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**Invited Paper**

# REGULAR AND CHAOTIC BEHAVIOR EXHIBITED BY COUPLED OSCILLATORS WITH FRICTION

*UDC 534.141+531+530.1*

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**Abstract**. *Using a new approach, the nonlinear behavior of an autonomous 2-DOF mechanical system with friction is investigated. The domains of a chaotic motion are obtained in various sections of a three-dimensional driving parameter space. Chaotic and regular motions are detected and classified as stick-slip or slip-slip ones.*

## 1. INTRODUCTION

Our attention is focused on the non-smooth dynamical system analysis that has been attracting a wide spectrum of both mathematicians [1] and applied oriented researchers [2]. However, many of the problems occurring in this field are still far from being well understood and satisfactorily explained. Although the presented method is applicable to any dynamical system governed by ordinary differential equations, our considerations will concentrate on a two degree-of-freedom mechanical system with friction, similar to the systems already investigated by us [3-6]. On the other hand, this paper can be treated as both application and extension of the novel method reported in [7], which is especially suitable for estimation of regular and chaotic motions.

## 2. THE STUDIED SYSTEM

Consider two masses $m_1$ and $m_2$ (as shown in Fig. 1) moving on a driving belt that has constant velocity $v_0$. Mass $m_1$ is attached to the inertial space by spring $k_1$, while masses $m_1$ and $m_2$ are coupled by spring $k_2$. Friction force $T_i$ acting between mass $m_i$ and the belt

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depends on relative velocity $w_i$ ($i$=1,2). These two-degree-of-freedom autonomous oscillations are governed by the following second-order set of differential equations:

$$\begin{cases} m_1\ddot{x}_1 = -k_1 x_1 - k_2 x_1 + k_2 x_2 + T_1(w_1) \\ m_2\ddot{x}_2 = -k_2 x_2 + k_2 x_1 + T_2(w_2), \end{cases} \tag{1}$$

where: $w_i = v_0 - \dot{x}_i$, ($i$=1, 2).

Let us consider the following friction model (see Fig. 2):

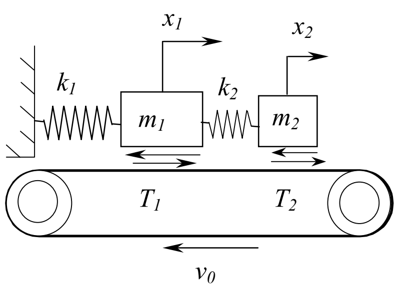



Fig. 1. Coupled oscillators with friction.

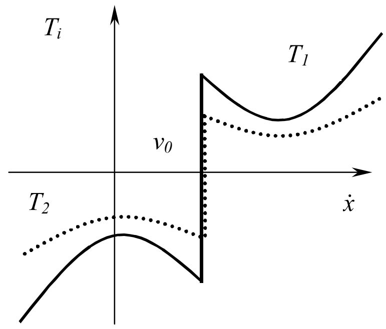



Fig. 2. Friction model.

$$T_i(w_i) = T_{0i} sign\ w_i - \alpha_i(T_{0i})w_i + \beta_i(T_{0i})w_i^3,$$

$$\alpha_i = \frac{3}{4}\frac{T_{0i}}{v_i^*}, \quad \beta_i = \frac{T_{0i}}{4(v_i^*)^3}$$

($i$=1, 2).

Here the maximum static friction force is denoted by $T_{oi}$ and $v_i^*$ is the velocity that corresponds to the local extremum value of $T_i(w_i)$ ($i$=1,2).

## 3. CONDITIONS FOR APPEARANCE OF CHAOS (STICK-SLIP AND SLIP-SLIP MOTIONS)

A chaotic behavior of nonlinear deterministic systems supposes wandering of the trajectories of motion around various equilibrium states. They are characterized by unpredictability and sensitive dependence on the initial conditions. By analyzing the trajectories of motion of these systems, it is possible to determine the regions of chaotic vibrations in the control parameter space.

Let a dynamical system be expressed as a set of ordinary differential equations

$$\dot{\mathbf{x}} = f(t, \mathbf{x}), \tag{2}$$

where $\mathbf{x} \in R^n$ is the state vector, $f(t,\mathbf{x})$ is defined in $R \times R^n$ and describes the time derivative of the state vector. It is supposed, that $f(t,\mathbf{x})$ is smooth enough to guarantee existence of a solution to set (2) as well as its uniqueness. The right-hand side of (2) can be discontinuous while the solution remains continuous. For instance, in the cases of discontinuous vector fields of "transversal intersection" and "attracting sliding mode" types a solution to set (2) exists and is unique. This property of the continuous dependence on the initial conditions $\mathbf{x}^{(0)} = \mathbf{x}(t_0)$ of the solution of set (2) will be used as follows: *for every initial condition* $\mathbf{x}^{(0)}$, $\tilde{\mathbf{x}}^{(0)} \in R^n$ *, for every number* $T > 0$*, no matter how large, and for every preassigned arbitrary small* $\varepsilon > 0$*, it is possible to indicate a positive number* $\delta > 0$ *such that if the distance* $\rho$ *between* $\mathbf{x}^{(0)}$ *and* $\tilde{\mathbf{x}}^{(0)}$ $\rho(\mathbf{x}^{(0)}, \tilde{\mathbf{x}}^{(0)}) < \delta$ *and* $|t| \le T$*, the following inequality*

$$\rho(\mathbf{x}(t), \tilde{\mathbf{x}}(t)) < \varepsilon$$

*is valid.*

This means that if the initial points are chosen close enough, then during the preassigned arbitrary large time interval $-T \le t \le T$ the distance between simultaneous positions of moving points will be smaller than a given positive number $\varepsilon$.

Being interested in tracing chaotic and regular dynamics, we shall suppose that with the increase of time all trajectories with remain in the closed bounded domain of a phase space, i.e.

$$\exists C_i \in R : \max_t |x_i(t)| \le C_i \qquad (i=1,2 \ldots n).$$

Note that if in any case a trajectory tends to infinity, it may be diagnosed easily.

To analyze the trajectories of motion of set (2), let us consider characteristic vibration amplitudes $A_i$ of the components of motion $x_i(t)$:

$$A_i = \frac{1}{2}\left| \max_{t_1 \le t \le T} x_i(t) - \min_{t_1 \le t \le T} x_i(t) \right|, \qquad (i=1,2 \ldots n).$$

Here $[t_1,T] \subset [t_0,T]$, $[t_0,T]$ is the time interval in the space of which the motion is considered, and $[t_0, t_1]$, $[t_0, t_1]$ is the time interval in the space of which all transient processes are damped.

From the many ways in which metric $\rho$ on $R^n$ can be assigned, for the sake of our investigations it seems the most convenient to use the embedding theorem and to consider an $n$-dimensional parallelepiped instead of a hyper-sphere with the center at point $\mathbf{x}$.

The *embedding theorem* may be expressed in the following way:

If $S_\varepsilon(\mathbf{x}) = \{\tilde{\mathbf{x}} \in R^n : \rho(\mathbf{x}, \tilde{\mathbf{x}}) < \varepsilon\}$ is a hyper-sphere with centre at point $\mathbf{x}$ and with radius $\varepsilon$ , and $P_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mathbf{x}) = \{\tilde{\mathbf{x}} \in R^n : |x_i - \tilde{x}_i| < \varepsilon_i\}$ is an $n$-dimensional parallelepiped, then for any $\varepsilon>0$ there is a parallelepiped $P_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mathbf{x})$ such that $P_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mathbf{x}) \subset S_\varepsilon(\mathbf{x})$. And conversely, for any parallelepiped $P_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mathbf{x})$ it is possible to indicate $\varepsilon>0$ such that $S_\varepsilon(\mathbf{x}) \subset P_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mathbf{x})$ .

In the parallelepiped $P_{\delta_1,\delta_2,\ldots,\delta_n}(\mathbf{x}^{(0)})$ let us choose two neighboring initial points $\mathbf{x}^{(0)}$ and $\tilde{\mathbf{x}}^{(0)}$ such that $|x_i^{(0)} - \tilde{x}_i^{(0)}| < \delta_i$, where $\delta_i$ is small in comparison with $A_i$ ($i$=1,2 … $n$). In the case of regular motion it is expected that $\varepsilon_i$ in inequality $|x_i(t) - \tilde{x}_i(t)| < \varepsilon_i$ is also small in comparison with $A_i$ ($i$=1,2 … $n$). The wandering orbits attempt to fill up some bounded domain of the phase space. At instant $t_0$ the neighboring trajectories diverge exponentially. Hence, for some instant $t_1$ the absolute values of differences $|x_i(t)-\tilde{x}_i(t)|$ can take any values in interval [0, $2A_i$] directly on the boundary values of this interval. If the differences $|x_i(t)-\tilde{x}_i(t)|$ are equal to zero for some instants $\{t_k^*\}$, ($t_k^* \in [t_1, T]$), then the trajectories $\mathbf{x}(t)$ and $\tilde{\mathbf{x}}(t)$ coincide at these instants. Obviously, $2A_i$ are the maximal values for these differences, and for some time instants this value is permissible. Let us introduce an auxiliary parameter $\alpha$, $0<\alpha<1$ and let $\alpha A_i$ be referred to as divergence measures of observable trajectories in the directions of generalized coordinates $x_i$ ($i$=1, 2, … $n$). By analyzing equation (2) and its equilibrium states it is easy to select parameter $\alpha$, $0<\alpha<1$, such that from the truth of the statement

$$\exists t^* \in [t_1, T] : |x_i(t^*) - \tilde{x}_i(t^*)| > \alpha A_i, \qquad (i=1, 2, \ldots n), \tag{3}$$

it follows that there is a time interval or a set of time intervals, for which the affixes of the trajectories $\mathbf{x}(t)$ and $\tilde{\mathbf{x}}(t)$, closed at the initial instant, move around various equilibrium states or these trajectories are sensitive to a change of the initial conditions. Thus, these trajectories are the wandering ones. Indeed, as it has already been mentioned, all trajectories are in the closed bounded domain of space $R^n$. With the aid of parameter $\alpha$ the divergence measures of the trajectories $\alpha A_i$ have been chosen, which is *inadmissible* for the case of 'regularity' of the motion. Note that this choice is non-unique and parameter $\alpha$ can take various values in interval (0, 1). It is clear, however, that if $\alpha$ is close to 0 and $|x_i(t)-\tilde{x}_i(t)| < \alpha A_i$ when $t \in [t_0, T]$, then the trajectories do not diverge and the trajectories are regular. There are values of parameter $\alpha$ that a priori correspond to the divergence measures of the trajectories $\alpha A_i$ ($i$=1, 2, … $n$) inadmissible in the sense of 'regularity'. For example, $\alpha \in \{1/3, 1/2, 2/3, 3/4\}$ or other choices are possible. If the representative points of the observable trajectories move chaotically, then for another choice $\alpha$ from the set of a priori 'appropriate' $\alpha$, the divergence of the trajectories will be recorded at another time instant $t^*$. As numerical experiments show, the obtained domains of chaotic behaviour with various a priori 'appropriate' values of $\alpha$ are practically congruent. Therefore, in this work figures for different values of $\alpha$ are not presented.

By varying parameters of the investigated space, and using condition (3), it is possible to find the domains of chaotic motions (including transient and alternating chaos) and those of regular motions.

To classify the motion of the considered oscillators as the one of stick-slip or slip-slip type, the following condition have been used:

$$v_0 max\Delta t_{st\,i} < \delta A_i \qquad (i = 1, 2). \tag{4}$$

Here $max\Delta t_{st\,i}$ is the maximum of "adhesion" times $\Delta t_{st\,i}$ ($i$ = 1, 2) of the first and the second oscillator to the belt during time interval $[t_1, T]$, in the space of which the motion is considered as a steady state. When the solution $\mathbf{x}(t)$ of equations (1) is obtained, the maximum "adhesion" times of the first and the second oscillators to the belt are determined from the conditions

$$\dot{x}_i = v_0, \qquad (i=1, 2)$$

for each selected trajectory. An auxiliary parameter $\delta < 1$ defines the smallness of the "stick"-segments on the phase plane in comparison with the characteristic vibration amplitudes $A_i$ of the components of motion $x_i(t)$ ($i$=1,2). Thus, if condition (4) holds, then the $i$-oscillator is in the slip-slip motion, and the other one is in the stick-slip motion.

## 4. DISCUSSION OF THE NUMERICAL SIMULATIONS

After a uniform coordinate sampling, the driving space defined by the maximum static friction force of the 1-st oscillator $T_{01}$, the maximum static friction force of the 2-nd oscillator $T_{02}$ , and the velocity of the belt $v_0$, have been investigated using conditions (3) and (4). The domains, in which chaotic behaviors of both the first and the second oscillator are possible, have been found, including the stick-slip and slip-slip motion. The coordinate sampling steps of the parameter space ($v_0$, $T_{01}$, $T_{02}$) are $\Delta v_0$= 0.05, $\Delta T_{01}$= 0.5, $\Delta T_{02}$= 0.5 in a rectangular parallelepiped (0< $v_0$ ≤4; 0< $T_{01}$ ≤50, 0< $T_{02}$ ≤50). The time period for the simulation is 240 time units. During computations, one half of the time period corresponds to the time interval [$t_0$,$t_1$], where transitional processes are damped. The integration step size is equal to $2.5\times10^{-3}$ time units. Initial conditions for the closed trajectories are distinguished by 0.5 percent in ratio to characteristic vibration amplitudes $A_i$ ($i$=1,2), and parameter $\alpha$ is equal to 1/3. Numerical calculations have been carried out for the following fixed values: $m_1$= 4, $m_2$= 2, $k_1$= 10, $k_2$= 7, $v_1^*$= 4, $v_2^*$= 3.

The regions displayed in section $T_{02}$=5 (a),(c) of parameter space ($v_0$, $T_{01}$, $T_{02}$) and in section $T_{01}$=15 (b),(d) in Fig. 3 are the domains, where chaotic vibrations (black color) of the first (a),(b) and of the second (c),(d) oscillators are possible including transient and alternating chaos. It is interesting that these regions for the first (a),(b) and for the second (c),(d) oscillators are almost congruent. In other words, during numerical simulations the situation when only one of the oscillators moves chaotically is not observed. The gray color presents the conditions for the stick-slip motion. In the present investigations parameter $\delta$ has been taken to equal 0.1. It is worth noting that in contrast to the chaotic behavior of the oscillators, there are conditions when only the first oscillator moves with "adhesion" to the belt and the other oscillator moves without "adhesion", and vice versa.

Fig. 4 shows the trajectories (a),(c) and Poincaré maps (b),(d) of the 1-st (a),(b) and the 2-nd (c),(d) oscillator on a phase plane ($v_0$=0.55, $T_{01}$=23.5, $T_{02}$=5) corresponding to the domains (see Fig. 3 (a), (c)) of chaotic vibrations and stick-slip motions. To obtain the Poincaré maps, the values ($x_k$, $v_k$), corresponding to the local minimums of the velocity of the oscillator, have been sampled. This has been done for the 1-st and for the 2-nd oscillator. The phase portraits and Poincaré maps plotted in Fig. 5 ($v_0$=0.85, $T_{01}$=15, $T_{02}$=11.5) correspond to the domains of chaotic behavior and stick-slip motions shown in Fig. 3 (b),(d).

Fig. 6 ($v_0$=3.35, $T_{01}$=38.5, $T_{02}$=5) and Fig. 7 ($v_0$=2.6, $T_{01}$=15, $T_{02}$=2.5) display transient chaos. From the beginning, the oscillators move chaotically and then alight to a periodic regime (limit cycle). In Fig. 6, the 1-st oscillator moves without a stick condition, the 2-nd oscillator is in the stick-slip motion. Vice versa, in Fig. 7: the 2-nd oscillator moves without "adhesion" to the belt and the 1-st oscillator is in the stick-slip motion.

Fig. 8 ((a) $v_0$=1.1, $T_{01}$=48, $T_{02}$=5; (b) $v_0$=2.72, $T_{01}$=15, $T_{02}$=46.58) displays periodic vibrations of both oscillators. In Fig. 8 (a) the 1-st oscillator is in the stick-slip motion and the 2-nd one moves without "adhesion" to the belt. In Fig. 8 (b) the situation is reversed. All these data demonstrate good agreement with the obtained chaotic and regular vibration domains and with the domains of stick-slip and slip-slip motion (see Fig. 3).

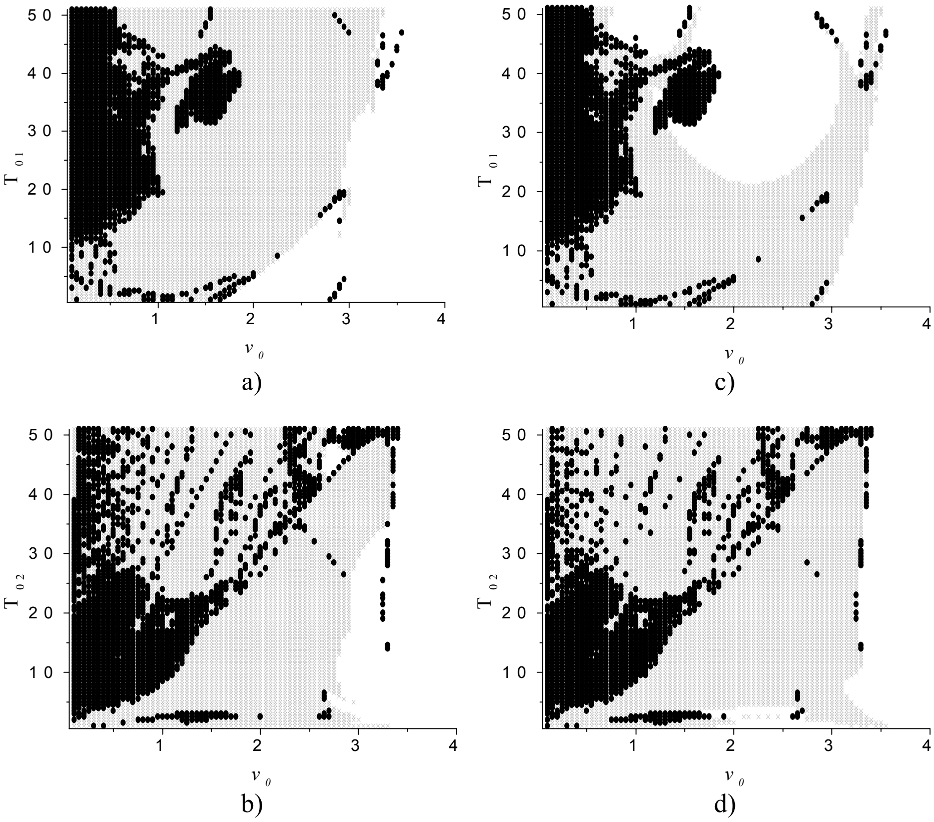
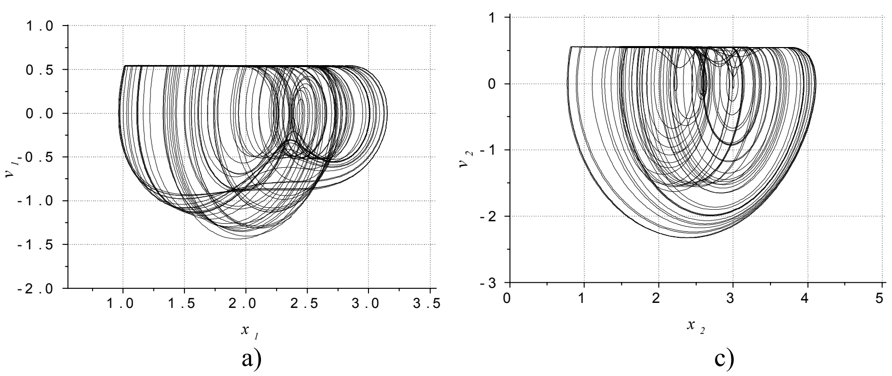
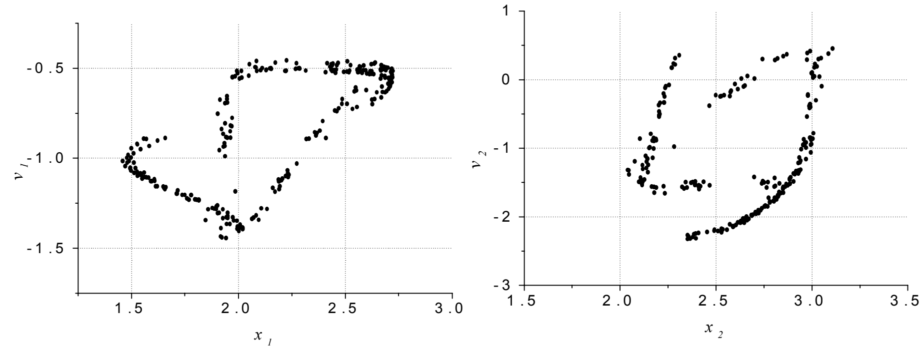


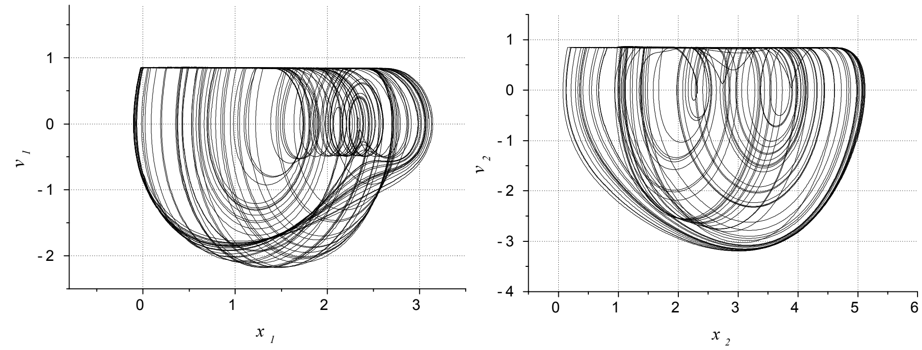

Fig. 3. Domains of chaotic (black) and stick-slip (gray) motion of the first (a), (b) and the second (c), (d) oscillator in sections of space ($v_0$, $T_{01}$, $T_{02}$): (a), (c) $T_{02}$=5, (b), (d) $T_{01}$=15.



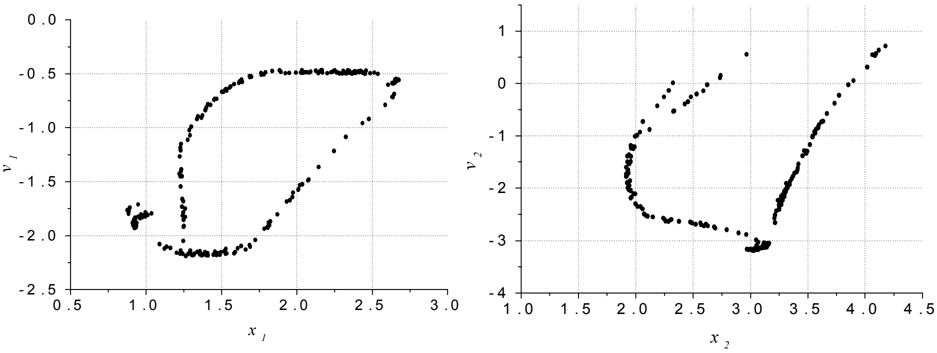



b) d)

Fig. 4. Phase portraits and Poincaré maps of chaotic trajectories of the first (a), (b) and the second (c), (d) oscillator for $v_0$=0.55, $T_{01}$=23.5, $T_{02}$=5.




a) c)




b) d)

Fig. 5. Phase portraits and Poincaré maps of chaotic trajectories of the first (a), (b) and the second (c), (d) oscillator for $v_0$=0.85, $T_{01}$=15, $T_{02}$=11.5.

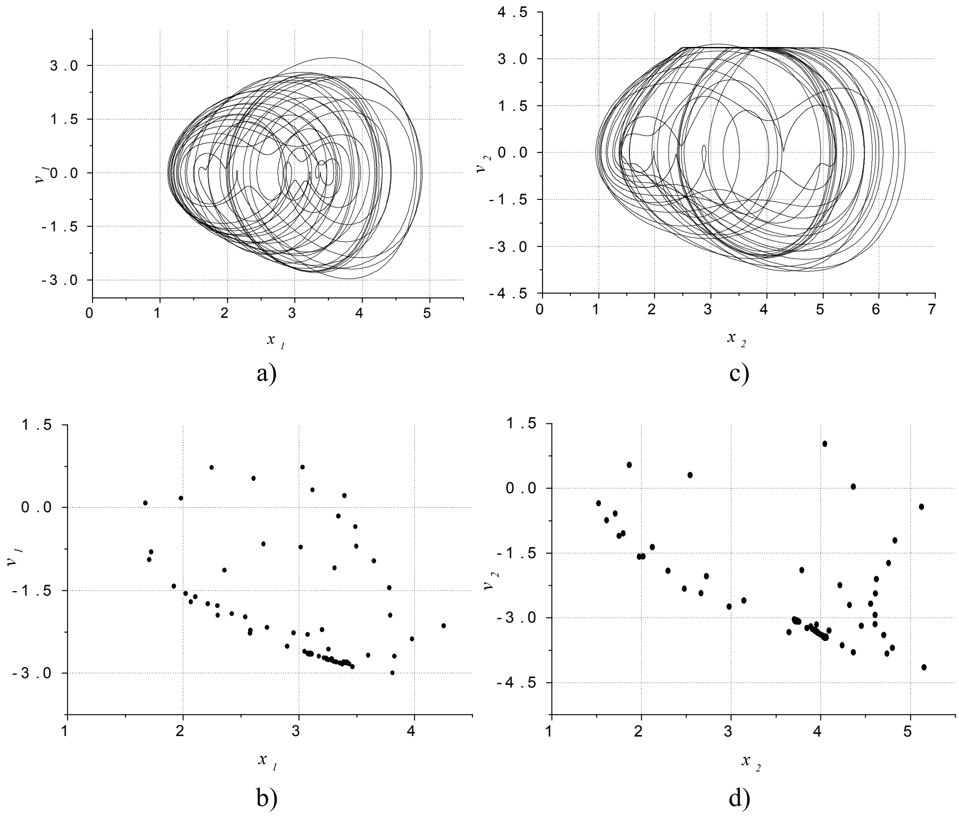
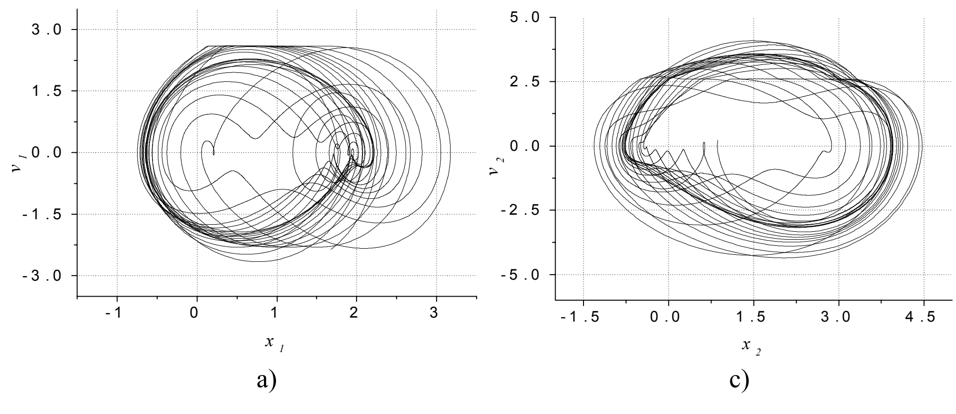
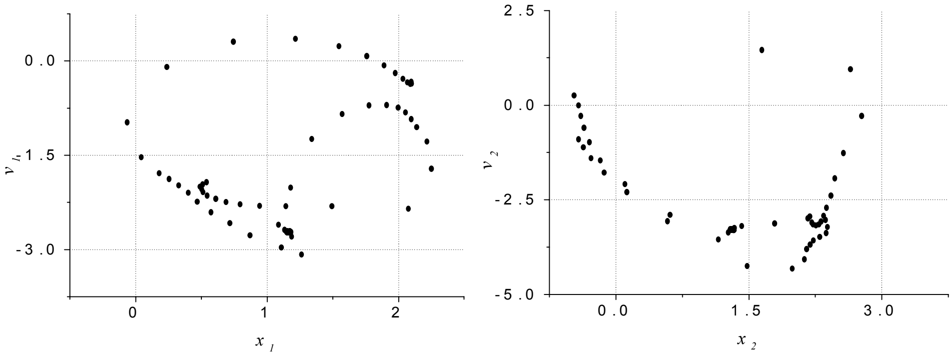


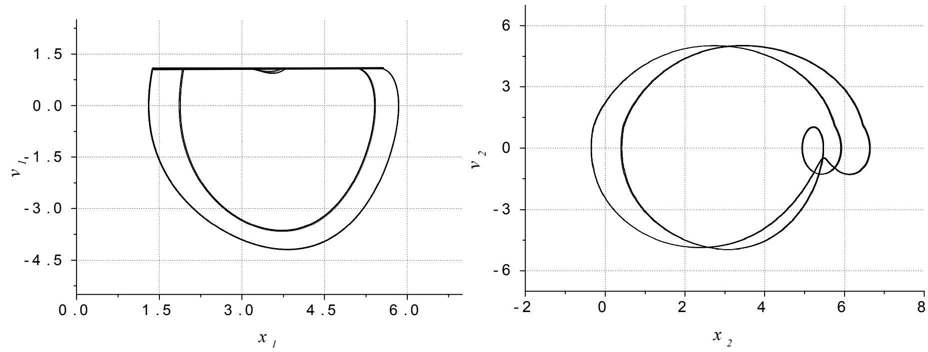

Fig. 6. Transient chaos; phase trajectories and Poincaré maps of the first (a), (b) and the second (c), (d) oscillator for $v_0$=3.35, $T_{01}$=38.5, $T_{02}$=5.



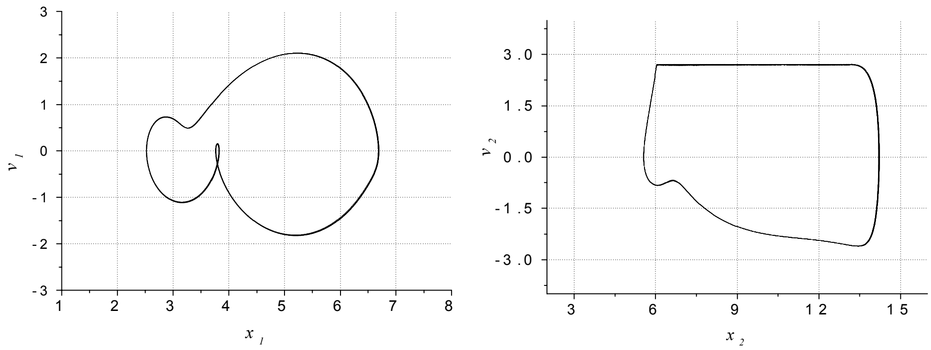



b) d)

Fig. 7. Transient chaos; phase trajectories and Poincaré maps of the first (a), (b) and the second (c), (d) oscillator for $v_0$=2.6, $T_{01}$=15, $T_{02}$=2.5.




(a)




(b)

Fig. 8. Phase portraits of regular trajectories:
(a) $v_0$=1.1, $T_{01}$=48, $T_{02}$=5; (b) $v_0$=2.72, $T_{01}$=15, $T_{02}$=46.58.

Thus, mainly the stick-slip chaos is exhibited by our system in the considered conditions.

## 5. CONCLUSIONS

This paper deals with two parts of the investigation. The first one presents a novel numerical approach which, in contrast to the standard numerical methods (including computations of Lyapunov exponents), is effective, convenient to use, requires much less computational time in comparison with other approaches, and can be applied to the investigation of both "smooth and non-smooth" problems.

In the second part, the regular and chaotic stick-slip and slip-slip dynamics of the studied system of two degrees of freedom with friction are reported and analyzed in some detail. It has been shown, among others: (i) how only the first oscillator moves with "adhesion" to the belt (Figure 3); (ii) how both oscillators exhibit stick-slip chaos (Figure 4, 5); (iii) transient chaos (Figure 6,7); (iv) stick-slip periodic motion of either the first or the second oscillator (Figure 8).

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# REGULARNO I HAOTIČNO PONAŠANJE IIZVEDENO POMOĆU SPREGUNTIH OSCILATORA SA TRENJEM

**J. Awrejcewicz, L. Dzyubak**

*Koristeći novi pristup nelinearno ponašanje autonomnog mehaničkog sistema sa dva stepena slobode kretanja i sa trenjem je istraživano. Dobijena je oblast haotičnog ponašanja u različitim oblastima trodimenzionalnog parametarskog prostora. Haotično i regularno kretanje je detektovano i klasikovano kao stick-slip (lepljivo-sklizavog) ili slip-slip (sklizavo-sklizavog) karaktera.*