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Invited Paper

LOCALLY CONFORMALLY KÄHLER MANIFOLDS OF CONSTANT TYPE AND J-INVARIAN CURVATURE TENSOR

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Abstract. *We find the metrics of all locally conformally Kähler manifolds of constant type and J-invariant curvature tensor and discuss the corresponding conformal invariants.*

Key words: *locally conformally Kähler manifold, almost Hermitian manifold of constant type, J-invarian curvature tensor, Bochner and generalized Bochner tensor, complex concircular curvature tensor.*

1. ALMOST HERMITIAN MANIFOLD OF CONSTANT TYPE AND J-INVARIANT CURVATURE TENSOR

Let (**M**,**g**,**J**) be an almost Hermitian manifold, where **g** is a Reimian metric and **J** is a complex structure. Than

$$dim\mathbf{M} = 2n, \quad \mathbf{J}^2 = -ident., \ \mathbf{g}(\mathbf{JX},\mathbf{JY}) = \mathbf{g}(\mathbf{X},\mathbf{Y}) \tag{1.1}$$

for all **X**,**Y**∈X(**M**), where X(**M**) is the Lie algebra of $\mathbf{C}^{\infty}$ vector fields on **M**. If **F**(**X**,**Y**) = **g**(**JX**,**Y**), then **F**(**X**,**Y**) = − **F**(**X**,**Y**).

The two-dimensional plane π of the tangent bundle $\mathbf{T}_p(\mathbf{M})$ at "$p \in \mathbf{M}$" is holomorphic if **J**π = π and antyholomorphic if **J**π ⊥ π.

The *Kähler* manifold is an almost Hermitian manifold satisfying the condition ∇**J** = **0** where ∇ denotes the operator of the covariant derivative with respect to the Levi-Civita connection Γ on (**M**,**g**). The condition ∇**J** = **0** implies

$$\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{JZ},\mathbf{JW}) = \mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})\,. \tag{1.2}$$

where **R**(**X**,**Y**,**Z**,**W**) is the Reimian curvature tensor of the metric **g**. The relation (1.2) is the *Kähler* identity.

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An almost Hermitian manifold is said to be of **J**-invariant curvature tensor if

$$\mathbf{R}(\mathbf{JX},\mathbf{JY},\mathbf{JZ},\mathbf{JW})=\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})\,. \tag{1.3}$$

Such manifolds, called also **RK**-manifolds, are investigated in [6].

To define the constant type of an almost Hermitian manifold, we first consider the tensor

$$\lambda(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})=\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})-\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{JZ},\mathbf{JW})\,.$$

So λ measures the defect from the *Kähler* identity. Now, we put

$$\lambda(\mathbf{X},\mathbf{Y})=\lambda(\mathbf{X},\mathbf{Y},\mathbf{Y},\mathbf{X})$$

and say that an almost Hermitian manifold (**M**,**g**,**J**) is of constant type at $p\in\mathbf{M}$ provided that for all $\mathbf{X}\in\mathbf{T}_p(\mathbf{M})$, we have

$$\lambda(\mathbf{X},\mathbf{Y})=\lambda(\mathbf{X},\mathbf{Z}) \tag{1.4}$$

whenever the planes $\{\mathbf{X},\mathbf{Y}\}$, $\{\mathbf{X},\mathbf{Z}\}$ are antyholomorphic and

$$\mathbf{g}(\mathbf{X},\mathbf{Y})=\mathbf{g}(\mathbf{X},\mathbf{Z})=0\,,\quad \mathbf{g}(\mathbf{Z},\mathbf{Z})=\mathbf{g}(\mathbf{Y},\mathbf{Y})\,,$$

$\mathbf{Y},\mathbf{Z}\in\mathbf{T}_p(\mathbf{M})$. If (1.4) hold for all $p\in\mathbf{M}$, (**M**,**g**,**J**) has pointwise constant type. (**M**,**g**,**J**) has global constant type if (1.4) is the constant function.

Now, let us put

$$\mathbf{L}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})=\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})-\alpha[\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{g}(\mathbf{Y},\mathbf{Z})-\mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{g}(\mathbf{X},\mathbf{W})]\,. \tag{1.5}$$

The following theorem is proved in [7].

The tensor (1.5) satisfies the *Kähler* identity

$$\mathbf{L}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})=\mathbf{L}(\mathbf{X},\mathbf{Y},\mathbf{JZ},\mathbf{JW})$$

if and only if the tensor **R** satisfies the following conditions

a) $\mathbf{R}(\mathbf{JX},\mathbf{JY},\mathbf{JZ},\mathbf{JW})=\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})$

b) **R** has constant type.

Thus, if the curvature tensor is **J**-invariant curvature tensor and has constant type, then

$$\begin{aligned}&\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})-\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{JZ},\mathbf{JW})=\\&\quad=\alpha[\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{g}(\mathbf{Y},\mathbf{Z})-\mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{g}(\mathbf{Y},\mathbf{W})-\mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{F}(\mathbf{Y},\mathbf{Z})+\mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{F}(\mathbf{Y},\mathbf{W})].\end{aligned} \tag{1.6}$$

With respect to the local coordinates, (1.6) can be expressed as follows

$$\mathbf{R}_{kjih}-\mathbf{R}_{kjab}\mathbf{J}_i^a\mathbf{J}_h^b=\alpha\,[\mathbf{g}_{kh}\mathbf{g}_{ji}-\mathbf{g}_{ki}\mathbf{g}_{jh}-\mathbf{F}_{kh}\mathbf{F}_{ji}+\mathbf{F}_{ki}\mathbf{F}_{jh}]\,. \tag{1.7}$$

2. LOCALLY CONFORMALLY *KÄHLER* MANIFOLDS

An almost Hermitian manifold $(\mathbf{M},\overline{\mathbf{g}},\mathbf{J})$ is a locally conformally *Kähler* manifold if the metric $\overline{\mathbf{g}}$ is conformally related to the metric **g** of a *Kähler* manifold $(\mathbf{M},\mathbf{g},\mathbf{J})$, i.e.

$$\overline{\mathbf{g}}_{ij} = e^{2f}\mathbf{g}_{ij} \tag{2.1}$$

where $f = f(x^i)$ is a scalar function.

Denoting by $\overline{\nabla}$ the operator of the covariant derivative with respect to the Levi-Chivita connection of the metric $\overline{g}$, we find

$$\overline{\nabla}_j J_i^s = \nabla_j J_i^s + \delta_j^s f_a J_i^a + \overline{g}_{ji} f^a J_a^s - J_j^s f_i - \overline{F}_{ij} f^s$$

where $f_i = \dfrac{\partial f}{\partial x^i}$, $f^s = \overline{g}^{st} f_t$, $\overline{F}_{ij} = J_i^t \overline{g}_{tj}$.

Thus, if $(\mathbf{M}, \mathbf{g}, \mathbf{J})$ is a Kähler manifold, i.e. if $\nabla_j \mathbf{J}_i^s = 0$, then

$$\overline{\nabla}_j J_i^s = \delta_j^s f_a J_i^a + \overline{g}_{ji} f^a J_a^s - J_j^s f_i - \overline{F}_{ij} f^s \tag{2.2}$$

The relation (2.2) characterizes the locally conformally Kähler manifolds [2].

From (2.2), using the Ricci identity, we get:

$$\begin{aligned}\overline{\mathbf{R}}_{kjih} - \overline{\mathbf{R}}_{kjab} J_i^a J_h^b = &-\overline{g}_{ji} P_{kh} + \overline{g}_{ki} P_{jh} - \overline{g}_{kh} P_{ji} + \overline{g}_{jh} P_{ki} - \overline{F}_{ji} P_{ka} J_h^a + \\ &+ \overline{F}_{ki} P_{ja} J_h^a - \overline{F}_{kh} P_{ja} J_i^a + \overline{F}_{jh} P_{ka} J_i^a\end{aligned}$$

where

$$P_{kh} = \overline{\nabla}_k f_h + f_k f_h - \frac{1}{2}\overline{g}_{kh} f_t f^t \tag{2.3}$$

and $\overline{\mathbf{R}}$ is the Riemannian curvature tensor of the metric $\overline{g}$.

Thus and in view of (1.7) $(\mathbf{M}, \overline{\mathbf{g}}, \mathbf{J})$ is locally conformally Kähler manifold of constant type and **J**-invariant curvature tensor if and only if

$$\begin{aligned}&-\overline{g}_{ji} P_{kh} + \overline{g}_{ki} P_{jh} - \overline{g}_{kh} P_{ji} + \overline{g}_{jh} P_{ki} - \overline{F}_{ji} P_{ka} J_h^a + \overline{F}_{ki} P_{ja} J_h^a - \overline{F}_{kh} P_{ja} J_i^a + \overline{F}_{jh} P_{ka} J_i^a = \\ &\qquad = \alpha[\overline{g}_{kh}\overline{g}_{ji} - \overline{g}_{ki}\overline{g}_{jh} - \overline{F}_{kh}\overline{F}_{ji} + \overline{F}_{ki}\overline{F}_{jh}]\end{aligned} \tag{2.4}$$

Transvecting (2.4) with $\overline{g}^{ji}$ and taking into account that P_{ja} is symmetric, while $J_i^a \overline{g}^{ji}$ is skew-symmetric, we find

$$-(2n-3)P_{kh} + P_{ab} J_k^a J_h^b - \overline{g}_{kh} P_{ji} \overline{g}^{ji} = 2(n-1)\alpha \overline{g}_{kh} \tag{2.5}$$

Transvecting (2.5) with $\overline{g}^{kh}$, we get:

$$P_{kh}\overline{g}^{kh} = -n\alpha,$$

because of which (2.5) reduces to:

$$-(2n-3)P_{kh} + P_{ab} J_k^a J_h^b = (n-2)\alpha \overline{g}_{kh} \tag{2.6}$$

The relation (2.6) implies

$$P_{kh} - (2n-3)P_{ab} J_k^a J_h^b = (n-2)\alpha \overline{g}_{kh},$$

which, together with (2.6), gives

$$-2(n-1)(n-2)P_{kh} = (n-1)(n-2)\alpha \overline{g}_{kh}$$

Thus, if $n > 2$, we have

$$P_{kh} = -\frac{\alpha}{2}\overline{g}_{kh} \qquad (2.7)$$

Conversely, if (2.7), (2.4) is identically satisfied.

In view of (2.7), (2.3) becomes

$$\overline{\nabla}_k f_h + f_k f_h = \Phi \overline{g}_{kh} \qquad (2.8)$$

where

$$\Phi = \frac{1}{2}(\overline{g}^{st} f_s f_t - \alpha)$$

Thus, we can state

Theorem: *The relation (2.8) is the necessary and sufficient condition for locally coformally Kähler manifold* $(\mathbf{M}, \overline{\mathbf{g}}, \mathbf{J})$, $\dim M > 4$ *to be of constant type and have* **J** *-invariant curvature tensor.*

The relation (2.8) is the condition with respect to the metric $\overline{g}$. With respect to the metric g, it is:

$$\nabla_k f_h - f_k f_h = \psi g_{kh} \qquad (2.9)$$

$$\psi = -\frac{1}{2}(\alpha e^{2f} + g^{st} f_s f_t).$$

Now, we shall give the simpler form of (2.9). To do this, we put

$$f = \log\frac{1}{G}.$$

Then

$$f_h = \frac{\partial f}{\partial x^h} = -\frac{G_h}{G}, \; \nabla_k f_h - f_k f_h = -\frac{1}{G}\nabla_k G_h, \quad G_h = \frac{\partial G}{\partial x^h},$$

and the condition (2.9) obtains the form:

$$\nabla_k G_h = \varphi g_{kh}, \; (\varphi = -G\psi) \qquad (2.10)$$

Thus, to find the metric $\overline{g}$ of a locally conformally *Kähler manifold* of constant type and **J**-invarian curvature tensor, we have to find the metric g of the *Kähler manifold* admiting the scalar function G such that (2.10) holds.

It is known ([8], p.95) that the Riemannian manifold (**M**,**g**), $\dim M = m$, admitting the scalar function G satisfying (2.10) has, with respect to the suitable local coordinates, the metric of the form

$$ds^2 = (dx^1)^2 + Q(x^1)\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta, \qquad (2.11)$$

where $Q(x^1)$ is nonconstant function of x^1 only and $\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta$, $\frac{\partial \tilde{g}_{\alpha\beta}}{\partial x^1} = 0$, $\alpha, \beta, \gamma, \delta = 2,3,...,m$ is a metric of $(m-1)$ - dimensional Riemannian manifold $(\widetilde{\mathbf{M}}, \widetilde{\mathbf{g}})$.

Indeed, with respect to the metric (2.11), the components of the Levi-Chivita connection are:

$$\Gamma^1_{11} = \Gamma^1_{\alpha 1} = \Gamma^\alpha_{11} = 0,$$

$$\Gamma^1_{\alpha\beta} = -\frac{Q'}{2}\tilde{g}_{\alpha\beta}, \qquad \Gamma^\alpha_{1\beta} = \frac{Q'}{2Q}\delta^\beta_\alpha, \qquad \Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma}$$

where $\tilde{\Gamma}^\alpha_{\beta\gamma}$ are the components of the Levi-Chivita connection with respect to the metric $\tilde{g}$. Then for

$$G = A\int\sqrt{Q}dx^1 + B, \qquad A, B \quad const$$

we have

$$\nabla_k G_h = \frac{A}{2}\frac{Q'}{\sqrt{Q}}g_{kh},$$

and this is just the condition (2.10).

The metric (2.11) is the metric of a *Kähler manifold* if and only if [3], $\dim M = 2n$, $Q = (x^1)^2$ and $(\tilde{\mathbf{M}}, \tilde{\mathbf{g}})$ is a Sasakian manifold. If $(\varphi, \xi, \eta, \tilde{g})$ is the Sasakian structure of the manifold $\tilde{\mathbf{M}}$, then the complex structure of the *Kähler manifold* (2.11) is given by

$$J^1_1 = 0, \quad J^\alpha_1 = \frac{1}{x^1}\xi^\alpha, \quad J^1_\alpha = -x^1\eta_\alpha, \quad J^\beta_\alpha = \varphi^\beta_\alpha \tag{2.12}$$

Thus, the metric of any locally conformally *Kähler manifold* $(\mathbf{M}, \bar{\mathbf{g}}, \mathbf{J})$, $\dim M > 4$, of constant type and **J**-invariant curvature tensor has, with respect to the suitable local coordinates, the form

$$d\bar{s}^2 = e^{2f}ds^2 = \frac{1}{\left[\frac{A}{2}(x^1)^2 + B\right]^2}[(dx^1)^2 + (x^1)^2\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta] \tag{2.13}$$

where $\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta$ is the metric of a $(2n-1)$- dimensional Sasakian manifold.

Conversely, ds^2 being the metric of a *Kähler manifold*, (2.13) is the metric of locally conformally *Kähler manifold.* Next, the components of the Levy-Civita connection $\bar{\Gamma}$ of (2.13) are the folowing:

$$\bar{\Gamma}^1_{11} = -\frac{Ax^1}{\frac{A}{2}(x^1)^2 + B}, \quad \bar{\Gamma}^1_{1\alpha} = \bar{\Gamma}^\alpha_{11} = 0, \quad \bar{\Gamma}^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma}$$

$$\bar{\Gamma}^1_{\alpha\beta} = \left[-x^1 + \frac{A(x^1)^3}{\frac{A}{2}(x^1)^2 + B}\right]\tilde{g}_{\alpha\beta}, \qquad \bar{\Gamma}^\alpha_{1\beta} = \left[\frac{1}{x^1} - \frac{Ax^1}{\frac{A}{2}(x^1)^2 + B}\right]\delta^\alpha_\beta$$

Thus, for

$$f = \log\frac{1}{\frac{A}{2}(x^1)^2 + B},$$

we have

$$f_1 = -\frac{Ax^1}{\frac{A}{2}(x^1)^2 + B}, \quad f_\alpha = 0, \quad \bar{\nabla}_1 f_1 = -\frac{A}{\frac{A}{2}(x^1)^2 + B},$$

$$\bar{\nabla}_1 f_\alpha = 0, \quad \bar{\nabla}_\beta f_\alpha = \left[\frac{1}{2}(Ax^1)^2 - AB\right]\tilde{g}_{\alpha\beta},$$

and therefore

$$\overline{\nabla}_k f_h + f_k f_h = \left[\frac{1}{2}(Ax^1)^2 - AB\right]\widetilde{g}_{kh} .$$

But this is just the condition (2.8).

Thus we can state

Theorem: *The metric*

$$d\widetilde{s}^2 = \frac{1}{\left[\frac{A}{2}(x_1)^2 + B\right]^2}[(dx^1)^2 + (x^1)^2\,\widetilde{g}_{\alpha\beta}dx^\alpha dx^\beta] \tag{2.13}$$

where A and B are constants, $\widetilde{g}_{\alpha\beta}dx^\alpha dx^\beta$, $\frac{\partial \widetilde{g}_{\alpha\beta}}{\partial x^1} = 0$, $\alpha,\beta = 2,3,...,2n$ is the metric of a $(2n-1)$-dimensional Sasakian manifold, is the metric of locally conformally *Kähler manifold* of constant type and **J**-invariant curvature tensor. Conversely, any locally conformally *Kähler manifold* of dimension $2n > 4$, of constant type and **J**-invariant curvature tensor has, with respect to some local coordinates (x^i), the form (2.13).

3. Holomorphic curvature tensor

Let (**M**,**g**,**J**) be an almost Hermitian manifold. Because of $\mathbf{JX} \perp \mathbf{X}$, any vector **X** determines the holomorphic section. The sectional curvature with respect to the holomorphic section,

$$\mathbf{H}(\mathbf{X}) = \frac{\mathbf{R}(\mathbf{X},\mathbf{JX},\mathbf{JX},\mathbf{X})}{\mathbf{g}(\mathbf{X},\mathbf{X})^2}$$

is *the holomorphic sectional curvature*.

We define the holomorphic curvature tensor $(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})$ as follows (see for ex. [1],[4])

$$\begin{aligned}(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) =& \\ = \frac{1}{16}\{3[&\mathbf{R}(\mathbf{X}.\mathbf{Y},\mathbf{Z},\mathbf{W}) + \mathbf{R}(\mathbf{JX},\mathbf{JY},\mathbf{Z},\mathbf{W}) + \mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{JZ},\mathbf{JW}) + \mathbf{R}(\mathbf{JX},\mathbf{JY},\mathbf{JZ},\mathbf{JW})] - \\ &- \mathbf{R}(\mathbf{X},\mathbf{Z},\mathbf{JW},\mathbf{JY}) - \mathbf{R}(\mathbf{JX},\mathbf{JZ},\mathbf{W},\mathbf{Y}) - \mathbf{R}(\mathbf{X},\mathbf{W},\mathbf{JY},\mathbf{JZ}) - \mathbf{R}(\mathbf{JX},\mathbf{JW},\mathbf{Y},\mathbf{Z}) + \\ &+ \mathbf{R}(\mathbf{JX},\mathbf{Z},\mathbf{JW},\mathbf{Y}) + \mathbf{R}(\mathbf{X},\mathbf{JZ},\mathbf{W},\mathbf{JY}) + \mathbf{R}(\mathbf{JX},\mathbf{W},\mathbf{Y},\mathbf{JZ}) + \mathbf{R}(\mathbf{X},\mathbf{JW},\mathbf{JY},\mathbf{Z})\}\end{aligned}$$

$$(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})$$

It is easy to see that $(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})$ is an algebraik curvature tensor, i.e. that

$$(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = -(\mathbf{HR})(\mathbf{Y},\mathbf{X},\mathbf{Z},\mathbf{W}) = -(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{W},\mathbf{Z}),$$

$$(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = (\mathbf{HR})(\mathbf{Z},\mathbf{W},\mathbf{X},\mathbf{Y}),$$

$$(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) + (\mathbf{HR})(\mathbf{Y},\mathbf{Z},\mathbf{X},\mathbf{W}) + (\mathbf{HR})(\mathbf{Z},\mathbf{X},\mathbf{Y},\mathbf{W}) = 0$$

But the tensor (3.1) has also the following remarkable properties

$$(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{JZ},\mathbf{JW}) = (\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}), \tag{3.2}$$

$$(\mathbf{HR})(\mathbf{X},\mathbf{JX},\mathbf{JX},\mathbf{X}) = \mathbf{R}(\mathbf{X},\mathbf{JX},\mathbf{JX},\mathbf{X}), \tag{3.3}$$

The relation (3.3) shows that the holomorphic sectional curvatures with respect to **R** and **HR** are the same. This is the reason to name **HR** *the holomorphic curvature tensor.*

The relation (3.2) shows that although the Riemannian curvature tensor of an almost Hermitian manifold in general does not satisfy the *Kähler* identity (1.2) , nevertheless there exists the tensor satisfying it: this is the tensor (3.1).

We underline that if (1.2) holds, then

$$(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = \mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) \tag{3.4}$$

Thus, for the *Kähler manifolds,* the holomorphic curvature tensor reduces to the Riemannian curvature tensor.

If $(\mathbf{M},\mathbf{g},\mathbf{J})$ is of **J**-invariant curvature tensor, i.e. if (1.3) holds, then

$$\begin{aligned}&(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) =\\ &= \frac{1}{8}\{3[\mathbf{R}(\mathbf{X}.\mathbf{Y},\mathbf{Z},\mathbf{W}) + \mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{JZ},\mathbf{JW})] - \mathbf{R}(\mathbf{X},\mathbf{Z},\mathbf{JW},\mathbf{JY}) -\\ &- \mathbf{R}(\mathbf{X},\mathbf{W},\mathbf{JY},\mathbf{JZ}) + \mathbf{R}(\mathbf{JX},\mathbf{Z},\mathbf{JW},\mathbf{Y}) + \mathbf{R}(\mathbf{JX},\mathbf{W},\mathbf{Y},\mathbf{JZ})\}\end{aligned}$$

If besides this $(\mathbf{M},\mathbf{g},\mathbf{J})$ is of constant type,i.e. if (1.6) holds, then

$$\begin{aligned}(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) &= \mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) - \frac{\alpha}{4}\{3[g(\mathbf{X},\mathbf{W})g(\mathbf{Y},\mathbf{Z}) - g(\mathbf{X},\mathbf{Z})g(\mathbf{Y},\mathbf{W})] -\\ &- [\mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{F}(\mathbf{Y},\mathbf{Z}) - \mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{F}(\mathbf{Y},\mathbf{W}) - 2\mathbf{F}(\mathbf{X},\mathbf{Y})\mathbf{F}(\mathbf{Z},\mathbf{W})]\}\end{aligned} \tag{3.5}$$

We define the Ricci tensor and Ricci tensor of $(\mathbf{M},\mathbf{g},\mathbf{J})$ by

$$\rho(x,y) = \sum_i \mathbf{R}(\mathbf{e_i},\mathbf{X},\mathbf{Y},\mathbf{e_i}) \qquad \overset{*}{\rho}(x,y) = \sum_i \mathbf{R}(\mathbf{e_i},\mathbf{X},\mathbf{JY},\mathbf{Je_i})$$

where $\{\mathbf{e_i}\}$ is an orthogonal basis. They have the properties:

$$\rho(\mathbf{X},\mathbf{Y}) = \rho(\mathbf{Y},\mathbf{X}) \qquad \overset{*}{\rho}(\mathbf{X},\mathbf{JY}) = -\overset{*}{\rho}(\mathbf{Y},\mathbf{JX}).$$

If (**M**,**g**,**J**) is a *Kähler manifold,* then

$$\rho(\mathbf{JX},\mathbf{JY}) = \rho(\mathbf{X},\mathbf{Y}) \qquad \overset{*}{\rho}(\mathbf{X},\mathbf{Y}) = \rho(\mathbf{X},\mathbf{Y}).$$

If (**M**,**g**,**J**) is **J**-invariant and has the constant type,i.s. if (1.6) holds, than

$$\rho(\mathbf{X},\mathbf{Y}) = \overset{*}{\rho}(\mathbf{X},\mathbf{Y}) + 2(n-1)\alpha\,\mathbf{g}(\mathbf{X},\mathbf{Y}) \tag{3.6}$$

The scalar curvature and * - scalar curvature are defined by

$$\tau = \sum_i \rho(\mathbf{e_i},\mathbf{e_i}), \qquad \overset{*}{\tau} = \sum_i \overset{*}{\rho}(\mathbf{e_i},\mathbf{e_i})$$

For the Kähler manifolds $\tau = \overset{*}{\tau}$.

We obtain from (3.6)

$$\alpha = \frac{1}{4n(n-1)}(\tau - \overset{*}{\tau})$$

Therefore, if (**M**,**g**,**J**) is **J**-invariant and has the constant type, the Ricci tensor and *-Ricci tensor are related as follows

$$\overset{*}{\rho}(\mathbf{X},\mathbf{Y}) = \rho(\mathbf{X},\mathbf{Y}) + \frac{1}{2n}(\tau - \overset{*}{\tau})\mathbf{g}(\mathbf{X},\mathbf{Y}) \tag{3.7}$$

while (3.5) has the form

$$(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = \mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) - \frac{(\tau - \overset{*}{\tau})}{16n(n-1)}\{3[g(\mathbf{X},\mathbf{W})g(\mathbf{Y},\mathbf{Z}) - g(\mathbf{X},\mathbf{Z})g(\mathbf{Y},\mathbf{W})] - \tag{3.8}$$
$$-[\mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{F}(\mathbf{Y},\mathbf{Z}) - \mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{F}(\mathbf{Y},\mathbf{W}) - 2\mathbf{F}(\mathbf{X},\mathbf{Y})\mathbf{F}(\mathbf{Z},\mathbf{W})]\}$$

In the same manner, we define Ricci tensor, *-Ricci tensor, scalar curvature and *-scalar curvature associated with holomorphic curvature tensor, namely,

$$\rho(\mathbf{HR})(\mathbf{X},\mathbf{Y}) = \sum_i (\mathbf{HR})(\mathbf{e_i},\mathbf{X},\mathbf{Y},\mathbf{e_i})$$
$$\overset{*}{\rho}(\mathbf{HR})(\mathbf{X},\mathbf{Y}) = \sum_i (\mathbf{HR})(\mathbf{e_i},\mathbf{X},\mathbf{JY},\mathbf{Je_i}) \tag{3.9}$$
$$\tau(\mathbf{HR}) = \sum_i \rho(\mathbf{HR})(\mathbf{e_i},\mathbf{e_i}),$$
$$\overset{*}{\tau}(\mathbf{HR}) = \sum_i \overset{*}{\rho}(\mathbf{HR})(\mathbf{e_i},\mathbf{e_i})$$

Because (**HR**) is an algebraic curvature tensor satisfying the Kähler identity, we have

$$\rho(\mathbf{HR})(\mathbf{X},\mathbf{Y}) = \rho(\mathbf{HR})(\mathbf{Y},\mathbf{X}) = \rho(\mathbf{HR})(\mathbf{JX},\mathbf{JY}) = \overset{*}{\rho}(\mathbf{HR})(\mathbf{X},\mathbf{Y})$$
$$\tau(\mathbf{HR}) = \overset{*}{\tau}(\mathbf{HR})$$

Explicitly,

$$\rho(\mathbf{HR})(\mathbf{X},\mathbf{Y}) = \frac{1}{8}[\rho(\mathbf{X},\mathbf{Y}) + 3\overset{*}{\rho}(\mathbf{X},\mathbf{Y}) + 3\overset{*}{\rho}(\mathbf{Y},\mathbf{X}) + \rho(\mathbf{JX},\mathbf{JY})],$$
$$\tau(\mathbf{HR}) = \frac{1}{4}[\tau + 3\overset{*}{\tau}] \tag{3.10}$$

For the Kähler manifolds we have

$$\rho(\mathbf{HR})(\mathbf{X},\mathbf{Y}) = \rho(\mathbf{X},\mathbf{Y}) \qquad \tau(\mathbf{HR}) = \tau \tag{3.11}$$

For (**M**,**g**,**J**) satisfying (1.6), we have

$$\rho(\mathbf{HR})(\mathbf{X},\mathbf{Y}) = \frac{1}{4}[\rho(\mathbf{X},\mathbf{Y}) + 3\overset{*}{\rho}(\mathbf{X},\mathbf{Y})] \tag{3.12}$$

or, taking into account (3.7)

$$\rho(\mathbf{HR})(\mathbf{X},\mathbf{Y}) = \rho(\mathbf{X},\mathbf{Y}) + \frac{3}{8n}(\overset{*}{\tau} - \tau)\mathbf{g}(\mathbf{X},\mathbf{Y}) \tag{3.13}$$

4. THE GENERALIZED BOCHNER CURVATURE TENSOR

Let us consider an almost Hermitian manifold (**M**,**g**,**J**), and conformal transformation (2.1) of its metric. Then

$$(\overline{\nabla}-\nabla)(\mathbf{X},\mathbf{Y})=\omega(\mathbf{X})\mathbf{Y}+\omega(\mathbf{Y})\mathbf{X}-\mathbf{g}(\mathbf{X},\mathbf{Y})\mathbf{U}$$

for any vector fields **X**, **Y**, where ω is 1-form defined by $\omega = df$ and **U** is a vector field satisfying $\mathbf{g}(\mathbf{U},\mathbf{X}) = \omega(\mathbf{X})$.

It is well known that the Riemannian curvature tensors of the metrics $\overline{\mathbf{g}}$ and **g** respectively, are related as follows

$$e^{-2f}\overline{\mathbf{R}}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})=\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})+\mathbf{g}(\mathbf{X},\mathbf{W})\sigma(\mathbf{Y},\mathbf{Z})+\mathbf{g}(\mathbf{Y},\mathbf{Z})\sigma(\mathbf{X},\mathbf{W})-$$
$$-\mathbf{g}(\mathbf{X},\mathbf{Z})\sigma(\mathbf{Y},\mathbf{W})-\mathbf{g}(\mathbf{Y},\mathbf{W})\sigma(\mathbf{X},\mathbf{Z})$$

where σ is the tensor field of type (0,2) defined by

$$\sigma(\mathbf{X},\mathbf{Y})=(\nabla_x\omega)\mathbf{Y}-\omega(\mathbf{X})\omega(\mathbf{Y})+\frac{1}{2}\omega(\mathbf{U})\mathbf{g}(\mathbf{X},\mathbf{Y}) \tag{4.1}$$

We note that $\sigma(\mathbf{X},\mathbf{Y}) = \sigma(\mathbf{Y},\mathbf{X})$. As for the corresponding holomorphic curvature tensors, we have, in view of (3.1)

$$8e^{-2f}(\mathbf{H}\overline{\mathbf{R}})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})=8(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})+\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{S}(\mathbf{Y},\mathbf{Z})+\mathbf{g}(\mathbf{Y},\mathbf{Z})\mathbf{S}(\mathbf{X},\mathbf{W})-$$
$$-\mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{S}(\mathbf{Y},\mathbf{W})-\mathbf{g}(\mathbf{Y},\mathbf{W})\mathbf{S}(\mathbf{X},\mathbf{Z})+\mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{S}(\mathbf{JY},\mathbf{Z})+\mathbf{F}(\mathbf{Y},\mathbf{Z})\mathbf{S}(\mathbf{JX},\mathbf{W})-$$
$$-\mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{S}(\mathbf{JY},\mathbf{W})-\mathbf{F}(\mathbf{Y},\mathbf{W})\mathbf{S}(\mathbf{JX},\mathbf{Z})-2\mathbf{F}(\mathbf{X},\mathbf{Y})\mathbf{S}(\mathbf{JZ},\mathbf{W})-2\mathbf{F}(\mathbf{Z},\mathbf{W})\mathbf{S}(\mathbf{JX},\mathbf{Y})$$

where

$$\mathbf{S}(\mathbf{X},\mathbf{Y})=\sigma(\mathbf{X},\mathbf{Y})+\sigma(\mathbf{JX},\mathbf{JY}) \tag{4.3}$$

Therefore

$$\mathbf{S}(\mathbf{JX},\mathbf{Y})=\sigma(\mathbf{JX},\mathbf{Y})-\sigma(\mathbf{X},\mathbf{JY})$$

Putting into (4.2) $\mathbf{X}=\mathbf{W}=\mathbf{e_i}$, summing up and taking into account (3.9), we obtain

$$8\rho(\mathbf{H}\overline{\mathbf{R}})(\mathbf{Y},\mathbf{Z})=8\rho(\mathbf{HR})(\mathbf{Y},\mathbf{Z})+2(n+2)\mathbf{S}(\mathbf{X},\mathbf{Z})+\mathbf{g}(\mathbf{Y},\mathbf{Z})\sum_i\mathbf{S}(\mathbf{e_i},\mathbf{e_i}) \tag{4.4}$$

Putting into (4.4) $\mathbf{Y}=\mathbf{Z}=\mathbf{e_i}$ and summing up, we get

$$\sum_i\mathbf{S}(\mathbf{e_i},\mathbf{e_i})=\frac{2}{n+2}[e^{2f}\tau(\mathbf{H}\overline{\mathbf{R}})-\tau(\mathbf{HR})]\,, \tag{4.5}$$

because of which (4.4) reduces to

$$\mathbf{S}(\mathbf{Y},\mathbf{Z})=\frac{4}{n+2}[\rho(\mathbf{H}\overline{\mathbf{R}})(\mathbf{Y},\mathbf{Z})-\rho(\mathbf{HR})(\mathbf{Y},\mathbf{Z})]-\frac{e^{2f}\tau(\mathbf{H}\overline{\mathbf{R}})-\tau(\mathbf{H}\overline{\mathbf{R}})}{(n+1)(n+2)}\mathbf{g}(\mathbf{Y},\mathbf{Z}) \tag{4.6}$$

Thus, (4.2) becomes

$$e^{-2f}\mathbf{B}(\mathbf{H}\overline{\mathbf{R}})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})=\mathbf{B}(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) \tag{4,7}$$

where

$$\begin{aligned}
&\mathbf{B(HR)(X,Y,Z,W)} = \\
&\mathbf{(HR)(X,Y,Z,W)} - \frac{1}{2(n+2)}\{\mathbf{g(X,W)}\rho\mathbf{(HR)(Y,Z)} + \mathbf{g(Y,Z)}\rho\mathbf{(HR)(X,W)} - \\
&- \mathbf{g(X,Z)}\rho\mathbf{(HR)(Y,W)} - \mathbf{g(Y,W)}\rho\mathbf{(HR)(X,Z)} + \mathbf{F(X,W)}\rho\mathbf{(HR)(JY,Z)} + \\
&+ \mathbf{F(Y,Z)}\rho\mathbf{(HR)(JX,W)} - \mathbf{F(X,Z)}\rho\mathbf{(HR)(JY,W)} - \mathbf{F(Y,W)}\rho\mathbf{(HR)(JX,Z)} - \\
&- 2\mathbf{F(X,Y)}\rho\mathbf{(HR)(JZ,W)} - 2\mathbf{F(Z,W)}\rho\mathbf{(HR)(JX,Y)}\} + \\
&+ \frac{\tau\mathbf{(HR)}}{\mathbf{4}(n+1)(n+2)}\{\mathbf{g(X,W)g(Y,Z)} - \mathbf{g(X,Z)g(Y,W)} + \mathbf{F(X,W)F(X,Z)} - \\
&- \mathbf{F(X,Z)F(Y,W)} - 2\mathbf{F(X,Y)F(Z,W)}\}
\end{aligned} \tag{4.8}$$

and $\mathbf{B(H\overline{R})}$ is constructed in the same way but using the tensor $\mathbf{H\overline{R}}$.

Thus, we see that: *For any almost Hermitian manifold, the tensor (4.8) satisfies (4.7).*

The tensor (4.8) is the generalized Bochner curvature tensor. In [1], it is obtained applying the decomposition on a 2n-dimensional Hermitian vector space into orthogonal components under the action of the unitary group. Here, we obtain it using the classical way.

For the Kähler manifolds, (3.4) and (3.11) hold, because of which (4.8) become the well known Bochner curvature tensor.

If (**M**,**g**,**J**) is **J**-invariant and has the constant type, then (3.8), (3.10) and (3.12) hold, and (4.8) reduces to

$$\begin{aligned}
&\mathbf{B(HR)(X,Y,Z,W)} = \mathbf{R(X,Y,Z,W)} - \frac{1}{8(n+2)}\{\mathbf{g(X,W)}[\rho\mathbf{(Y,Z)} + 3\overset{*}{\rho}\mathbf{(Y,Z)}] + \\
&+ \mathbf{g(Y,Z)}[\rho\mathbf{(X,W)} + 3\overset{*}{\rho}\mathbf{(X,W)}] - \mathbf{g(X,Z)}[\rho\mathbf{(Y,W)} + 3\overset{*}{\rho}\mathbf{(Y,W)}] - \\
&- \mathbf{g(Y,W)}[\rho\mathbf{(X,Z)} + 3\overset{*}{\rho}\mathbf{(X,Z)}] + \mathbf{F(X,W)}[\rho\mathbf{(JY,Z)} + 3\overset{*}{\rho}\mathbf{(JY,Z)}] + \\
&+ \mathbf{F(Y,Z)}[\rho\mathbf{(JX,W)} + 3\overset{*}{\rho}\mathbf{(JX,W)}] - \mathbf{F(X,Z)}[\rho\mathbf{(JY,W)} + 3\overset{*}{\rho}\mathbf{(JY,W)}] - \\
&- \mathbf{F(Y,W)}[\rho\mathbf{(JX,Z)} + 3\overset{*}{\rho}\mathbf{(JX,Z)}] - 2\mathbf{F(X,Y)}[\rho\mathbf{(JZ,W)} + 3\overset{*}{\rho}\mathbf{(JZ,W)}] - \\
&- 2\mathbf{F(Z,W)}[\rho\mathbf{(JX,Y)} + 3\overset{*}{\rho}\mathbf{(JX,Y)}] + \frac{\tau + 3\overset{*}{\tau}}{16(n+1)(n+2)}\{\mathbf{g(X,W)g(Y,Z)} - \\
&- \mathbf{g(X,Z)g(Y,W)} + \mathbf{F(X,W)F(Y,Z)} - \mathbf{F(X,Z)F(Y,W)} - 2\mathbf{F(X,Y)F(Z,W)} + \\
&+ \frac{\overset{*}{\tau} - \tau}{16n(n-1)}\{3[\mathbf{g(X,W)g(Y,Z)} - \mathbf{g(X,Z)g(Y,W)}] - \\
&- [\mathbf{F(X,W)F(Y,Z)} - \mathbf{F(X,Z)F(Y,W)} - 2\mathbf{F(X,Y)F(Z,W)}]\}
\end{aligned} \tag{4.9}$$

This is just the tensor find in [5], but using the decomposition of the vector space.

In view of (3.13), (4.9) can be rewritten in the form

$$\mathbf{B}(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = \mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) - \frac{1}{2(n+2)}\{\mathbf{g}(\mathbf{X},\mathbf{W})\rho(\mathbf{Y},\mathbf{Z}) + \mathbf{g}(\mathbf{Y},\mathbf{Z})\rho(\mathbf{X},\mathbf{Z})$$
$$- \mathbf{g}(\mathbf{X},\mathbf{Z})\rho(\mathbf{Y},\mathbf{W}) - \mathbf{g}(\mathbf{Y},\mathbf{W})\rho(\mathbf{X},\mathbf{Z}) + \mathbf{F}(\mathbf{X},\mathbf{W})\rho(\mathbf{JY},\mathbf{Z}) + \mathbf{F}(\mathbf{Y},\mathbf{Z})\rho(\mathbf{JX},\mathbf{W}) -$$
$$- \mathbf{F}(\mathbf{X},\mathbf{Z})\rho(\mathbf{JY},\mathbf{W}) - \mathbf{F}(\mathbf{Y},\mathbf{W})\rho(\mathbf{JX},\mathbf{Z}) - 2\mathbf{F}(\mathbf{X},\mathbf{Y})\rho(\mathbf{JZ},\mathbf{W}) - 2\mathbf{F}(\mathbf{Z},\mathbf{W})\rho(\mathbf{JX},\mathbf{Y})\} +$$
$$+ \frac{1}{8(n+2)}\left[\frac{\tau + 3\overset{*}{\tau}}{2(n+1)} - \frac{3}{4}(\overset{*}{\tau} - \tau)\right]\{\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{g}(\mathbf{Y},\mathbf{Z}) - \mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{g}(\mathbf{Y},\mathbf{W}) +$$
$$+ \mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{F}(\mathbf{Y},\mathbf{Z}) - \mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{F}(\mathbf{Y},\mathbf{W}) - 2\mathbf{F}(\mathbf{X},\mathbf{Y})\mathbf{F}(\mathbf{Z},\mathbf{W})\} +$$
$$+ \frac{\overset{*}{\tau} - \tau}{16n(n-1)}\{3[\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{g}(\mathbf{Y},\mathbf{Z}) - \mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{g}(\mathbf{Y},\mathbf{W})] -$$
$$- [\mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{F}(\mathbf{Y},\mathbf{Z}) - \mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{F}(\mathbf{Y},\mathbf{W}) - 2\mathbf{F}(\mathbf{X},\mathbf{Y})\mathbf{F}(\mathbf{Z},\mathbf{W})]\}$$

The components of the curvature tensor of the metric (2.13) which do not vanish, are the following

$$\overline{\mathbf{R}}_{\alpha 1 1 \beta} = LP\tilde{g}_{\alpha\beta}$$

$$\overline{\mathbf{R}}_{\alpha\beta\gamma\delta} = (x^1)^2 L\{\tilde{\mathbf{R}}_{\alpha\beta\gamma\delta} + (P-1)(\tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma} - \tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta})\} \tag{4.11}$$

Where

$$L = \frac{1}{\left[\frac{A}{2}(x^1)^2 + B\right]^2}, \qquad P = \frac{2A(x^1)^2}{\left[\frac{A}{2}(x^1)^2 + B\right]} - \frac{A^2(x^1)^4}{\left[\frac{A}{2}(x^1)^2 + B\right]^2}.$$

The Ricci tensor $\overline{\rho}$ of the metric (2.13) has the components

$$\overline{\rho}_{11} = \frac{(2n-1)P}{(x^1)^2}, \qquad \overline{\rho}_{1\alpha} = 0$$

$$\overline{\rho}_{\alpha\beta} = \tilde{\rho}_{\alpha\beta} + [(2n-1)P - 2(n-1)]\tilde{g}_{\alpha\beta}$$

where $\tilde{\rho}_{\alpha\beta}$ is the Ricci tensor of the Sasakian manifold $\tilde{\mathbf{M}}$. For the scalar curvature $\overline{\tau}$ and *-scalar curvature $\overset{*}{\overline{\tau}}$, we have

$$\overline{\tau} = \frac{1}{(x^1)^2 L}[\tilde{\tau} + 2n(2n-1)P - 2(n-1)(2n-1)] \tag{4.12}$$

$$\overset{*}{\overline{\tau}} = \frac{1}{(x^1)^2 L}[\tilde{\tau} + 2nP - 2(n-1)(2n-1)] \tag{4.13}$$

where $\tilde{\tau}$ is the scalar curvature of the Sasacian manifold $\tilde{\mathbf{M}}$.

Substituting this into the formula corresponding to (4.10) for the metric $\overline{g}$, we find that the components of the generalized Bochner curvature tensor $\mathbf{B}(\mathbf{H}\overline{\mathbf{R}})$ of the metric (2.13), which do not vanish, are the following

$$\left[\frac{A}{2}(x^1)^2+B\right]^2 \mathbf{B}(\mathbf{H}\overline{\mathbf{R}})_{\alpha 1 1 \beta} = -\frac{1}{2(n+2)}\widetilde{\rho}_{\alpha\beta} + \frac{\widetilde{\tau}}{4(n+1)(n+2)}(\widetilde{g}_{\alpha\beta}+3\eta_\alpha\eta_\beta) + \\ + \frac{3(n-1)}{2(n+1)(n+2)}[\widetilde{g}_{\alpha\beta}-(2n-1)\eta_\alpha\eta_\beta] \tag{4.14}$$

$$\left[\frac{A}{2}(x^1)^2+B\right]^2 \mathbf{B}(\mathbf{H}\overline{\mathbf{R}})_{1\alpha\beta\gamma} = x^1\left\{-\frac{1}{2(n+2)}(\eta_\gamma\varphi^\sigma_\alpha\widetilde{\rho}_{\sigma\beta}-\eta_\beta\varphi^\sigma_\alpha\widetilde{\rho}_{\sigma\gamma}-2\eta_\alpha\varphi^\sigma_\beta\widetilde{\rho}_{\sigma\gamma})\right\}+ \\ + x^1\frac{\widetilde{\tau}+6(n-1)}{4(n+1)(n+2)}[\eta_\gamma\varphi_{\alpha\beta}-\eta_\beta\varphi_{\alpha\gamma}-2\eta_\alpha\varphi_{\beta\gamma}]$$

$$\left[\frac{A}{2}(x^1)^2+B\right]^2 \mathbf{B}(\mathbf{H}\overline{\mathbf{R}})_{\alpha\beta\gamma\delta} = (x^1)^2\{\widetilde{\mathbf{R}}_{\alpha\beta\gamma\delta}-(\widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\gamma}-\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\delta})\} - \\ -\frac{(x^1)^2}{2(n+2)}[\widetilde{g}_{\alpha\delta}\widetilde{\rho}_{\beta\gamma}+\widetilde{g}_{\beta\gamma}\widetilde{\rho}_{\alpha\delta}-\widetilde{g}_{\alpha\gamma}\widetilde{\rho}_{\beta\delta}-\widetilde{g}_{\beta\delta}\widetilde{\rho}_{\alpha\gamma}]+ \\ -\frac{(x^1)^2}{2(n+2)}[\varphi_{\alpha\delta}\varphi^\lambda_\beta\widetilde{\rho}_{\lambda\gamma}+\varphi_{\beta\gamma}\varphi^\lambda_\alpha\widetilde{\rho}_{\lambda\delta}-\varphi_{\alpha\gamma}\varphi^\lambda_\beta\widetilde{\rho}_{\lambda\delta}-\varphi_{\beta\delta}\varphi^\lambda_\alpha\widetilde{\rho}_{\lambda\gamma}-2\varphi_{\alpha\beta}\varphi^\lambda_\gamma\widetilde{\rho}_{\lambda\delta}-2\varphi_{\gamma\delta}\varphi^\lambda_\alpha\widetilde{\rho}_{\lambda\beta}]+ \\ +(x^1)^2\frac{\widetilde{\tau}+2(n-1)(2n+5)}{2(n+1)(n+2)}[\widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\gamma}-\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\delta}+\varphi_{\alpha\delta}\varphi_{\beta\gamma}-\varphi_{\alpha\gamma}\varphi_{\beta\delta}-2\varphi_{\alpha\beta}\varphi_{\gamma\delta}]$$

where $\varphi_{\alpha\beta}=\varphi^\lambda_\alpha\widetilde{g}_{\lambda\beta}$.

But the expression of the right hand sides of (4.14) are the components $B_{\alpha 1 1\beta}$, $B_{1\alpha\beta\gamma}$, $B_{\alpha\beta\gamma\delta}$ of the Bochner tensor of the Kähler metric (2.11) ($Q=(x^1)^2$), respectively, that is, we have

$$e^{-2f}\mathbf{B}(\mathbf{H}\overline{\mathbf{R}})(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})=\mathbf{B}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})$$

as was to be expected.

5. CONCIRCULAR TRANSFORMATIONS

A geodesic circle of a Riemannian manifold (**M**,**g**), dim **M** = **m** is defined as a curve whose first curvature is constant and other are identically zero. If a conformal transformation (2.1) transforms every geodesicx circle unto geodesic circle, then the function f must satisfy the partial differential equation [9]

$$\sigma(\mathbf{X},\mathbf{Y})=\theta\, g(\mathbf{X},\mathbf{Y}) \tag{5.1}$$

where $\sigma(\mathbf{X},\mathbf{Y})$ is defined by (4.1) and θ is a scalar function.

The conformal transformation satisfying (5.1) is called *concircular* one. The tensor

$$\mathbf{C}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})=\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})-\frac{\tau}{m(m-1)}[\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{g}(\mathbf{Y},\mathbf{Z})-\mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{g}(\mathbf{Y},\mathbf{W})] \tag{5.2}$$

satisfies the following relation

$$e^{-2f}\overline{\mathbf{C}}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})=\mathbf{C}(\mathbf{X},\mathbf{Y}.\mathbf{Z}.\mathbf{W})$$

and is called *concircular curvature tensor* [9].

Now, we consider an almost Hermitian manifold and tray to find the complex concircular curvature tensor, i.e. the complex analog of the tensor (5.2).

From (5.1) , it follows $\sigma(\mathbf{JX},\mathbf{JY}) = \theta g(\mathbf{X},\mathbf{Y})$. Therefore

$$\mathbf{S}(\mathbf{X},\mathbf{Y}) = \sigma(\mathbf{X},\mathbf{Y}) + \sigma(\mathbf{JX},\mathbf{JY}) = 2\theta g(\mathbf{X},\mathbf{Y}),$$

because of which (4.2) becomes

$$2e^{-2f}(\mathbf{H\overline{R}})(\mathbf{X},\mathbf{Y},\mathbf{Z}.\mathbf{W}) = 2(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z}.\mathbf{W}) +$$
$$+\theta[\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{g}(\mathbf{Y},\mathbf{Z}) - \mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{g}(\mathbf{Y},\mathbf{W}) + \mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{F}(\mathbf{Y},\mathbf{Z}) - \mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{F}(\mathbf{Y},\mathbf{W}) - 2\mathbf{F}(\mathbf{X},\mathbf{Y})\mathbf{F}(\mathbf{Z},\mathbf{W})]$$

From (5.3), we find

$$\rho(\mathbf{H\overline{R}})(\mathbf{Y},\mathbf{Z}) = \rho(\mathbf{HR})(\mathbf{Y},\mathbf{Z}) + (n+1)\theta\mathbf{g}(\mathbf{Y},\mathbf{Z})$$

and

$$\theta = \frac{e^{2f}\tau(\mathbf{H\overline{R}}) - \tau(\mathbf{HR})}{2n(n+1)}$$

because of which (5.3) reduces to

$$e^{-2f}\overline{\mathbf{C}}(\mathbf{H\overline{R}})(\mathbf{X},\mathbf{Y},\mathbf{Z}.\mathbf{W}) = \mathbf{C}(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z}.\mathbf{W})$$

where

$$\begin{aligned}\mathbf{C}(\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z}.\mathbf{W}) &= (\mathbf{HR})(\mathbf{X},\mathbf{Y},\mathbf{Z}.\mathbf{W}) - \\ &- \frac{\tau(\mathbf{HR})}{4n(n+1)}[\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{g}(\mathbf{Y},\mathbf{Z}) - \mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{g}(\mathbf{Y},\mathbf{W})] - \\ &- \frac{\tau(\mathbf{HR})}{4n(n+1)}[\mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{F}(\mathbf{Y},\mathbf{Z}) - \mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{F}(\mathbf{Y},\mathbf{W}) - 2\mathbf{F}(\mathbf{X},\mathbf{Y})\mathbf{F}(\mathbf{Z},\mathbf{W})]\end{aligned} \tag{5.4}$$

Thus, (5.4) is complex concircular curvature tensor of an almost Hermitian manifold $(\mathbf{M},\mathbf{g},\mathbf{J})$. If $(\mathbf{M},\mathbf{g},\mathbf{J})$ is a Kähler manifold (5.4) becomes

$$\begin{aligned}(\mathbf{CK})(\mathbf{X},\mathbf{Y}.\mathbf{Z}.\mathbf{W}) &= \mathbf{R}(\mathbf{X},\mathbf{Y}.\mathbf{Z}.\mathbf{W}) - \\ &- \frac{\tau}{4n(n+1)}[\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{g}(\mathbf{Y},\mathbf{Z}) - \mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{g}(\mathbf{Y},\mathbf{W})] - \\ &- \frac{\tau}{4n(n+1)}[\mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{F}(\mathbf{Y},\mathbf{Z}) - \mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{F}(\mathbf{Y},\mathbf{W}) - 2\mathbf{F}(\mathbf{X},\mathbf{Y})\mathbf{F}(\mathbf{Z},\mathbf{W})]\end{aligned} \tag{5.5}$$

If (**M**,**g**,**J**) is of constant type and **J**-invariant curvature tensor, (5.4) becomes

$$\begin{aligned}\mathbf{C}(\mathbf{HR})(\mathbf{X},\mathbf{Y}.\mathbf{Z}.\mathbf{W}) &= \mathbf{R}(\mathbf{X},\mathbf{Y}.\mathbf{Z}.\mathbf{W}) - \\ &- \frac{\tau + 3\overset{*}{\tau}}{16n(n+1)}[\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{g}(\mathbf{Y},\mathbf{Z}) - \mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{g}(\mathbf{Y},\mathbf{W})] - \\ &- \frac{\tau + 3\overset{*}{\tau}}{16n(n+1)}[\mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{F}(\mathbf{Y},\mathbf{Z}) - \mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{F}(\mathbf{Y},\mathbf{W}) - 2\mathbf{F}(\mathbf{X},\mathbf{Y})\mathbf{F}(\mathbf{Z},\mathbf{W})] + \\ &+ \frac{\overset{*}{\tau} - \tau}{16n(n+1)}3[\mathbf{g}(\mathbf{X},\mathbf{W})\mathbf{g}(\mathbf{Y},\mathbf{Z}) - \mathbf{g}(\mathbf{X},\mathbf{Z})\mathbf{g}(\mathbf{Y},\mathbf{W})] - \\ &- \frac{\overset{*}{\tau} - \tau}{16n(n+1)}[\mathbf{F}(\mathbf{X},\mathbf{W})\mathbf{F}(\mathbf{Y},\mathbf{Z}) - \mathbf{F}(\mathbf{X},\mathbf{Z})\mathbf{F}(\mathbf{Y},\mathbf{W}) - 2\mathbf{F}(\mathbf{X},\mathbf{Y})\mathbf{F}(\mathbf{Z},\mathbf{W})]\end{aligned} \tag{5.6}$$

We have seen in §2 that the Kähler manifold admitting the concircular transformation has, with respect to the suitable local coordinates, the metric of the form (2.11). The corresponding locally conformally Kähler manifold has the metric of the form (2.13).

This last metric is the metric of the manifold of constant type and $\mathbf{J}$-invariant curvature tensor. Thus, to find its complex concircular curvature tensor, we have to use the formula (5.6). Indeed, according to (4.11), (4.12) and (4.13), the non-zero components of the tensor (5.6) are the following

$$\left[\frac{A}{2}(x^1)^2+B\right]^2\mathbf{C}(\mathbf{H}\overline{\mathbf{R}})_{1\alpha\beta\gamma}=-\frac{x^1}{4n(n+1)}[\tilde{\tau}-2(n-1)(2n-1)](\eta_\gamma\varphi_{\alpha\beta}-\eta_\beta\varphi_{\alpha\gamma}-2\eta_\alpha\varphi_{\beta\gamma}),$$

$$\left[\frac{A}{2}(x^1)^2+B\right]^2\mathbf{C}(\mathbf{H}\overline{\mathbf{R}})_{\alpha 11\beta}=-\frac{\tilde{\tau}-2(n-1)(2n-1)}{4n(n+1)}(\tilde{g}_{\alpha\beta}+3\eta_\alpha\eta_\beta),$$

$$\left[\frac{A}{2}(x^1)^2+B\right]^2\mathbf{C}(\mathbf{H}\overline{\mathbf{R}})_{\alpha\beta\gamma\delta}=\left\{\tilde{\mathbf{R}}_{\alpha\beta\gamma\delta}-\frac{\tilde{\tau}+2(5n-1)}{4n(n+1)}(\tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma}-\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta})\right\}-$$

$$-\frac{\tilde{\tau}-2(n-1)(2n-1)}{4n(n+1)}[\varphi_{\alpha\delta}\varphi_{\beta\gamma}-\varphi_{\alpha\gamma}\varphi_{\beta\delta}-2\varphi_{\alpha\beta}\varphi_{\gamma\delta}].$$

But, on the right hand side we have just the components $(\mathbf{CK})_{1\alpha\beta\gamma}$, $(\mathbf{CK})_{\alpha 11\beta}$, and $(\mathbf{CK})_{\alpha\beta\gamma\delta}$, of the tensor (5.5).

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LOKALNO KONFORMNE KÄHLEROVE MNOGOSTRUKOSTI TIPA KONSTANTE I J-INVARIJANTNOG TENZORA KRIVINE

Mileva Prvanović

U radu su odredjene metrike svih lokalno konformnih Kählerovih mnogostrukosti tipa konstante i J-invarijantnog tenzora krivine i diskutovane odgovarajuće conformalne invarijante.

Ključne reči: Lokalno konformna Kählerova mnogostrukost, skoro Hermitian mnogostrukost tipa konstante, J-invarijantni tenzor krivine, Bochnerov i generalisani Bochner-ov tenzor, kompleksni kocirkularni tenzor krivine.