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DISCRETE FOURIER TRANSFORM IN THE PROBLEM OF THE WAVE PACKET DYNAMICS

UDC 531

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Abstract. *The method of the wave packet dynamic description for Tollmien-Schlichting waves in boundary layer flow of incompressible fluid is suggested. The method is based on combined using of the one-mode spectral wave components equation and the wave packet envelope equation. This approach is available when splitting of nonlinear equations is used for linear and nonlinear parts at each time step. The linear part can be solved with the use of the wave packet spectral component equation and then we transform the field from the wave number space to the physical space. In the physical space we solve the system of ordinary differential equations with the subsequent inverse Fourier transformation in the wave number space. As a procedure of discrete Fourier transformation a standard FFT is used.*

1. INTRODUCTION

A boundary layer (BL) in incompressible fluid on a plate is considered. The dynamic of finite spectral size disturbances in BL is of great interest lately. The matter is that the phenomena in the laminar part of BL can have analogies in the turbulent part of BL. In addition to that the models of development of disturbance can be of great interest to transition prediction.

Experiment shows [1] that in real cases the transition of the laminar motion of gas to the turbulent motion is connected with an appearance of WP (Tollmien-Schlichting waves) of a finite spectral size in the laminar part of BL which develops in downstream direction in a linear and then in a weakly nonlinear manner. The appearance of the strong nonlinearity really corresponds to the transition point. The paper [2] is devoted to the linear dynamics of disturbances in BL. The nonlinear dynamics at an early stage is reflected in [3-6]. The results of the papers [2, 5] can be used for comparison of the results when the effectiveness of the models is assessed.

Disturbances of the fluid motion have some components which correspond with the wave types excited in BL: the Tollmien-Schlichting and the Squire waves of descrete and

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continuous spectrum. The influence of the Tollmien-Schlichting waves of continuous spectrum.on the disturbance dynamics is considered in papers [7, 8]. Further the waves of continuous spectrum will not be taken into consideration for simplicity.

The solution of Navier-Stokes equations is not convenient for a wave packet (WP) description because of the small value of disturbance amplitude in comparison with the base flow. So it is interesting to highlight the dynamics of WP. It is possible if we take into account that only one Tollmien-Schlichting mode is excited as a rule.

Even in a short-cut form the nonlinear equations in three-wave resonance approximation are difficult to solve numerically because of integrodifferential equation for "0-packet" (the new element of the three-wave resonance dynamics which is a set of harmonics at wave number space origin) that arises due to the finite size of WP. But it is very easy to solve a linear problem WP dynamics in the wave number space. In physical space we can calculate amplitude distribution with the help of Fourier transform. So the method of solution of general problem arises as the splitting of the whole operator in the linear and the nonlinear part. The linear part must be solved in wave number space, the nonlinear part – in physical space. We can realize the connection between this stages with the help of fast Fourier transform (FFT).

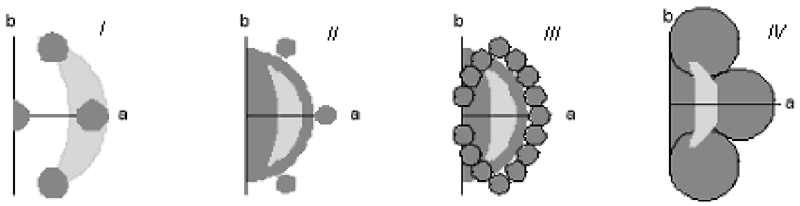

Fig. 1.

The solution of the spectral problem for the Tollmien-Schllichting waves of discrete spectrum shows [9] that the set of unstable wave numbers in the wave number space is a compact region near wave number space origin. The following variants of WP with amplitude downstream increase are possible (see Fig.1, k=(a, b), the instability region is tagged by lighter gray level): **I**, the resonant triplet with base harmonics in unstable region plus 0-harmonics (the singular part of WP); **II**, the singular region covers the unstable region (we can describe the amplification of the wave by the integral operator) and the resonant triplet is placed out of the unstable region; **III**, the singular region covers the unstable region and discrete modes correspond to the multiple three-wave resonance; **IV**, the singular region, the resonance triplet and the unstable region are overlapped.

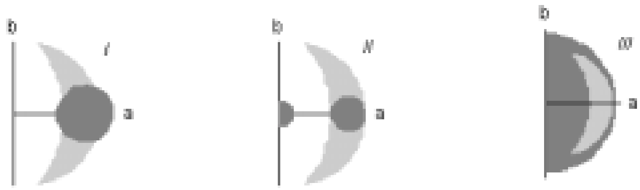

Fig. 2.

Further simpler situations are considered (Fig.2) for the illustration of the method: **I**, WP covers the unstable region, the linear problem; **II**, WP has two different components: the singular part and a separate harmonics in the unstable region, the nonlinear problem; **III**, WP covers the singular region and the unstable region, the linear problem.

2. EQUATIONS FOR HARMONICS AND EQUATIONS FOR THE ENVELOPE OF WP

In papers [3, 6] the equation which describes space-time dynamics of spectral components f_k of disturbance localized in space is deduced.

$$\left(\frac{\partial}{\partial \overline{t}} + i[\overline{\omega}^R(\vec{k}) - (\dot{X}_0\vec{k})]\right) f_{\vec{k}} - \varepsilon^2 \frac{\partial}{\partial \alpha}\left(\frac{\partial \omega^R}{\partial \overline{X}_0} f_{\vec{k}}\right) - \varepsilon^2 (\overline{\omega}^I(\vec{k}) + Q(\vec{k})) f_{\vec{k}} + \\ = \varepsilon \int H_{\vec{k}\vec{k}'} f_{\vec{k}'} f_{\vec{k}-\vec{k}'} d\vec{k}' + \ldots$$

Here $\overline{\omega}(\mathbf{k}) = \overline{\omega}^R(\mathbf{k}) + i\overline{\omega}^I(\mathbf{k})$ is a non-dimensional eigenvalue of the Orr-Sommerfeld spectral problem (frequency of the oscillation of the Tollmien-Schlichting waves) which has a region of instability in the wave number space, $\mathbf{k} = (\alpha,\beta)$ is a wave vector, $\overline{X}_0(t)$ is a point in the flow which moves together with the flow and from which we look at a disturbance (the point of origin of moving system of reference), the values Q and H are defined in [3, 6], ε is a small parameter. It is interesting to construct an approximation to $\overline{\omega}(k)$ in the vicinity of the wave number space origin because the instability region is placed near the origin ($k \approx 0.1 \div 0.3$). In [3,6,9] the formula is obtained:

$$\overline{\omega}(\vec{k}) = \overline{\omega}^R(\vec{k}) + i\varepsilon^2\overline{\omega}^I(\vec{k}) = \alpha a(\overline{X}_0) + b(\overline{X}_0)\alpha k + i\varepsilon^2 d(\overline{X}_0)$$

The parameters of it are defined in the Table:

R	a	b	$\varepsilon^2 d$
500	0.24610402	0.96970043	−0.00256494
1000	0.21921957	0.94623006	−0.00177655
1500	0.20552628	0.91792750	−0.00145604
2000	0.19858133	0.88027801	−0.00122656
3000	0.19144657	0.85439075	−0.00100397

Some initial configurations of spectral components against a background instability region are suggested in Fig.1. The spectral dimensions of the distinct initial WP can be small (variant I). The dynamics of such a WP in physical space is defined by its small neighborhood in wave number space. Otherwise, the amplification/attenuation of a disturbance in physical space is defined by the integral operator, and it acts on all the region of WP location (variant IV). There exist intermediate cases (variants II and III).

In connection with these variants the problem of number modeling arises. A system of partial differential equations for small spectral size WP can be deduced with additional integrodifferential equation for WP which is placed in the vicinity of wave number space origin (0 - packet). It complicates the solution of the problem, because the linear integral operator is singular in physical space. But in wave number space this task is not singular. That is why the idea of splitting the whole operator into two parts arises: linear and nonlinear.

3. NARROW WP

In papers [3, 6] the dynamic equations of WP envelope in the laminar part of BL on the plate are deduced. They are composed of integrodifferential equation for 0-packet which is subjected to the interaction with three-wave resonant harmonics and selfinteraction. On the other hand, 0-packet influence three- wave resonant harmonics which is described by the nonlinear Schredingerian-type equation. These equations are as follows:

$$\frac{\partial \Psi^{(0)}}{\partial t}(t,\vec{r}_1) - \varepsilon(\dot{\bar{X}}_0 - a(\bar{X}_0))\frac{\partial \Psi^{(0)}}{\partial x_1}(t,\vec{r}_1) = \varepsilon^2 I^{(0)};$$

$$\frac{\partial \Psi_j^{(n)}}{\partial t} + i(q(\hat{P}_j^{(n)}) - q(n\vec{k}_j))\tilde{\Psi}_j^{(n)} + i\varepsilon x_1 \frac{\partial \omega^R}{\partial \bar{X}_0}(\hat{P}_j^{(n)})\tilde{\Psi}_j^{(n)} = \varepsilon^2 I_j^{(n)}$$

where

$$I_j^{(n)} = (\bar{\omega}(\hat{P}_j^{(n)}) + Q(\hat{P}_j^{(n)}))\tilde{\Psi}_j^{(n)}(t,\vec{r}_1) + ix_1^2 P^{(2)}(\hat{P}_j^{(n)})(n\alpha)\tilde{\Psi}_j^{(n)}(t,\vec{r}_1) +$$

$$+\sum_{l=-\infty}^{\infty} H(n\vec{k}_j, l\vec{k}_j)\tilde{\Psi}_j^{(n)}(t,\vec{r}_1{}')\tilde{\Psi}_j^{(n)}(t,\vec{r}_1{}'')e_J^{(l)}e_J^{(n-l)}e_J^{(-n)} + V_J^{(n)}$$

$$\hat{P}_j^{(n)} = n\vec{k}_j + \varepsilon\frac{\nabla}{i},\ q(\vec{k}) = \bar{\omega}^R(\vec{k}) - \dot{\bar{X}}_0\alpha;\ \Psi_j^{(n)} = e_j^{(n)}\tilde{\Psi}_j^{(n)};$$

$$\vec{s} = (\xi,\eta),\ e_j^{(n)} = \exp\{-i\int_0^t q(n\vec{k}_j)dt\},$$

In the last equation $V_j^{(n)}$ are defined as follows:

$$V_1^{(n)} = \{H(n\vec{k}_1, n\vec{k}_2) + H(n\vec{k}_1, n\vec{k}_3)\}\tilde{\Psi}_2^{(n)}(\vec{r})\tilde{\Psi}_3^{(n)}(\vec{r});$$
$$V_2^{(n)} = \{H(n\vec{k}_2, n\vec{k}_1 + H(n\vec{k}_2, n\vec{k}_3))\}\tilde{\Psi}_1^{(n)}(\vec{r})\tilde{\Psi}_3^{(-n)}(\vec{r});$$
$$V_3^{(n)} = \{H(n\vec{k}_3, n\vec{k}_1) + H(n\vec{k}_3, n\vec{k}_2)\}\tilde{\Psi}_1^{(n)}(\vec{r})\tilde{\Psi}_2^{(-n)}(\vec{r});$$

The value of $H_{0,0}$ was defined as the limit of $H_{\vec{k},\ \vec{k}-\vec{k}_1}$ where $\vec{k}$ is directed along longitudinal axis in the wave number space $\vec{k}$ and $\vec{k}-\vec{k}_1$ are its subharmonics at $\vec{k}\to 0$. It turned out that and behave like power functions $\alpha^{0.72}$ and $\alpha^{0.45}$ at $\alpha\to 0$. The renormalisation of the initial equation ($f_{\vec{k}} = k^{\mu}\ \ \tilde{f}_{\vec{k}}$, $\mu \cong -0.72$), results in finite and real, and gives regularization to model equations from [3, 6]. The left part the equation operator with accuracy to within $O(\varepsilon^2)$ can be presented in the following way:

$$\frac{\partial\tilde{\phi}_j^{(n)}(\mathbf{r},t)}{\partial t} + \varepsilon\left[\left.\frac{\partial\bar{\omega}^R}{\partial\mathbf{k}}\right|_{\mathbf{k}=n\mathbf{k}_j}\cdot\nabla - \dot{X}_0\frac{\partial}{\partial x}\right]\tilde{\phi}_j^{(n)}(\mathbf{r},t) - i\frac{\varepsilon^2}{2}\left(\nabla\cdot\frac{\partial}{\partial\mathbf{k}}\right)^2\bar{\omega}^R\Big|_{\mathbf{k}=n\mathbf{k}_j}\tilde{\phi}_j^{(n)}(\mathbf{r},t),$$

$$\nabla = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right),\ \frac{\partial}{\partial\mathbf{k}} = \left(\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\beta}\right)$$

4. SOME SPECIAL CASES

In the case I (fig. 2) the instability region can be approximated with the help of polinomials. In this case the dynamic of Gaussian WP at the initial moment is described by the quadrature:

$$\Psi(x,z,t)=\int_{-\infty}^{\infty}d\beta\int_{-\infty}^{\infty}d\alpha\exp(-i\Phi_1+t\Phi_2-\Phi_3+i((\alpha-0.3)x+\beta z))\cong$$
$$\cong 0.293\exp(-0.25i(12+(0.3+0.6ib)t+ix)^2)/((b-i)t-20i)-$$
$$-(0.0125+0.09ib)t-0.3ix+0.5iz^2/((b-0.4i)t-40i)/$$
$$/\sqrt{(20+i+ibt)(6.366+0.0.637t+0.159ibt)}$$

$$\Phi_1=(2b(\alpha-0.3)^2+b\beta^2)/2,\quad \Phi_2=0.01-(\alpha-0.15)^2-\beta^2/5,\quad \Phi_3=((\alpha-0.3)^2+\beta^2)/0.05$$

The same problem can be solved in linear approximation with the help of discrete Fourier transformation. The comparison of this result with the result of [2, 5] shows the quality correspondence between them. A similar solution can be found for case III.

The numerical solution easily reproduces the case when the WP support contains the origin of wave number space and the instability region (case II). Selfinteraction is taken into consideration at the same time. The splitting scheme can be viewed as:

$$\frac{\partial\Psi_0}{\partial t}=H_{00}\Psi_0^2,\quad \frac{\partial f}{\partial t}=-i\Omega f$$

where $i\Omega=(i[\overline{\omega}^R(\vec{k})-(\dot{X}_0\vec{k})])-\varepsilon^2(\overline{\omega}^I(\vec{k})+Q(\vec{k}))$. The result shows that the initial Gaussian WP transforms into WP located in the instability region, and then begins the stage of dispersion of the WP in the physical space. This result can explain the phenomena, described in [10, 11, 4], in which the transformation of the initial long wave WP into short wave WP is observed. The solution of the nonlinear problem of the 0-packet selfinteraction is submitted in fig. 3. Different fragments correspond to the dimensionless time showed in fig.3.

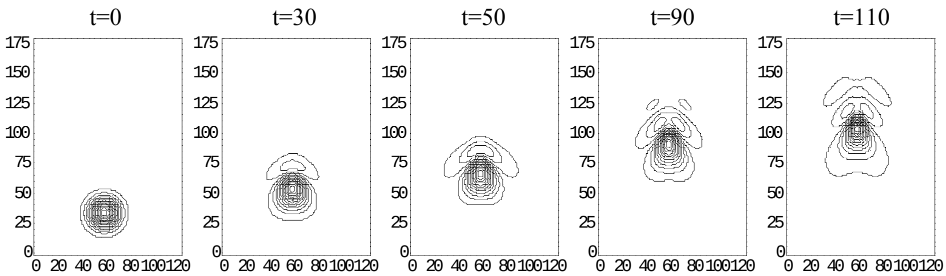

Fig. 3.

These results demonstrate the possibilities of the numerical solution of the weakly nonlinear WP dynamics which can be used for the cases with more complicated configuration of the WP localization.

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DISKRETNA FOURIER-OVA TRANSFORMACIJA U PROBLEMIMA DINAMIKE SKUPA (SVEŽNJA) TALASA

V.A. Zharov

Metoda dinamike talasnog paketa opisana za Tolmien –Schlichting-ove talese u struji graničnog sloja u nestišljivom fluidu je predložena. Metoda se bazira na kombinovanom korišćenju jednačina za jedno-modne spektralne talasne komponente i jednačine obvojnice svežnja talasa. Taj pristup je pogodan za razdvajanje nelinearnih jednačina i koristi se za linearne i nelinearne delove u svakom stepenu vremena.

U proceduri je korišćena diskretna Fourier-va trasformacija u obliku brze Fourier-ve transformacije.