FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol.3, N° 15, 2003, pp. 1127 - 1132

VECTOR OPTIMIZATION AND DIFFERENTIAL GAMES

UDC 681.5.01

Dragovan M. Perišić

Faculty of transport and traffic engineering, University of Belgrade
Vojvode Stepe 305, Belgrade, Yugoslavia
dragovan@sf.sf.bg.ac.yu

Abstract. *The problem of optimal control with a vector criterion, reduced to non-cooperative differential game is defined. For a given problem the necessary and sufficient conditions are given, satisfying equilibrium strategies for measurable equilibrium controls and the conditions for the existence of equilibrium minimizing controls. In the problem under consideration there are constraints to the phase coordinate, controls and unknown parameters; to every participant in the game a part of these values belongs and particular control components can take unlimited values.*

Key words: *vector criterion, differential games, equilibrium controls, minimizing process.*

1. INTRODUCTION

The basic problem to be considered here regards the optimal control of dynamical systems with a continuous change of time with many unknown control vectors determined on the basis of the extremum of different criteria. This problem is typical for the multiple player differential games, or in the system with incomplete information on the state and disturbances in the system, where the unknown disturbances are determined from the extremum condition of one or many criteria [2]. The problem of the existence of the optimal equilibrium control in the class of measurable time functions (measurable according to Lebesgue) or appropriate series of equilibrium controls when the appropriate criteria tend to extreme values is particularly considered. Somewhat different approach is given in papers [2] and [3].

Received June 21, 2002

2. PROBLEM FORMULATION

The dynamics of nonlinear systems, for which we consider the problem of the optimal control with many criteria, is defined by the following system of ordinary differential equations

$$\dot{x}_i = f_i(x(t),u(t),w,t),\ (i=1,\dots,n) \tag{1}$$

$$t_0 \le t \le t_1,\ x=(x_1,\dots,x_n)\in E^n$$
$$u=(u_1,\dots,u_r)=(u^1,\dots,u^k,\dots,u^m)\in U\subset E^r,$$
$$w=(w_1,\dots,w_q)=(w^1,\dots,w^k,\dots,w^m)\in W\subset E^q,$$
$$u^k\in U^k\subset E^{r_k},\ w^k\in W^k\subset E^{q_k},\ k=1,\dots,m,$$

where x ,u, w are the vectors of phase coordinates, control and parameters; indices k designate those values (vectors) which are determined on the basis of the extremum of the kth criterion , when the following expressions are valid

$$r_1+\cdots+r_k+\cdots+r_m=r \quad i \quad q_1+\cdots+q_k+\cdots+q_m=q.$$

There are also the limits for the values of unknown parameters in the system, of initial, final coordinates and time

$$(x(t_0),x(t_1),w,t_0,t_1)\in G\subset E^{2n+q+2} \tag{2}$$

where on the basis of the kth criterion some components of unknown values from expression (2) are determined, the other being fixed and not belonging to the following sets

$$(x^k(t_0),x^k(t_1),w^k,t_0^k,t_0^k)\in G^k\subset G, \tag{3}$$

where the sum of all unfixed values in sets G^k is equal to the total number of unfixed values in set G.

Phase coordinates (x) and those of control satisfy the following constraints:

$$(x(t),u(t))\in V(t),\ t_0\le t\le t_1, \tag{4}$$

where, in a general case, set V can be a function of coordinates, time and unknown parameters. D will designate the set of permitted processes $x(t)$, $u(t)$, w, to, t_1; in order that the problem observed has a sense we take that $D \neq \phi$,i.e., that there is at least one measurable control and the appropriate trajectory (time functions of limited variations) and all unknown parameters allowing given limitations. It is necessary to determine the permitted controls u^k and the appropriate parameters satisfying conditions (3), from the extremum condition (minimum) of criteria

$$J=\phi_0^k(x(t_0),x(t_1),w,t_0,t_1)+\int_{t_0}^{t_1} f_0^k(x(t),u(t),w,t)dt,\ k=1,\dots,m. \tag{5}$$

The permitted process, minimizing the above criteria with respect to the quoted unknown values represents a solution of the vector optimization problem or of the respective non-cooperative differential game.

The constraints which the permitted processes satisfy ((2) and (4)) are defined by the following equalities and inequalities:

$$h_i(x(t_0),x(t_1),w,t_0,t_1)=0,\ i=(1,\dots,n_h)\ , \tag{6}$$

$$s_i(x(t_0),x(t_1),w,t_0,t_1)\le 0,\ i=(1,\dots,n_s)\ , \tag{7}$$

$$\begin{aligned} &g_i(x(t),w,u'(t),t)\le 0,\ \ i=(1,\dots,n_g),\ \ t_0\le t\le t_1,\\ &u'\in E^{r'},\ u''\in E^{r''},\ t_0\le t\le t_1,\ r'+r''=r, \end{aligned} \tag{8}$$

where u' denotes the control components limited, and u'' - the control components which can have even unlimited values. The set of permitted processes, which the kth player can select will be denoted by D^k; then the remaining undefined parameters and those of control , belonging to other players, fixed and determined from the extreme condition correspond to the criterion (equilibrium values).

In the following text some assumptions will be adduced to which we will refer when deriving the respective conditions satisfying the equilibrium process. Functions f_i are of the following shape

$$\begin{gathered} f_i(x,u,w,t)=a_i(x,u,w,t)+b_{ij}(x,u',w,t)\cdot u''_j,\ \ f_{n+k}=f_0^k\,,\\ k=1,\dots,m,\ \ i=1,\dots,n+m,\ j=r-r''+1,\dots,r. \end{gathered} \tag{9}$$

The case when functions a_i depend only on u' will be called linear (with respect to u'') and in other cases quasilinear (or nonlinear). The basic assumptions are:

A) functions $a_i, b_{ij}, i = 1,\dots,n + m, j = r - r'' + 1,\dots,r,\dots$ and functions ϕ_0, $h = (h_1,\dots,h_{n_h})$, $s = (s_1,\dots,s_{n_s})$, $g = (g_1,\dots,g_{n_g})$, are continuous and continuously differential with respect to the set of their arguments.

B) the derivatives with respect to x and t functions b_{ij} satisfy locally The Lipschitz condition with respect to x,

C) functions b_{ij} satisfy the appropriate conditions of Frobenijus.

D) for functions a_i and b_{ij} the following inequalities are valid

$$|\,a_i\,|\le C\,|\,1+|\,x\,|\,|^{m_i}, |\,b_{ij}\,|\le K\,|\,1+|\,x\,|\,|^{m_i\,m_i},\dots$$

where C and K are constants and m_i=1 for i=1,..n, and for $i > n$, $m_i = m_0 \ge 1$, (a natural number)

E) the set of permitted values $x(t_0)$ w, t_0, t_1 is a compact set.

The problem of the existence of impulsive controls at the beginning and at the end of the process causes definite difficulties in defining the conditions satisfying equilibrium controls; this problem will be solved by introducing additional parameters w and appropriate restrictions for which it is not hard to show that they are not disturbing the above introduced assumptions, provided the permitted processes are the functions of a limited variation.

3. Equilibrium Controls

The conditions satisfying equilibrium controls will be defined by means of respective Krotov's functions. Every deviation from equilibrium strategy (control) is less favorable for every participant in the game, provided the other participants cling to equilibrium strategies [2]. Appropriate Krotov's method in its integral form is used here, as it is suitable for the case when the permitted trajectories have interruptions of the first order.

Let us assume that there are functions (*x, t*) with the respective generalized derivatives and that the following functions are defined as well as their time integrals

$$R^k(x,u,w,t)dt = \frac{\partial \varphi^k}{\partial t} + \sum_{i=1}^{n} \frac{\partial \varphi^k}{\partial x_i} \cdot f_i + f_0^k \,, \tag{10}$$

$$\begin{aligned} &\int_{t_0}^{t_1} \left(\frac{\partial \varphi^k}{\partial t} + \sum_{i=1}^{n} \frac{\partial \varphi^k}{\partial x_i} \cdot f_i + f_0^k \right) dt = \int_{t_0}^{t_1} R^k(x,u,w,t)dt = \\ &= \varphi^k(x(t_1),w,t_1) - \varphi^k(x(t_0),w,t_0) + \int_{t_0}^{t_1} f_0{}^k(x,u,w,t)dt \,, \end{aligned} \tag{11}$$

Let us define the following functions, too:

$$S^k(x(t_0),x(t_1),w,t_0,t_1) = \phi_0{}^k(x(t_0),x(t_1),w,t_0,t_1) + \varphi^k(x(t_0),w,t_0) - \varphi^k(x(t_1),w,t_1), \tag{12}$$

so that at present the criteria are

$$J^k(x(\cdot),u(\cdot),w,t_0,t_1) = S^k(x(t_0),x(t_1),w,t_0,t_1) + \int_{t_0}^{t_1} R^k(x,u,w,t)dt, \quad k = 1...m \,. \tag{13}$$

Now we can estimate from below the values of the criterion J^k in the set of permitted processes D^k and function S^k in the set of permitted limitations G^k where other controls and parameters are fixed and correspond to equilibrium strategies. These estimates are

$$J_*{}^k = \inf_{D^k} J^k \,, \; S_*{}^k = \inf_{G^k} S^k \tag{14}$$

so that for the case when all the players, except the *k*th, cling to equilibrium strategies, the *k*th player having his arbitrary strategy, the following estimate is valid:

$$0 \le J^k(x(\cdot),u(\cdot),w,t_0,t_1) - J_*{}^k \le S^k(x(t_0),x(t_1),w,t_0,t_1) - S_*{}^k + \int_{t_0}^{t_1} R^k(x,u,w,t)dt \,. \tag{15}$$

The equilibrium controls satisfy the conditions defined by the following theorem.

Theorem 1. *When the equilibrium controls are measurable time functions and when the conditions A,B,C,D,E are fulfilled, then there are respective functions* $\varphi^k(k,w,t)$ *satisfying conditions* (1) *and in the equilibrium process they satisfy equalities*

$$\int_{t_0}^{t_1} R^k(x(t),u(t),w,t)dt = 0\,, \tag{16}$$

$$S^k(x(t_0),x(t_1),w,t_0,t_1) \ = S_*^k, \quad k = 1...m\,. \tag{17}$$

Proof: On the basis of estimates (15) the proof is obvious when the respective functions exist. In the case when all permitted controls are limited time functions condition (16) can be written in the form

$$\frac{\partial \varphi^k}{\partial t} + \sum_{i=1}^{n} \frac{\partial \varphi^k}{\partial x_i} \cdot f_i + f_0^k = 0\,, \tag{18}$$

almost for every $t_0 < t < t_1$.

When all permitted controls are not limited time functions, then the equilibrium controls (equilibrium process) satisfy the integral principle of maximum[1], so that the respective linearized functions φ^k can be obtained around any permitted trajectory with respect to x, using the appropriate constrained variables. If integral (16) is not equal to zero, it could be achieved by adding some time function. The same can be attained in the quasilinear case, too[5].

The case when equilibrium controls exist but are not measurable time functions then the conditions for the existence of equilibrium strategies are defined by the following theorem:

Theorem2. *For the equilibrium control and the respective equilibrium process to exist, it is sufficient that a series of permitted processes* $(x^s(t_0), x^s(t_1), w^s, t_0^s, t_1^s)$ *exists and functions* φ^{ks}, R^{ks} *and* S^{ks} *for which equalities* (11) *are valid and expressions*

$$\left(\int_{t_0}^{t_1} R^{ks}(x^s(t),u^s(t),w^s,t)dt \right) \underset{s\to\infty}{\longrightarrow} 0\,, \tag{19}$$

$$S^{ks}(x^s(t_0),x^s(t_1),w^s,t_0^s,t_1^s) \underset{s\to\infty}{\longrightarrow} S_*^k, \quad k = 1...m \tag{20}$$

The proof of this theorem is fully analogous to the previous and it can be applied even to the case of the existence of equilibrium measurable controls.

4. Conclusion

Sufficient conditions are given for the existence of equilibrium controls when they are measurable time functions which has not been often examined in conflict problems. While in a class of generalized functions we can expect that the majority of previously considered problems have a solution [1], [4], this problem, however, requires separate consideration.

REFERENCES

1. Yu V. Orlov, (1988) *Teorija optimaljnih sistem s obobščenim upralenijam*, Moscow, 'Nauka'
2. E.M.Vajsbord,V.I. Žukovskij, (l980) *Vvedenie v differencialjnie igri neskoljkih lic i ih priloženija*, Moscow.
3. F.H. Clarke,Yu S.Ledyaev and A.I. Subotin, (1997) *The Synthesis of Universal Feedback Pursuit Strategies in Differential Games*. SIAM J.Control and Optimization,Vol.35, No 2, pp.552-561
4. J. Warga, (1972) *Optimal Control of Differential and Functional Equations*, Academic Press, New York.
5. D.M.Perišić: *Optimal Control of a Dynamic System* (In Serbian, in print), Naučnotehnički pregled.

VEKTORSKA OPTIMIZACIJA I DIFERENCIJALNE IGRE

Perišić M. Dragovan

Definiše se problem optimalnog upravljanja sa vektorskim kriterijumom, koji se svodi na bezkoalicionu diferencijalnu igru. Za dati problem definišu se potrebni i dovoljni uslovi, koje zadovoljavaju ravnotežne strategije za merljiva ravnotežna upravljanja i uslovi postojanja ravnotežnih minimizirajućih upravljanja. U posmatranom problemu postoje ograničenja na fazne koordinate, upravljanja i nepoznate parametre, od kojih svakom učesniku u igri pripada jedan deo tih veličina, a pojedine komponente upravljanja mogu uzimati i neograničene vrednosti.