FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol.3, N° 15, 2003, pp. 1133 - 1138

THE GENERALIZED LAGRANGE'S EQUATIONS OF THE SECOND KIND AND THE FIELD METHOD FOR THEIR INTEGRATION

UDC 531.3

Ivana Kovačić

Faculty of Technical Sciences, Trg Dositeja Obradovica 6, Department of Mechanics, 21121 Novi Sad, Serbia and Montenegro

Abstract. *This paper presents the complete process of obtaining motion of a mechanical system with variable mass subject to non-holonomic constraints which are non-linear or of the first or higher order. Firstly, the generalized Lagrange's equations of the second kind are extended to a non-holonomic system with variable mass by introducing generalized reactive forces. Then, the field method is applied to these equations of motion to find their solution. Finally, an illustrative example showing the use of this algorithm is given.*

1. INTRODUCTION

The field method, primary developed as a method for integrating the equations of motion of holonomic non-conservative systems [1, 2], pertains to the Hamilton-Jacobi theory, limited in applications to conservative holonomic systems. Lately, it was shown that the field method is also suitable for integrating the equations of motion of non-holonomic systems [3, 4], while the Hamilton-Jacobi theory has very strict restrictions for their study [5, 6]. The application of the field method to non-holonomic problems comprises the generalization to non-holonomic systems whose configurations are determined by generalized coordinates and motions are modeled by Lagrange's equations with multipliers, being subjected to the non-holonomic constraint equations: linear, of the first order [4] or higher order of Chetaev's type [3]. Since the process of obtaining multipliers can be considerable difficult it is more suitable to model motion of non-holonomic systems with generalized Lagrange's equations of the second kind. What is more, generalized Lagrange's equations of the second kind enable us to take into consideration systems subject to non-linear constraints and those of a higher order. Therefore, in this paper the field method is applied and extended to the study of motion of systems with variable mass subject to such kind of constraints and modeled by generalized Lagrange's equations of the second kind.

Received April 20, 2003

2. THE GENERALIZED LAGRANGE'S EQUATIONS OF THE SECOND KIND

Let the position of a mechanical system is defined by n generalized coordinates q_j (j=1,…, n), while its motion is subject to m non-holonomic constraints:

$$q_{s+\nu}^{(k)} = \varphi_\nu(t, q_j, \dot{q}_j, ..., q_j^{(k-1)}, q_i^{(k)}), \tag{1}$$

where $\nu = 1,...,m$; s is the number of degrees of freedom $s = n - m$; $i = 1,...,s$; $()^{(k)} = \partial^k()/\partial t^k$; t is time; $k \geq 1$.

The constraints (1) are of Chaplygin's type [7], since q_i are independent coordinates and $q_{s+\nu}$ are dependent ones.

On the basis of the results of the paper [8], the equations of motion of this system can be written in the form:

$$(k+1)\frac{\partial L^{(k+1)}}{\partial q_i^{(k+1)}} - (k+2)\frac{\partial L^{(k)}}{\partial q_i^{(k)}} = -\Lambda_\nu \frac{\partial \varphi_\nu}{\partial q_i^{(k)}}, \quad (k+1)\frac{\partial L^{(k+1)}}{\partial q_{s+\nu}^{(k+1)}} - (k+2)\frac{\partial L^{(k)}}{\partial q_{s+\nu}^{(k)}} = \Lambda_\nu, \tag{2}$$

where L is a Lagrangian of the system and λ_ν are unknown Lagrange's multipliers.

Extending them to the system of N point with variable mass, one obtains:

$$\begin{aligned}(k+1)\frac{\partial L^{(k+1)}}{\partial q_i^{(k+1)}} - (k+2)\frac{\partial L^{(k)}}{\partial q_i^{(k)}} &= -\Lambda_\nu \frac{\partial \varphi_\nu}{\partial q_i^{(k)}} + R_i, \\ (k+1)\frac{\partial L^{(k+1)}}{\partial q_{s+\nu}^{(k+1)}} - (k+2)\frac{\partial L^{(k)}}{\partial q_{s+\nu}^{(k)}} &= \Lambda_\nu + R_{s+\nu}.\end{aligned} \tag{3}$$

Generalized reactive forces $R_j = \sum_{I=1}^{N} \dot{M}_I \left(\vec{V}_I - \frac{d\vec{r}_I}{dt}\right)\frac{\partial \vec{r}_I}{\partial q_j}$ appear as a consequence of mass variation $\dot{M}_I$ of I-th point and relative velocity of their change $(\vec{V}_I - d\vec{r}_I/dt)$, where $\vec{V}_I$ is the absolute velocity of an added or separated particle [9]. Note that the assumption of the reactive force in this form requires treating mass as a constant during the differentiation of the Lagrange's function.

After eliminating the multipliers the previous system can be transformed into:

$$(k+1)\frac{\partial L_*^{(k+1)}}{\partial q_i^{(k+1)}} - (k+2)\frac{\partial L_*^{(k)}}{\partial q_i^{(k)}} = \left(R_i + R_{s+\nu}\frac{\partial \varphi_\nu}{\partial q_i^{(k)}}\right)_*, \tag{4}$$

where $(\ )_*$ denotes the terms obtained after excluding $q_{s+\nu}^{(k)}$ and $q_{s+\nu}^{(k+1)}$ from ( ).

3. THE GENERALIZATION OF THE FIELD METHOD

The system (4) consists of m differential equations of the second order, which together with m constraints (1) enable us to find motion q_j (t). This system is of the general form:

$$q_j^{(u)} = X_j(t, q_j, \dot{q}_j, ..., q_j^{(u-1)}), \quad u \geq 2. \tag{5}$$

In order to write it down in the form suitable for applying the field method, i.e. in the form of the first order differential equations, the new variables are introduced:

$$q_j = x_j, \dot{q}_j = x_{n+j}, \ddot{q}_j = x_{2n+j}, ..., q_j^{(u-1)} = x_{(u-1)n+j}, \tag{6}$$

and the system becomes:

$$\dot{x}_j = x_{n+j}, \dot{x}_{n+j} = x_{2n+j}, ..., \quad \dot{x}_{(u-1)n+j} = X_j(t, x_j, x_{n+j}, ..., x_{(u-1)n+j}). \tag{7}$$

So, for the non-holonomic system of Chaplygin's type which is modeled by generalized Lagrange's equations of the second kind, the number of state variables is $s = u \cdot n$. The value of u for the non-holonomic constraints of the first and second order corresponds to the order of the left side of the equation (4), which is equal to two. In the case of non-holonomic constraints of an order $u \geq 2$, u is equal to the order of constraints.

System (7) can be considered as an "extended" holonomic problem, whose initial conditions satisfy the constraints (1). Further, according to the basic supposition of the field method, one of the state variables can be expressed as a function of time and the rest of variables:

$$x_1 = \Phi(t, x_A), \quad A = 2, ..., u \cdot n. \tag{8}$$

By differentiating (8) with respect to time and using (7), the basic equation is obtained:

$$\frac{\partial \Phi}{\partial t} + \sum_{A=2}^{u \cdot n} \frac{\partial \Phi}{\partial x_A} X_A(t, \Phi, x_A) - X_1(t, \Phi, x_A) = 0. \tag{9}$$

The field method does not look for the asked solution directly, but finds it through a complete solution of this quasi-linear partial equation of the first order. The solution of the basic equation can be represented in the form:

$$\Phi = f_1(t) + \sum_{A=2}^{u \cdot n} f_A(t) x_A, \tag{10}$$

where f_1 and f_A are unknown functions of time, which will be determined by substituting (9) into (10) and collecting and equating to zero free terms and terms containing x_A. It leads to solution for the field which depends on the arbitrary constants C_1, C_A:

$$\Phi = \Phi(t, x_A, C_1, C_A). \tag{11}$$

In accordance with the initial conditions $x_1(0) = x_{10}, x_A(0) = x_{A0}$, one of the constants, say C_1, can be expressed in terms of the initial conditions and the rest of constants. Consequently, the conditioned form solution is obtained:

$$\overline{\Phi} = \overline{\Phi}(t, x_A, C_1(0, x_{10}, x_{A0}, C_A), C_A). \tag{12}$$

The fact that the conditioned form solution should not depend on the value of the additional constants C_A produces:

$$\frac{\partial\overline{\Phi}}{\partial x_A}=0, \tag{13}$$

assuming that $\det(\partial^2\overline{\Phi}/(\partial C_A\partial x_B))\neq 0,\quad B=2,...,u\cdot n$.

So, the solution for motion of the original non-holonomic problem follows from (12), $(u\cdot n-1)$ algebraic equations (13) and the constraints equations for the initial values of state variables and it contains $(u\cdot n-m)$ constants.

4. EXAMPLE

A point whose mass varies exponentially $M=M_0\exp(-\alpha t)$, where M_0 and α are positive constants, moves on a plane, while its motion is subject to rheonomic non-holonomic linear constraint:

$$\dot{q}_2=t\,\dot{q}_1. \tag{14}$$

Since the Lagrangian of this system is $L=\frac{1}{2}M(\dot{q}_1^2+\dot{q}_2^2)$, the equation (4) becomes:

$$2\frac{\partial L_*^{(2)}}{\partial q_1^{(2)}}-3\frac{\partial L_*^{(1)}}{\partial q_1^{(1)}}=R_1+tR_2, \tag{15}$$

where, according to (14), q_1 is independent coordinate and q_2 is dependent one. Supposing that the absolute abandoned velocity of the particle is zero, which means that $\vec{V}=-\dot{\vec{r}}$, calculating necessary differentials, and using (14) and its differential, it is obtained:

$$\ddot{q}_1=\alpha\dot{q}_1-\frac{t}{1+t^2}\dot{q}_1,\quad \ddot{q}_2=\alpha t\dot{q}_1+\frac{1}{1+t^2}\dot{q}_1. \tag{16}$$

Introducing the substitutions $q_1=x_1, q_2=x_2, x_3=\dot{q}_1, x_4=\dot{q}_2$, it follows:

$$\dot{x}_1=x_3,\ \dot{x}_2=x_4,\ \dot{x}_3=\alpha x_3-\frac{t}{1+t^2}x_3,\ \dot{x}_4=\alpha t x_3+\frac{1}{1+t^2}x_3. \tag{17}$$

The basic equation (9) for the field $\Phi\equiv x_1=\Phi(t,x_2,x_3,x_4)$ is as follows:

$$\frac{\partial\Phi}{\partial t}+\frac{\partial\Phi}{\partial x_2}x_4+\frac{\partial\Phi}{\partial x_3}x_3\left(\alpha-\frac{t}{1+t^2}\right)+\frac{\partial\Phi}{\partial x_4}x_3\left(-\alpha t+\frac{1}{1+t^2}\right)-x_3=0. \tag{18}$$

In accordance with (10) its solution has the form:

$$\Phi=f_1(t)+f_2(t)x_2+f_3(t)x_3+f_4(t)x_4. \tag{19}$$

After substituting it into (18) and collecting the free term and the terms containing x_2, x_3, x_4, the following system is derived:

$$\dot{f}_1 = 0,\ \dot{f}_3 + f_3\left(\alpha - \frac{t}{1+t^2}\right) + \frac{1}{1+t^2} f_4 - 1 = 0,$$
$$\dot{f}_2 = 0,\ \dot{f}_4 + f_2 + \alpha f_4 = 0. \tag{20}$$

Its integration gives:

$$f_1 = C_1, f_3 = \sqrt{1+t^2}\exp(-\alpha t)\left[\int \frac{\exp(\alpha\tau)}{\sqrt{1+\tau^2}} d\tau + \frac{C_2}{\alpha}\int \frac{\exp(\alpha\tau)}{(1+\tau^2)^{3/2}} d\tau - \frac{C_4 t}{\sqrt{1+t^2}} + C_3\right],$$

$$f_2 = C_2, f_4 = -\frac{C_2}{\alpha} + C_4 \exp(-\alpha t). \tag{21}$$

According to the initial conditions $x_i(0) = x_{i0}$, $i = 1,...,4$ one finds:

$$x_{10} = C_1 + C_2 x_{20} + f_{30}(0, C_2, C_3, C_4) x_{30} + \left(\frac{-C_2}{\alpha} + C_4\right) x_{40}. \tag{22}$$

which enable us to express the constant C_1 as a function of the rest of constants and the initial conditions.

Finally, the applications of (13) yield:

$$\frac{\partial\overline{\Phi}}{\partial C_4} = 0 \Rightarrow -x_{40} - t\exp(-\alpha t)x_3 + \exp(-\alpha t)x_4 = 0,$$
$$\frac{\partial\overline{\Phi}}{\partial C_3} = 0 \Rightarrow -x_{30} + \sqrt{1+t^2}\exp(-\alpha t)x_3 = 0,$$
$$\frac{\partial\overline{\Phi}}{\partial C_2} = 0 \Rightarrow -x_{20} - \frac{x_{30}}{\alpha}\int_0^t \frac{\exp(\alpha\tau)}{(1+\tau^2)^{3/2}} d\tau + \frac{x_{40}}{\alpha} + x_2 - \frac{x_4}{\alpha} = 0. \tag{23}$$

After solving these equations, the field (19) gives the equation of motion:

$$x_1 = x_{10} + x_{30}\int_0^t \frac{\exp(\alpha\tau)}{\sqrt{1+\tau^2}} d\tau, \tag{24}$$

while the constraint (14) imposes the restriction:

$$x_{40} = 0. \tag{25}$$

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GENERALIZACIJA LAGRANGE-OVIH JEDNAČINA DRUGE VRSTE I METODA POLJA ZA NJIHOVU INTEGRACIJU

Ivana Kovačić

Ovaj rad prezentuje kompletan proces dobijanja rešenja kretanja mehaničkog sistema sa promenljivom masom i neholonomnim vezama koje su nelinearne, te prvog ili višeg reda. U radu su najpre Lagranževe jednačine druge vrste proširene na neholonomne sisteme sa promenljivom masom uvođenjem generalisanih reaktivnih sila. Zatim je na ove jednačine primenjena metoda polja u cilju nalaženja njihovog rešenja. Konačno, dat je primer koji ilustruje prezentovan algoritam rešavanja.