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LINE SOURCE ABOVE NON-LINEAR INFINITE PLATE

UDC 621

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Abstract. *Very large steady uniform line charge is placed in two-layer medium, in the upper air layer, on the known height from the separating surface. The lower layer is isotropic, but non-linear, having known constitutive relation. Using Maxwell's equations electric potential and electric field are determined as solution of Poisson's equations in both layers, including existing boundary conditions that the potential and normal electric induction components are unchanged on the separating surface. Afterwards integral transform method is used. The working line and working point of non-linear layer are determined numerically and graphically, using one new and simple approximation of the non-linear medium characteristics.*

1. INTRODUCTION

The solution of electromagnetic field problems in non-linear media is very complicated, while in that case Maxwell's equations are non-linear partial differential equations, which solution depends on the initial conditions and earlier media situation, so the solutions are not unique. The powerful modern numerical approaches are also not useful in these purposes, because then are often without convergence and stability of the solutions. The choice of initial values in appropriate iterative procedures is of large influence on the validity and the accuracy of the final solution.

This paper is aimed to proposing one semi-analytical approach for solving non-linear electrostatic problem, when very large uniform line steady charge is placed in two-layer medium and one of them is isotropic, but non-linear. Using Maxwell's equations and existing boundary conditions on the separating surface electric scalar potential functions are determined as the solution of Laplace's equations in both layers. Then integral transform method is used and working line and working point of non-linear media are determined graphically and numerically, using one very simple, but useful approximation of polarization characteristic and depolarization curve of non-linear layer.

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2. SHORT THEORETICAL APPROACH

Very large line steady charge of uniform density q' is placed in two-layer medium in the upper layer on the height h from the separating surface, as in Fig. 1. The lower layer is isotropic, but non-linear, having known constitutive relation

$$D = \varepsilon_0 E + P = f(E)\,, \tag{1}$$

where:
- D is electric induction;
- E is electric field intensity;
- P is polarization; and
- ε_0 is free space permittivity.

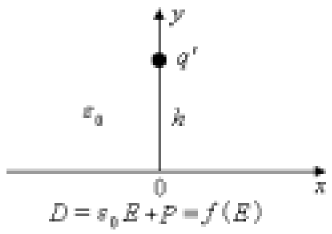


Fig. 1. Uniform line charge over non-linear infinite media.

Electric potentials, φ_0, in the upper layer, and in non-linear layer, φ, satisfy Poisson's equations,

$$\Delta\varphi_0 = \frac{\partial^2\varphi_0}{\partial x^2} + \frac{\partial^2\varphi_0}{\partial y^2} = -\frac{q'}{\varepsilon_0}\delta(x)\delta(y-h) \tag{2}$$

and

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{\operatorname{div} P}{\varepsilon_0}\,, \tag{3}$$

where $\delta(x)$ and $\delta(y-h)$ are one-dimensional Dirac's δ - functions.

After determining potential functions, the vectors of the electric field and electric inductions can be put as

$$E_0 = -\operatorname{grad}\varphi_0\,, \tag{4}$$

$$D_0 = \varepsilon_0 E_0 \tag{5}$$

and

$$E = -\operatorname{grad}\varphi\,, \tag{6}$$

$$D = \varepsilon_0 E + P\,, \tag{7}$$

where zero index is related to the value in upper air layer.

The presented Poisson's equations should be solved using following boundary conditions:

$$\varphi_0(y=+0) = \varphi(y=-0) \tag{8}$$

and

$$\mathrm{D}_{0y}(y=+0)=D_y(y=-0)\,, \tag{9}$$

including the requirement that the potential is finite on the large distance from separating surface, when $y \to \pm\infty$.

Using integral transform method, the potential in the upper layer can be expressed as

$$\varphi_0=\begin{cases}\displaystyle\int_0^\infty\left[\frac{q'}{2\pi\varepsilon_0 \mathrm{p}}\mathrm{e}^{p(y-h)}+D_p e^{-py}\right]\cos(px)dp, & 0\le y\le h\\ \displaystyle\int_0^\infty\left[D_p+\frac{q'}{2\pi\varepsilon_0 p}e^{ph}\right]e^{-py}\cos(px)dp, & h\le y\le\infty,\end{cases} \tag{10}$$

where D_p is unknown depending on the value p.

In regard to this solution, the polarization and the potential in lower layer, $-\infty < y \le 0$, can be supposed in the forms

$$P=\int_0^\infty[A_\mathrm{p}\cos(px)+B_\mathrm{p}\sin(px)]e^{py}dp \tag{11}$$

and

$$\varphi=\int_0^\infty F_p(x)e^{py}dp\,. \tag{12}$$

Substituting (11) and (12) into Poisson's equation (3), the value F_p satisfies the following differential equation

$$F_p''+p^2F_p=p[R_p\cos(px)+S_p\sin(px)]\,, \tag{13}$$

where

$$R_p=(B_{px}+A_{py})/\varepsilon_0\,, \tag{14}$$

$$S_p=(B_{py}-A_{px})\;\;/\varepsilon_0\,, \tag{15}$$

$$A_p=A_{px}\hat{x}+A_{py}\hat{y}\,, \tag{16}$$

$$B_\mathrm{p}=B_{px}\hat{x}+B_{py}\hat{y} \tag{17}$$

and $\hat{x}$ and $\hat{y}$ are unit vectors associated to the x and y co-ordinate lines.

Integrating differential equation (13) by help of the variation constants method, the function F_p can be presented as

$$F_p=\left(D_{1p}+\frac{R_p}{4p}\right)\cos(px)+\left(D_{2p}+\frac{S_p}{4p}\right)\sin(px)+\frac{x}{2}[R_p\sin(px)-S_p\cos(px)]\,. \tag{18}$$

Using boundary conditions (8) and (9), the unknown constants can be determined as

$$R_p = 0\,,\; S_p = 0\,,\; D_{2p} = 0\,,\; D_{1p} = D_p + \frac{q'}{2\pi\varepsilon_0 p} e^{-ph}\,,$$

$$B_{px} = -A_{py} \text{ and } B_{py} = A_{px}\,. \tag{19}$$

Finally,

$$\varphi = \int_0^\infty \left(D_p + \frac{q'}{2\pi\varepsilon_0 p} e^{-ph} \right) \cos(px) e^{py} dp\,, \tag{20}$$

$$P_x = -2\varepsilon_0 \int_0^\infty pD_p \sin(px) e^{py} dp\,, \tag{21}$$

$$P_y = 2\varepsilon_0 \int_0^\infty pD_p \cos(px) e^{py} dp\,, \tag{22}$$

$$\mathrm{E_x} = \int_0^\infty \left(pD_p + \frac{q'}{2\pi\varepsilon_0 p} e^{-ph} \right) \sin(px) e^{py} dp\,, \tag{23}$$

$$E_y = -\int_0^\infty \left(pD_p + \frac{q'}{2\pi\varepsilon_0 p} e^{-ph} \right) \cos(px) e^{py} dp\,, \tag{24}$$

$$D_x = \int_0^\infty \left(\frac{q'}{2\pi} e^{-ph} - p\varepsilon_0 D_p \right) \sin(px) e^{py} dp \tag{25}$$

and

$$D_y = \int_0^\infty \left(p\varepsilon_0 D_p - \frac{q'}{2\pi} \mathrm{e}^{-ph} \right) \cos(px) e^{py} dp\,. \tag{26}$$

In order to create the working line equation the following expressions will be observed:

$$D_x + \varepsilon_0 E_x = \frac{\mathrm{q}'}{\pi} \int_0^\infty \sin(px) e^{p(y-h)} dp = \frac{q'}{\pi} \frac{x}{\sqrt{x^2 + (h + |\,y\,|)^2}} \tag{27}$$

and

$$D_y + \varepsilon_0 E_y = -\frac{\mathrm{q}'}{\pi} \int_0^\infty \cos(px) e^{p(y-h)} dp = -\frac{q'}{\pi} \frac{x + |\,y\,|}{\sqrt{x^2 + (h + |\,y\,|)^2}}\,. \tag{28}$$

Since the lower layer is isotropic, the electric field and induction vectors are collinear, so the following expressions exist:

$$D_x = D\cos\theta\,,\; D_y = D\sin\theta\,,\; E_x = E\cos\theta \text{ and } E_y = E\sin\theta\,, \tag{29}$$

and the working line equation is

$$D + \varepsilon_0 E = \frac{q'}{\pi\sqrt{x^2 + (h + |\,y\,|)^2}}\,, \tag{30}$$

where electric field vector direction is determined with

$$\theta = -\operatorname{arctg}\frac{h+|y|}{x}. \tag{31}$$

All points of the circle line

$$x^2 + (h+|y|)^2 = d^2, \tag{32}$$

with radius $d \geq h$ and centre when point charge is placed, $x = 0$, $y = h$, have the some working line,

$$D + \varepsilon_0 E = \frac{q'}{\pi d}. \tag{33}$$

The working point position is determined as the crossing of the working line and the curve describing non-linear dependence between electric induction and electric field, $D = f(E)$. Then two special cases can be separately observed:

a) For the first polarization of non-linear material the primary polarization characteristic is useful, as Fig. 2 shows.

Working line "a" corresponds to the point $x = 0$, $y = 0$ and has the largest values of the electric field and electric induction, for $d = h$. All other lines are with $d > h$ (Working line "b").

The position of the working point can be determined analytically, if the polarization curve is approximated as

$$D = \frac{1}{2}D_\mathrm{s}\left[1 + \frac{2}{\pi}\operatorname{arctg}\left(a\frac{E_\mathrm{s}}{E_\mathrm{s} - E} - b\frac{E_\mathrm{s}}{E}\right)\right],\ 0 \leq E \leq E_\mathrm{s} \text{ and}$$

$$D = \varepsilon_0 (E - E_s) + D_s,\ E_s \leq E \tag{34}$$

where E_s and D_s are intensity of electric field and electric induction of saturated polarization (All electric dipoles are oriented in the direction of the motive electric field.). Presented approximation (34) contents two characteristic points, $E = 0$, $D = 0$ and $E = E_s$, $D = D_s$. Using condition

$$\left.\frac{dD}{dE}\right|_{E=E_s-0} = \left.\frac{dD}{dE}\right|_{E=E_s+0} \tag{35}$$

it determines

$$a = \frac{1}{\pi}\varepsilon_{rs},\ \varepsilon_{rs} = \frac{\varepsilon_s}{\varepsilon_0} \text{ and } \varepsilon_s = \frac{D_s}{E_s}. \tag{36}$$

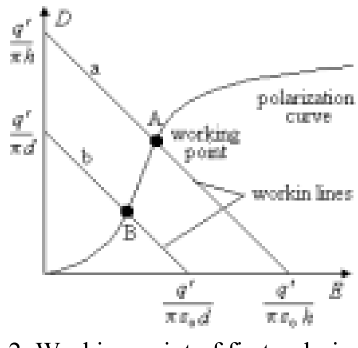
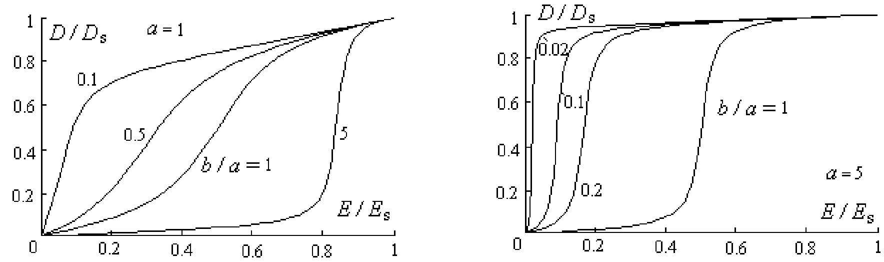


Fig. 2. Working point of first polarization



a) $a = 1$, $b = 0.1a$, $0.5a$, a and $5a$. b) $a = 5$, b $0.02a$, $0.1a$, $0.2a$ and a.

Fig. 3. Approximation of polarization curve.

So the value b determines the shape of the approximation (34), as Figs. 3a and 3b show.

b) The polarization process is finished and line charge is eliminated. Then $q' = 0$ and the working line equation is the same for the whole lower layer,

$$D + \varepsilon_0 E = 0\,. \tag{34}$$

Then working point is in cutting of working line and depolarization curve, Fig. 4.

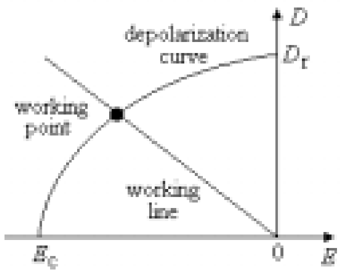


Fig. 4. Working line in the case of permanent polarization.

The following simple expression is useful in depolarization curve approximation,

$$\left|\frac{E}{E_c}\right|^a + \left(\frac{D}{D_r}\right)^a = 1\,,\ E_c \le E \le 0\,,\ 0 \le D \le D_r\,, \tag{35}$$

where:

E_c is coercive electric field;

D_r is remnant electric induction; and

$a \ge 1$ defines the shape of the depolarization curve (Fig. 5).

In order to determine the value of a the following non-linear equation is solved,

$$\left|\frac{E_1}{E_c}\right|^a + \left(\frac{D_1}{D_r}\right)^a = 1\,, \tag{36}$$

where $E = E_1$ and $D = D_1$ define one selected point on the depolarization curve (Table I).

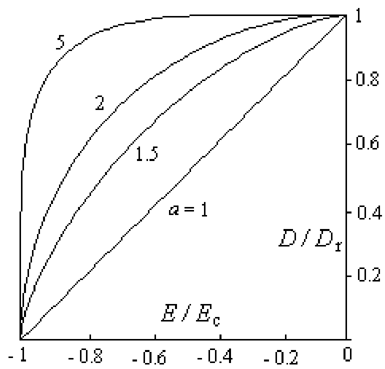


Fig. 5. Depolarization curves

Table I

a for different ratio D_1/D_r, when $E_1 = 0.5E_c$.

D_1/D_r	a
0.5	1
0.6	1.16072
0.7	1.37072
0.8	1.67852
0.9	2.24636
1	∞

In order to determine the electric field components in the upper layer the following expressions will be observed:

$$E_{0x} = \int_0^\infty \left(pD_p e^{-py} + \frac{q'}{2\pi\varepsilon_0} \mathrm{e}^{-p|y-h|} \right) \sin(px)dp \tag{37}$$

and

$$E_{0y} = \int_0^\infty \left(pD_p e^{-py} \mp \frac{q'}{2\pi\varepsilon_0} \mathrm{e}^{-p|y-h|} \right) \cos(px)dp, \tag{38}$$

where the sign minus is valid for $0 \le y \le h$ and plus is valid for $h \ge y < \infty$.

Eliminating the value D_p from these expressions and using the results obtained for electric field in the lower non-linear layer, it is obtained

$$E_{0x}(x, y \ge 0) = E_x(x,-y) + \frac{q'}{2\pi\varepsilon_0} \left[\frac{x}{x^2 + (y-h)^2} - \frac{x}{x^2 + (y+h)^2} \right] \tag{39}$$

and

$$E_{0y}(x, y > 0) = E_y(x,-y) + \frac{q'}{2\pi\varepsilon_0} \left[\frac{y-x}{x^2 + (y-h)^2} - \frac{y+x}{x^2 + (y+h)^2} \right]. \tag{40}$$

3. CONCLUSION

One semi-analytical approach for solving non-liner electrostatic problem, when very large uniform line steady charge is placed in two-layer medium, one of which is isotropic, but non-linear, is presented in this paper. Using Maxwell's equations and existing boundary conditions on the separating surface electric scalar potential functions are determined as the solution of Laplace's equations in both layers. Then integral transform method is used and working line and working point of non-linear media are determined graphically and numerically, using one very simple, but useful approximation of polarization characteristic and depolarization curve of non-linear layer.

With convenient modification, but using Maxwell's equations, existing boundary conditions and integral transform method, presented approach is useful to solve following similar problems of the electromagnetic field theory:

- Point steady charge in two layers non-linear media;
- Steady line current conductor above non-linear magnetic half space; and
- Steady current loop in two layers non-linear media.

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LINIJSKI IZVOR IZNAD NEOGRANIČENE NELINEARNE PLOČE

D. M. Veličković

Stalno linijsko podužno naelektrisanje nalazi se u vazduhu iznad neograničene ploče načinjene od izotropnog, ali nelinearnog materijala, za koji su konstitutivni izrazi poznati. Koristeći metod integralnih transformacija rešene su Maksvelove jednačine uz postojeće granične uslove i određen potencijal i električno polje u oba sloja. Zatim je obrazovana jednačina radne prave i određena radna tačka nelinearne sredine. Pri tome je za aproksimaciju polarizacionih karakteristika nelinearne sredine predložen jedan novi jednostavan izraz.