complex displacement w_n of (16) is found by interchanging the positions of z and $\bar{z}$ in that function, i.e., symbolically

$$w_n = w(z,\bar{z}) \rightarrow \overline{w}_n = w(\bar{z}, z)\,. \tag{24}$$

It follows immediately from (24) that the real and imaginary parts of w_n are respectively

$$\mathrm{Re}[w_n] = (w_n + \overline{w}_n)/2\,, \qquad \mathrm{Im}[w_n] = (w_n - \overline{w}_n)/2\,. \tag{25}$$

Because of the symmetry with respect to z and $\bar{z}$ in both (2) and (24), $\overline{w}_n$ will also satisfy the reduced wave equation. In terms of two new complex functions α^* and β^*, defined below,

$$\begin{aligned} \alpha &= \alpha(z,\bar{z}) \rightarrow \alpha^* = \alpha(\bar{z},z)\,, \\ \beta &= \beta(z,\bar{z}) \rightarrow \beta^* = \beta(\bar{z},z)\,, \end{aligned} \tag{26}$$

the complex conjugate of w_n is expressed as

$$\overline{w}_n = C_n J_n(\alpha^*) Y_n(\beta^*)\,, \tag{27}$$

assuming real C_n.

In Figure 1, the Im $[w_n]$ of (16) is plotted along $y = 0^+$ for several integer values of the Bessel function order n. Other parameters used in this plot, which are not shown on the figure, are $C_n = a = 1$. This figure clearly indicates that the boundary conditions of zero displacement are met at the crack tips, $x = \pm 1$, $y = 0^+$ for the values of n chosen. Also note that curves for even values of n are odd functions of x; whereas, the curves for odd values of n are even functions of x. All of the curves are continuous; however, the curve corresponding to $n = 1$ fails to be smooth at the w-axis. Solutions for negative integer values of n produce similar curves as their positive integer counterparts and are not marked on the figure. The curve corresponding to $n = 0$ has been omitted from the figure as it is discontinuous along the w-axis and proves unsuitable for the series expansion to follow.

Solutions of (16) for non-integer values of n are of little interest physically as they fail to satisfy the boundary condition of zero displacement at both crack tips. This is the consequence of identity (19), which fails to be satisfied for non-integer values of n.

The substitution of (16) and (27) into the second equation of (25) generates a form that readily allows the determination of real engineering shear strains from the partial derivatives of displacement, i.e.,

$$\gamma_x = \frac{\partial\, \mathrm{Im}[w_n]}{\partial x}\,, \quad \gamma_y = \frac{\partial\, \mathrm{Im}[w_n]}{\partial y}\,. \tag{28}$$

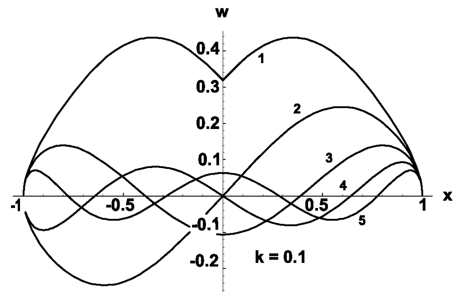

Fig. 1. Mode Shapes of Displacement for Solutions of the Reduced Wave Equation along Upper Crack Surface for Bessel Function Order n.

In Figure 1 all of the displacements exhibit vertical slopes at the crack tips. This illustrates graphically that strain singularities in γ_x exist at the crack tips.

Let us now examine the asymptotic behavior of the imaginary portion of w_n as $k \to \infty$, i.e., as $\alpha \to 0$, $\beta \to 0$, where for the time being the constant C_n will be set equal to one. Using the asymptotic formulas found in [31], we find upon substitution into (16) that

$$\operatorname{Im}[J_n(\alpha)Y_n(\beta)]\big|_{k\to\infty} = \operatorname{Im}\left[-\frac{1}{\pi}\frac{\Gamma(n)}{\Gamma(n+1)}\left(\frac{\alpha}{\beta}\right)^n\right],\quad n \neq 0\,, \tag{29}$$

where the ratio of gamma functions $\Gamma(n)$ / $\Gamma(n + 1)$ may be set equal to $1/n$ provided n is a positive integer [24].

Expressed in terms of elementary complex variables, this expansion (29) is equivalent to

$$\operatorname{Im}[J_n(\alpha)Y_n(\beta)]\big|_{k\to\infty} = \operatorname{Im}\left[-\frac{1}{n\pi}\left(\frac{z}{a}-\sqrt{\left(\frac{z}{a}\right)^2-1}\right)^n\right],\quad n \neq 0\,. \tag{30}$$

Note the asymptotic form of w_n in (30) is a function only of z. Therefore as k grows large, w_n behaves harmonically, as is the case for the imaginary part of any analytical function of z alone [30].

Toward removing the radical in (30), let us now introduce the elliptical coordinate system (ξ,η), i.e.,

$$z = a\cosh(\xi + i\eta) \tag{31}$$

In this coordinate system, ξ and η represent an orthogonal family of confocal ellipses and confocal hyperbolas. The transformation back to Cartesian coordinates from elliptical follows by equating real and imaginary parts of (31) from opposite sides of the equal sign, i.e.,

$$x = a\cosh\xi\cos\eta,\quad y = a\sinh\xi\sin\eta. \tag{32}$$

Upon substituting z from (31) into (30), one finds the asymptotic form of displacement reduces to

$$\text{Im}[J_n(\alpha)Y_n(\beta)]\Big|_{k\to\infty} = \frac{1}{n\pi}e^{-n\xi}\sin(n\eta), \quad n \neq 0 . \tag{33}$$

For n = 1, relationship (33) is similar to the linear elastic displacement found for a mode III crack in equilibrium with loads of uniform traction applied along the crack faces. Along the upper crack surface $y = 0^+$, the displacement (33) can be rewritten in Cartesian coordinates as

$$\text{Im}[J_n(\alpha)Y_n(\beta)]\Big|_{k\to\infty,\, y=0^+} = \frac{1}{n\pi}\sin\left(n \arccos\frac{x}{a}\right), \quad -a \le x \le a, \quad n \neq 0 . \tag{34}$$

Alternatively, the displacement in (34) can be expressed in terms of the Chebyshev polynomials [32] of the second kind $U_n(\,)$ as

$$\text{Im}[J_n(\alpha)Y_n(\beta)]\Big|_{k\to\infty,\, y=0^+} = \frac{1}{n\pi}\sqrt{1-(x/a)^2}\,U_{n-1}(x/a), \quad -a \le x \le a, \quad n \neq 0 . \tag{35}$$

Chebyshev polynomials are introduced because of their orthogonality properties, which allow series expansions of arbitrary functions.

The solution w_n can also be expressed in integral form as

$$J_n(\alpha)Y_n(\beta) = \frac{1}{2n\pi}\sqrt{\frac{\bar{z}}{k}}\int_{-a}^{a}\sqrt{\frac{1-(t/a)^2}{z-t}}\,U_{n-1}(t/a)\,Y_1\left(\sqrt{\bar{z}(z-t)/k}\right)dt, \tag{36}$$

which can be derived using the specific case below

$$J_n(\alpha)Y_n(\beta) = \frac{\alpha\beta}{n\pi}\int_0^{\pi}\frac{\sin^2\phi\, C_{n-1}^1(\cos\phi)}{\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos\phi}}\,Y_1\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos\phi}\right)d\phi, \tag{37}$$

of a more general integral found in [33], together with an identity [24] for the special Gegenbauer polynomial $C_n^1(\cos\phi)$, i.e.,

$$C_n^1(\cos\phi) = \frac{\sin(n+1)\phi}{\sin\phi} . \tag{38}$$

It is interesting to note that the harmonic function given in (35) as the asymptotic solution of displacement as k grows large is contained within the integrand of the exact solution (36) of the reduced wave equation. In [26] it is proved that every solution of the reduced wave equation is uniquely associated with a particular harmonic function provided the region to which it is applies is simply connected. This property proved useful in the derivation of equation (11) of [8] for a first-term expansion of the gradient elasticity solution. The consequence of retaining only the first term of the solution is that the domain changes effectively from a multiply connected region to a simply connected region. Although (36) applies to a multiply connected region, its form somewhat resembles equation (11) of [8].

In [32] the following expansion of an arbitrary function $f(x)$ is given in terms of Chebyshev polynomials of the second kind

$$f(x) = \sqrt{1-x^2}\sum_{n=1}^{\infty} p_n U_{n-1}(x) , \tag{39}$$

$$p_n = \frac{2}{\pi}\int_{-1}^{1} f(x) U_{n-1}(x) dx \,. \tag{40}$$

Together with (36), one may conclude using linear superposition that solution of the reduced wave equation for an internal crack of length $2a$ has the form

$$w = \sum_{n=1}^{\infty} n p_n J_n(\alpha) Y_n(\beta) = \frac{1}{2\pi}\sqrt{\frac{\bar{z}}{k}}\int_{-a}^{a}\frac{f(t/a)}{\sqrt{z-t}} Y_1\left(\sqrt{\bar{z}(z-t)/k}\right) dt \,. \tag{41}$$

By taking the imaginary part of (41), one obtains a real-valued displacement.

3. Gradient Elasticity Application

In the case of the Aifantis strain gradient elasticity theory, the governing equation for the displacement of the mode III crack problem assumes the form of an inhomogeneous reduced wave equation

$$w + k\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = w_0 \,, \tag{42}$$

where w_0 is the corresponding displacement of the analogous linear elastic crack problem. As w_0 itself is a harmonic function, the solution of (42) can be decomposed into the sum of two parts

$$w = w_0 + w_+ \,, \tag{43}$$

where w_+ is a solution of the homogeneous reduced wave equation (1) with w_0 satisfying Laplace's equation. As mentioned earlier in the introduction, the displacement from the Aifantis gradient elasticity theory requires additional boundary conditions over those of linear elasticity, as it is a solution of a fourth-order equation rather than a second-order equation. Provided $k < 0$ the adoption of the boundary condition $\partial w_{nn} = 0$ along the crack surface guarantees uniqueness of solution and generates a solution for which the strain remains bounded at the crack tip

For demonstration purposes, a simpler procedure will be adopted here as was followed previously in [6] and [8].

An appropriate solution for displacement w_+ is sought which when added to the solution for displacement w_0, eliminates the strain singularity. Traction is unaffected by w_+ terms in the Aifantis theory. In [6] and [8] the small-scale yielding or first term of the linear elastic solution was used. Here the analogous operation will be performed for a finite length crack using the exact linear elastic solution rather than the asymptotic solution. As has been mentioned earlier, the solution is not unique for $k > 0$ whether or not the additional boundary condition $\partial w_{nn} = 0$ is satisfied.

The form of complex displacement w_0 for the classical mode III crack problem and its derivative are

$$w_0 = \frac{\tau_\infty}{G}\sqrt{z^2 - a^2} \rightarrow \frac{dw_0}{dz} = \frac{\tau_\infty}{G}\frac{z}{\sqrt{z^2 - a^2}} \,, \tag{44}$$

where the physical or real-valued displacement is found from (44) by taking the Im $[w_0]$.

By comparing (22) and second equation of (44), it is concluded that for odd-valued integers n, the condition necessary to eliminate strain singularities at both cracks tips simultaneously is

$$C_n = -\tau_\infty \pi a/G\ . \tag{45}$$

Similarly, from (22) and (44) one can conclude that it is impossible to eliminate both singularities simultaneously for even values of n, using a single term of the gradient elasticity solution, as the partial derivatives of the functions ∂w_n with respect to z are even-valued functions of z while the derivative of w_0 with respect to z is an odd-valued function of z.

Therefore simple solutions of real displacements having finite-valued strains at the crack tips are of the form

$$w = \mathrm{Im}\left[\frac{\tau_\infty}{G}\left(\sqrt{z^2 - a^2} - \pi a J_n(\alpha) Y_n(\beta)\right)\right], \quad n \text{ odd integer.} \tag{46}$$

In Figure 2, three different displacement curves of (46) are plotted along the crack surface $y = 0^+$ for three different odd values of n and the parameters $a = 1$ and $\tau_\infty = G/\pi$. Note how the slopes of displacement at the crack tips have changed dramatically from Figure 1. On closer inspection of the crack tip regions (not evident from the scale used on Figure 2 except for case n = 3), one finds that the slope is zero, which indicates that the strain γ_x is zero there.

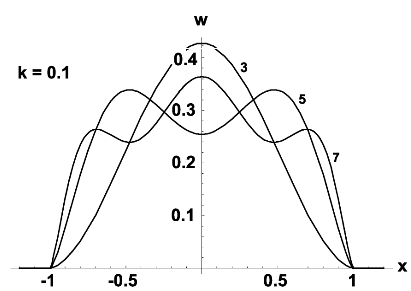

Fig. 2. Displacement along Upper Crack Surface for Variations in Bessel Function Order n for Gradient Elasticity Theory.

In Figure 3 the effect of variations in the parameter k on displacement (46) are shown for the case n = 3, a = 1, and $\tau_\infty = G/\pi$. Observe how the slopes of the all curves tend toward zero near the crack tips regardless of the value of k.

Displacement beyond the immediate crack region is shown in Figure 4 for the parameters $a = 1$ and $\tau_\infty = G/\pi$. Figure 4 shows the antisymmetric pattern of displacement relative to the x-axis, which is indicative of the mode III crack problem. The figure also shows how the crack surfaces meet to form cusps at the crack tips, a behavior reminiscent of the Barenblatt or Dugdale models [30], as has been pointed out previously in a gradient elasticity context [10-13]. Also note that far from the crack line, the displacement varies linearly in the y-w plane. This characteristic relates to the asymptotic behavior of the linear elastic component (44) of the total displacement (46). In contrast, the component of displacement from gradient elasticity decays as one recedes

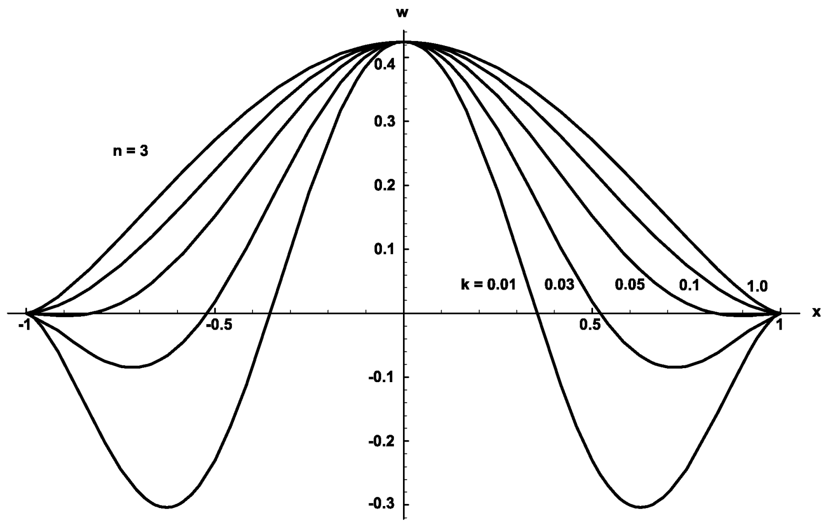

Fig. 3. Displacement along Upper Crack Surface for Variations in Parameter k under Gradient Elasticity Theory.

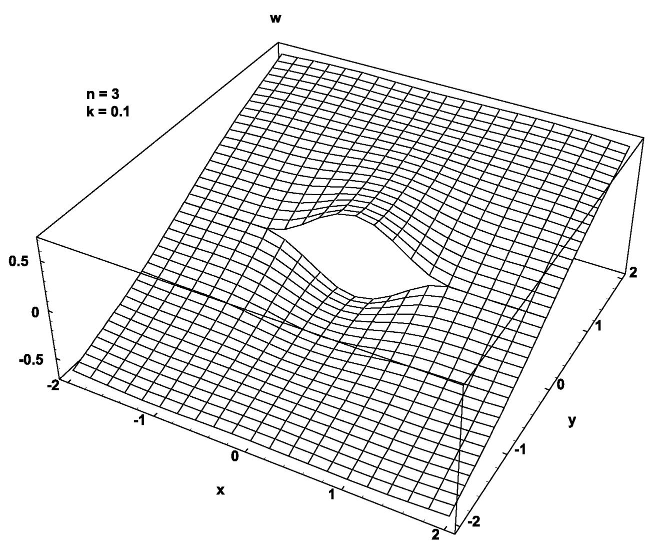

Fig. 4. Three-Dimensional View of Displacement for Gradient Elasticity Theory.

from the crack region. One can also see from Figure 4 that the value of γ_y will not be zero at the crack tips as in the case of γ_x , but will instead have a finite value corresponding to the slope of the surface in the y-direction at $x = \pm 1$, $y = 0$. This finite, but non-zero value at the crack tips, comes from the $\partial w/\partial \bar{z}$ contribution to strain.

Solution (46) represents a solution analogous to the asymptotic gradient elasticity solutions provided in [6] and [8]. Unlike the near crack tip or asymptotic solution, the entire solution demonstrates that for finite length cracks different mode shapes appear naturally. The entire solution also differs from the asymptotic in that γ_y is non-zero at the crack tip.

Boundary value problems for gradient elastic crack problems, which fulfill the extra boundary conditions $\partial w_{nn} = 0$ and where $k < 0$, have yet to be studied using solutions of the form (16). However, from previous analyses in coordinate systems other than (α,β), it is known that if both conditions are satisfied then the strain remains bounded at the crack tip under this gradient elasticity theory. It should also be pointed out that the strain remains finite under this theory without recourse to extraneous forces, as are introduced in the Barenblatt cohesive force model of the crack-tip process zone [30] and the Dugdale strip model [30] of crack-tip plasticity.

In the context of material modeling, the Barenblatt-Dugdale type approach has been used previously to simulate twinning [35], a special type of plastic deformation where mirror images of the crystal structure form along boundaries. Because of the similarities that exist between crack models using the Aifantis gradient elasticity theory and those of Barenblatt and Dugdale, it is anticipated that twinning can also be modeled using gradient elasticity theory.

4. Closing

The utility of general integral (41) and the associated series of Bessel functions in that equation have yet to be explored for dynamics problems in linear elasticity. For example, the integral in (41) can be inverted to yield $f(z/a)$ as a complex integral of w. This has been done for the analogous case of a simply connected region in [34] by solving a Volterra integral equation. Once this inversion is performed, the function $f(z/a)$ can be determined from the specific form that w assumes along the length of the crack or those of its first derivative along the length of the crack. These constitute either a Dirichlet or Neumann type boundary value problem. The Dirichlet problem for the reduced wave equation results from imposing a desired displacement profile representing a steady-state vibrations problem in the form of a standing wave along the crack surface. Similarly, the Neumann boundary value problem is associated with the imposition of the first derivative of displacement normal to the crack surface. In the latter case, Hooke's Law can be used to relate the derivative of displacement to stress and subsequently traction along the crack faces, which vary as a sinusoidal function of time.

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NOVA KLASA REŠENJA REDUKOVANE TALASNE JEDNAČINE PRIMENLJIVE NA PROBLEME PRSLINA U VEZI SA GRADIJENT ELASTIČNOŠĆU

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Korišćenjem kompleksnih promenljivih, dvodimenzionalna redukovana talasna jednačina transformise se u jednačinu koja se sastoji od dva Hankel operatora. Usled simetričnosti ovih operatora, rešenja transformisane jednačine javljaju se kao proizvodi Besel funkcija, gde je svaki pojedinačni argument funkcije vezan za jedan od novoodredjenih nezavisnih promenljivih. Kada se odredjena klasa ovih rešenja identifikuje sa linearnim elastičnim pomeranjem za prsline konačne dužine, stvaraju se naponi na vrhovima koji su karakteristični po obrnutoj vrednosti kvadratnog korena. Kao primena ove posebne klase rešenja, postavlja se i rešava problem unutrašnje prsline koji podleže suštinskim pretpostavkama Aifantis elastične teorije gradijenta napona. Unutar ove gradijentske teorije elastičnosti, analogno linearno elastično pomeranje i napon uračunati su u rešenje odgovarajućeg problema gradijenta elastičnosti.