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Invited Paper

THE CONNECTION BETWEEN SET AND FUZZY DIFFERENTIAL EQUATIONS

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Abstract. *The study of fuzzy differential equations (FDEs) forms a suitable setting for mathematical modeling of real world problems in which uncertainties or vagueness pervades. In recent years, the theory of FDEs has been investigated extensively in the original formulation as well as in an alternative framework, which leads to ordinary multivalued differential inclusions. It has recently been realized that initiating the study of set differential equations in a metric space has several advantages, in addition to providing a natural setting for considering FDEs. In this talk, we present some interesting results in this direction with the necessary background material.*

Key words: *Set differential equations, fuzzy differential equations*

1 Preliminaries

Let $K(R^n)(K_c(R^n))$ denote the collection of all nonempty, compact (compact, convex) subsets of R^n. Define the Hausdorff metric

$$D[A,B] = \max\left[\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\right] \tag{1.1}$$

where $d[x,A] = \inf[d(x,y) : y \in A]$, A, B are bounded sets in R^n. We note that $K(R^n), (K_c(R^n))$, with the metric is a complete metric space.

It is known that if the space $K_c(R^n)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_c(R^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space [1, 14, 15].

The Hausdorff metric (1.1) satisfies the following properties.

$$D[A+C, B+C] = D[A,B] \text{ and } D[A,B] = D[B,A], \tag{1.2}$$

$$D[\lambda A, \lambda B] = \lambda D[A,B], \tag{1.3}$$

$$D[A,B] \le D[A,C] + D[C,B], \tag{1.4}$$

for all $A, B, C \in K_c(R^n)$ and $\lambda \in R^+$.

Let $A, B \in K_c(R^n)$. The set $C \in K_c(R^n)$ satisfying $A = B + C$ is known as the Hukuhara difference of the sets A and B and is denoted by the symbol $A - B$. We say that the mapping $F : I \to K_c(R^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if there exists an element $D_H F(t_0) \in K_c(R^n)$ such that the limits

$$\lim_{h\to 0^+} \frac{F(t_0+h) - F(t_0)}{h}, \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(t_0) - F(t_0-h)}{h}$$

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exist in the topology of $K_c(R^n)$ and are equal to $D_H F(t_0)$. Here I is any interval in R.

By embedding $K_c(R^n)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, we find that if

$$F(t) = X_0 + \int_0^t \Phi(s)\, ds, \quad X_0 \in K_c(R^n), \tag{1.5}$$

where $\Phi : I \to K_c(R^n)$ is integrable in the sense of Bochner, then $D_H F(t)$ exists and the equality

$$D_H F(t) = \Phi(t), \quad \text{a.e on} \quad I, \tag{1.6}$$

holds. Also, the Hukuhara integral

$$\int_I F(s)\, ds = \left[\int_I f(s)\, ds : \ f \text{ is a continuous selector of } F \right],$$

for any compact set $I \subset R_+$. With these preliminaries, we consider the set differential equation

$$D_H U = F(t, U), \quad U(t_0) = U_0 \in K_c(R^n), \quad t_0 \geq 0, \tag{1.7}$$

where $F \in C\ [R_+ \times K_c(R^n), K_c(R^n)]$.

The mapping $U \in C^1\ [J, K_c(R^n)]$, $J = [t_0, t_0 + a]$ is said to be a solution of (1.7) on J if it satisfies (1.7) on J. Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^t D_H U(s)\, ds, \quad t \in J. \tag{1.8}$$

Thus we associate with the initial value problem(IVP)(1.7) the following

$$U(t) = U_0 + \int_{t_0}^t F(s, U(s)) ds, \quad t \in J, \tag{1.9}$$

where the integral is the Hukuhara integral [6, 7]. Observe also that $U(t)$ is a solution of (1.7) iff it satisfies (1.9) on J.

The investigation of set differential equation (1.7) as an independent subject has some advantages. For example, when $U(t)$ is a single valued mapping, it is easy to see that Hukuhara derivative and the integral reduce to the ordinary vector derivative and the integral, and therefore, the results obtained in this new framework of (1.7) become the corresponding results of ordinary differential systems. Also, we have only semilinear complete metric space to work with, in the present setup, compared to the complete normed linear space one employs in the study of ordinary differential systems. Furthermore, set differential equations, that are generated by multivalued differential inclusions, when the multivalued functions involved do not possess convex values, can be used as a tool for studying multivalued differential inclusions. See Tolstonogov [15]. Moreover one can utilize set differential equations of the type (1.7) to investigate profitably fuzzy differential equations, since the original formulation of which suffers from grave disadvantages and does not reflect the rich behavior of corresponding differential equations without fuzziness [11]. This is due to the fact that the diameter of any solution of fuzzy differential equation increases as time increases because of the necessity of the fuzzification of the derivative involved.

In this paper, we plan to employ the results of set differential equations obtained in [12] as another alternative for discussing fuzzy differential equations and show that this approach captures both vagueness and rich properties of set differential equations in one and the same method.

2 Fuzzy Differential Equations

Consider the fuzzy sets with respect to base space R^n. To each $x \in R^n$, we assign a membership grade $u(x)$ taking values in $[0, 1]$ with $u(x) = 0$ corresponding to non membership, $0 < u(x) < 1$ to partial membership and $u(x) = 1$ to full membership. For example, a fuzzy set $u \in E^n$ is a function $u : R^n \to [0, 1]$ satisfying (i) to (iv) below.

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is, for $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min(u(x), u(y));$$

(iii) u is upper semi-continuous(usc);

(iv) $[u]^0 = cl[u \in E^n : u(x) > 0]$ is compact.

For $0 < \alpha \leq 1$, we denote $[u]^\alpha = [x \in R^n : u(x) \geq \alpha]$. Then from (i) to (iv), it follows that the α - level set $[u]^\alpha \in K_c(R^n)$ for $0 \leq \alpha \leq 1$. We set

$$D_0[u, v] = \sup_{0 \leq \alpha \leq 1} D\left[[u]^\alpha, [v]^\alpha\right] \tag{2.1}$$

which defines a metric in E^n and (E^n, D_0) is also a semilinear complete metric space [4, 10]. Also, $D_0[u, v]$ satisfies similar properties as $D[A, B]$ listed in (1.2) to (1.4). One can also define Hukuhara derivative $D_H F(t) \in E^n$ for a fuzzy map $F : J \to E^n$, where $J = [t_0, t_0 + a], a > 0$, similarly as before, taking limits in the metric space (E^n, D_0). Moreover, if $F : J \to E^n$ is continuous, it is integrable, the integral $G(t) = \int_{t_0}^t F(s)\, ds$ is differentiable and $D_H G(t) = F(t)$. Furthermore $F(t) = F(t_0) + \int_{t_0}^t D_H F(s)\, ds,\ t \in J$.

In this setup, the IVP for fuzzy differential equations originally proposed is of the type

$$D_H u = f(t, u), \quad u(t_0) = u_0 \in E^n, \tag{2.2}$$

where $f \in C[R_+ \times E^n, E^n]$, for which basic results are discussed. See [8, 9, 13]. This approach is based on the fuzzification of the differential operator, whose values are in E^n and therefore suffers from the disadvantage, since the solution u(t) of (2.2) has the property that $diam[u(t)]$ is nondecreasing as t increases. Consequently, this formulation can not fully reflect the rich behavior of solutions of corresponding ordinary differential equations.

Recently, Hüllermeier [5] has suggested an alternative formulation of fuzzy IVPs by replacing the R.H.S of a system of ordinary differential equation by a fuzzy function

$$f : R_+ \times R^n \to E^n, \tag{2.3}$$

and with the initial fuzzy set $x_0 \in E^n$, so that one can consider the fuzzy multivalued differential inclusion

$$x' \in f(t, x), \quad x(t_0) = x_0 \in E^n, \tag{2.4}$$

on J, where now f is defined from $R_+ \times R^n \to E^n$ rather than $R_+ \times E^n \to E^n$ as in (2.2). However, instead of (2.4), a sequence of multivalued differential inclusions

$$x_\beta' \in F(t, x_\beta; \beta), \quad x_\beta(t_0) \in [x_0]^\beta, \quad 0 \leq \beta \leq 1, \tag{2.5}$$

is investigated on J, where $F(t, x; \beta) \equiv [f(t, x)]^\beta$ and $F(t, x, 0) = \overline{supp}\ (f(t, x))$. The idea is that the set of all solutions $S_\beta(x_0, T),\ t_0 \leq t \leq T$, would be β - level of a fuzzy set $S(x_0, T)$, in the sense that all attainable sets $A_\beta(x_0, t),\ t_0 < t \leq T$, are levels of a fuzzy set on R^n. Considering $S(x_0, T)$ to be the solution of (2.2) thus captures both uncertainty and the rich behavior of differential inclusion in one and the same technique.

For this purpose, the standard results of multivalued differential inclusions, under the usual conditions on F in (2.5) yield that the attainable set $A_\beta(x_0, t)$ is a compact subset of R^n. If F is assumed to be quasiconcave in.addition, one can conclude, under reasonable assumptions, utilizing the representation theorem, the existence of a fuzzy set $u(t)$ such that $[u(t)]^\beta = A_\beta(x_0, t)$ with a similar relation for the solution set $S_\beta(x_0, T)$. See for details [2, 3, 13].

Let us consider the often quoted simple example to show the advantage gained by the alternative approach when compared to the original one.

Consider the crisp initial value problem with unknown initial value x_0, that is,

$$x' = -x,\ x(0) = x_0 \in [-1, 1]. \tag{2.6}$$

The solution of (2.6) when restricted to the interval $[-1,1]$ is $x(t) = [-e^{-t}, e^{-t}]$, $t \geq 0$. The fuzzy differential equation corresponding to (2.6) in E^1 is

$$D_H x = -x, \ x(0) = x_0 = [-1,1], \ x_0 \in E^1. \tag{2.7}$$

Suppose that $[x]^\beta = [x_1^\beta, x_2^\beta]$, $[D_H x]^\beta = [\frac{dx_1^\beta}{dt}, \frac{dx_2^\beta}{dt}]$, are the β - level sets for $0 \leq \beta \leq 1$. By extension principle, (2.7) becomes

$$\frac{dx_1^\beta}{dt} = -x_2^\beta, \quad \frac{dx_2^\beta}{dt} = -x_1^\beta, \quad 0 \leq \beta \leq 1. \tag{2.8}$$

The solution of (2.8) is given by $x_1^\beta(t) = -e^t, x_2^\beta(t) = e^t$ and therefore the fuzzy function $x(t)$ solving (2.7) is $x(t) = [-e^t, e^t]$, $t \geq 0$, which shows that the $diam(x(t)) \to \infty$ as $t \to \infty$.

In the framework of Hüllermeier, on the other hand, fuzzy differential equation (2.7) is replaced by the family of inclusions

$$x'_\beta \in F(t, x_\beta; \beta) = -[x_2^\beta, x_1^\beta], \quad x(0) = [-1,1]; \tag{2.9}$$

which has a fuzzy solution set $S([-1,1],T), 0 \leq t \leq T$ and fuzzy attainable set $A_\beta([-1,1],t)$ respectively are defined by β-level sets

$$S_\beta([-1,1],T) = [x(\cdot) : x(t) \in [-e^{-t}, e^{-t}]] \quad 0 \leq t \leq T, \tag{2.10}$$

$$A_\beta[[-1,1],t) = [-e^{-t}, e^{-t}], \tag{2.11}$$

which matches the kind of behavior a fuzzyfications of the crisp differential equation (2.6) should have.

3 Results in Set Differential Equations

In order to propose another alternative formulation based on set differential equations (not inclusions), we need some known results [12, 15] relative to such equations. We start with a comparison result.

Theorem 3.1 *Assume that $F \in C[R_+ \times K_c(R^n), K_c(R^n)]$ and for $t \in R_+, A, B \in K_c(R^n)$*

$$D[F(t,A), F(t,B)] \leq g(t, D[A,B]),$$

where $g \in C[R_+^2, R_+]$. Suppose further that the maximal solution $r(t, t_0, w_0)$ of the scalar differential equation

$$w' = g(t,w), \quad w(t_0) = w_0 \geq 0,$$

exists for $t \geq t_0$. Then, if $U(t) = U(t, t_0, U_0)$, $V(t) = V(t, t_0, V_0)$ are any two solutions of equation (1.7)such that $U(t_0) = U_0, V(t_0) = V_0$, $U_0, V_0 \in K_c(R^n)$ existing for $t \geq t_0$, we have:

$$D[U(t), V(t)] \leq r(t, t_0, w_0), \quad t \geq t_0,$$

provided $D[U_0, V_0] \leq w_0$.

The next result is an existence and uniqueness Theorem more general than Lipschitz type condition the proof of which exhibits the idea of comparison principle.

Theorem 3.2 *Assume that*

(a) $F \in C[R_0, K_c(R^n)]$, where $R_0 = J \times B(U_0, b), J = [t_0, t_0 + a], a > 0, B(U_0, b) = [U \in K_c(R^n) : D[U, U_0] \leq b]$ and $D[F(t,U), \theta] \leq M_0$ on R_0, where θ is the zero element of R^n regarded as a one point set;

(b) $g \in C[J \times [0,2b], R+], g(t,w) \leq M_1$ *on* $J \times [0,2b]$, $g(t,0) \equiv 0$, $g(t,w)$ *is nondecreasing in* w *for each* $t \in J$ *and* $w(t) \equiv 0$ *is the only solution of*

$$w' = g(t,w), \quad w(t_0) = 0;$$

(c) $D[F(t,U), F(t,V)] \leq g(t, D[U,V])$ *on* R_0.

Then the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^{t} F(s, U_n(s))ds, \quad n = 0,1,2,\ldots,$$

exist on $J_0 = [t_0, t+\alpha]$, $\alpha = \min(a, \frac{b}{M}]$, $M = max(M_0, M_1)$, *as continuous functions and converge uniformly to the unique solution* $U(t) = U(t,t_0,U_0)$ *of IVP (1.7) on* J_0.

The following global existence result is a special case of Theorem 5.2 in [12] which serves our purpose.

Theorem 3.3 *Assume that*

(1) $F \in C[R_+ \times K_c(R^n), K_c(R^n)]$ *and for* $(t,A) \in R_+ \times K_c(R^n)$,

$$D[F(t,A), \theta] \leq q(t, D[A,\theta]),$$

where $q \in C[R_+^2, R_+]$, $q(t,w)$ *is nondecreasing in* w *for each* $t \in R_+$ *and the maximal solution* $r(t,t_0,w_0)$ *of*

$$w' = q(t,w), \quad w(t_0) = w_0,$$

exists for $t \geq t_0$ *and for every* $w_0 > 0$;

(2) there exists a local solution $U(t) = U(t,t_0,U_0)$ *of (1.7) for every* $(t_0,U_0) \in R_+ \times K_c(R^n)$. *Then for every* $U_0 \in K_c(R^n)$ *such that* $D[U_0,\theta] \leq w_0$, *the IVP (1.7) possesses a solution* $U(t) = U(t,t_0,U_0)$ *defined for* $t \geq t_0$, *satisfying*

$$D[U(t),\theta] \leq r(t,t_0,w_0), t \geq t_0.$$

A result that relates the solution of set differential equation to the attainable set of multivalued differential inclusion is the following result. See [15].

Theorem 3.4 *Assume that* $F \in C[R_+ \times R^n, K_c(R^n)]$;

$$D[F(t,x), F(t,y)] \leq g(t, \|x-y\|), \quad (t,x,y) \in R_+ \times R^n \times R^n,$$

and $D[F(t,x),\theta] \leq q(t,\|x\|)$, $(t,x) \in R_+ \times R^n$, *where* g *and* q *satisfy the assumptions listed in Theorem 3.2 and Theorem 3.3 respectively, except that conditions relative to* g *hold for* $R_+ \times R_+$. *Then there exist a unique solution* $U(t) = U(t,t_0,U_0)$ *on* $[t_0,\infty)$ *of IVP (1.7) and the attainable set* $A(U_0,t)$ *of differential inclusion*

$$x' \in F(t,x), \quad x(t_0) \in U_0,$$

satisfying $A(U_0,t) \subset U(t)$, $t_0 \leq t < \infty$.

Finally, we need the following representation result [4, 10, 13].

Theorem 3.5 *Let* $Y_\beta \subset R^n$, $0 \leq \beta \leq 1$ *be a family of compact subsets satisfying*

(a) $Y_\beta \in K(R^n)$ *for all* $0 \leq \beta \leq 1$;

(b) $Y_\beta \subseteq Y_\alpha$ *whenever* $\alpha \leq \beta, \quad \alpha, \beta \in [0,1]$*;*

(c) $Y_\beta = \prod_{i=1}^{\infty} Y_{\beta_i}$, *for any nondecreasing sequence* $\beta_i \to \beta$ *in* $[0,1]$. *Then there is a fuzzy set* $u \in D^n$, *such that* $[u]^\beta = Y_\beta$. *If* Y_β *is also convex, then* $u \in E^n$. *(Here* D^n *denotes the set of usc normal fuzzy sets with compact support and thus* $E^n \subset D^n$*). Conversely, the level sets* $[u]^\beta$, *of any* $u \in E^n$, *are convex and satisfy these conditions.*

This Theorem is from [10], see also [5, 13]. It should be noted that Theorem 3.5 can easily be generalized from R^n to a Banach space.

4 Main Results

We propose, in this section, another formulation of fuzzy differential equation (2.2) by a set differential equation which is generated by β-level set of the R.H.S. of (2.2), where $f : R_+ \times R^n \to E^n$, as before. For this purpose, consider the level set for each β, $0 \leq \beta \leq 1$, and write

$$F(t,x;\beta) = [f(t,x)]^\beta \in K_c(R^n).$$

Next generate the mapping $H : R_+ \times K_c(R^n) \times I \to K_c(R^n)$, $I = [0,1]$ by defining

$$H(t,A;\beta) = \bar{C}o\, F(t,A;\beta) \tag{4.1}$$

for each $A \in K_c(R^n)$. Then consider the sequence of set differential equations given by

$$D_H U_\beta = H(t, U_\beta; \beta), \quad U_\beta(t_0) = U_{0\beta} \in K_c(R^n), \tag{4.2}$$

on $[t_0, T], T > t_0$, where $D_H U_\beta$ is the Hukuhara derivative for each β.

Let us list the following conditions.

(1) $F(t,x;\beta)$ is quasi-concave, that is,

  (a) for $(t,x) \in R_+ \times R^n$, $\alpha, \beta \in I$, $F(t,x;\alpha) \supseteq F(t,x,\beta)$ whenever $\alpha \leq \beta$

  (b) if β_n is a nondecreasing sequence in I, converging to β, then for $(t,x) \in R_+ \times R^n$, $\bigcap_{n=1}^{\infty} F(t,x;\beta_n) = F(t,x;\beta)$.

(2) $D[H(t,A;\beta),\ H(t,B;\beta)] \leq g(t, D[A,B])$ for $t \in R_+$, $A, B \in K_c(R^n)$, $\beta \in I$;

(3) $g \in C[R_+^2, R_+]$, $g(t,0) \equiv 0$, $g(t,w)$ is nondecreasing in w for each $t \in R_+$ and $w(t) \equiv 0$ is the only solution of

$$w' = g(t,w), \quad w(t_0) = 0,$$

for $t \geq t_0$;

(4) $D[H(t,A;\alpha), H(t,A;\beta)] \leq L|\alpha - \beta|$, $\alpha, \beta \in I, (t,A) \in R_+ \times K_c(R^n)$, $L > 0$.

We are now in a position to prove the following result.

Theorem 4.1 *Suppose that the assumptions (1) to (4) are satisfied. Then there exists a unique solution* $U_\beta(t) = U_\beta(t, t_0, U_{0\beta}) \in K_c(R^n)$, $\beta \in I$ *of (4.2) and* $U_\beta(t)$ *is quasiconcave in* β *for* $t \geq t_0$. *Moreover, there exists a fuzzy set* $u(t) \in E^n$ *such that*

$$[u(t)]^\beta = U_\beta(t), \quad t \geq t_0.$$

Proof: Since f is continuous on $R_+ \times R^n$, $F(t,x;\beta)$ is also continuous for $(t,x,\beta) \in R_+ \times R^n \times I$. This implies that $H(t,A;\beta)$ is a continuous map for $(t,A,\beta) \in R_+ \times K_c(R^n) \times I$. Consequently, by Theorems 3.2 and 3.3 it follows that there exists a unique solution $U_\beta(t) = U(t,t_0,U_{0\beta}) \in K_c(R^n)$ for $t \geq t_0$ of (4.2). We first show that $U_\beta(t) \subseteq U_\alpha(t)$ if $\alpha \leq \beta$ for $t \geq t_0$. From the definition of quasi-concavity of $F(t,x;\beta)$, it follows that $H(t,A;\beta)$ is also quasi-concave in β. Let $U_\alpha(t), U_\beta(t)$ be the solution of

$$D_H U_\alpha = H(t,U_\alpha;\alpha), \quad U_\alpha(t_0) = U_0 \in K_c(R^n),$$

$$D_H U_\beta = H(t,U_\beta;\beta), \quad U_\beta(t_0) = U_0, \quad \alpha \leq \beta,$$

then we find, using quasi-concavity

$$D_H U_\alpha = H(t,U_\alpha;\alpha) \supseteq H(t,U_\alpha;\beta), \quad \alpha \leq \beta.$$

But $H(t,U_\alpha;\beta) = H(t,U_\beta;\beta)$ because of $U_\alpha(t_0) = U_0 = U_\beta(t_0)$ and therefore $U_\alpha(t) \equiv U_\beta(t)$ by uniqueness of solutions of (4.2). Thus it is clear that $U_\beta(t) \subseteq U_\alpha(t),\ \alpha \leq \beta,\ t \geq t_0$.

We shall next prove that if β_n is a nondecreasing sequence, $\beta_n \in I$, converging to β, then $U_{\beta_n}(t) \to U_\beta(t)$, uniformly on compact subsets of $[t_0,\infty)$. For this purpose, set $m(t) = D[U_{\beta_n}(t), U_\beta(t)]$ and note that $D[U_{0\beta_n}, U_{0\beta}] = m(t_0)$. We shall assume that $U_{0\beta_n} \to U_{0\beta}$ as $n \to \infty$. Then employing the properties of the metric D, the definition of Hukuhara derivative and the conditions (2) and (4), we arrive at the scalar differential inequality

$$D^+ m(t) \leq g(t,m(t)) + L|\beta_n - \beta|,\ m(t_0) = D[U_{0\beta_n}, U_{0\beta}],\ t \geq t_0.$$

Hence by Lemma 1.3.1 in [11], we obtain

$$m(t) \leq r_n(t,t_0,\eta_n)$$

on any compact set $J \subset [t_0,\infty)$, where $\eta_n = D[U_{0\beta_n}, U_{0\beta}]$ and $r_n(t,t_0,\eta_n)$ is the maximal solution of

$$w' = g(t,w) + L|\beta_n - \beta|, \qquad w(t_0) = \eta_n, \text{ on } J.$$

By assumption (3) $r_n(t,t_0,\eta_n) \to r(t,t_0,0) \equiv 0$, uniformly on J as $n \to \infty$. Since $\beta_n \to \beta$ as $n \to \infty$, it follows that $m(t) \equiv 0$ on J, which in turn implies that $D[U_{\beta_n}(t), U_\beta(t)] \to 0$ as $n \to \infty$. Thus $U_\beta(t)$ is quasi-concave in $\beta \in I$ for $t \geq t_0$. Consequently, by Theorem 3.5, there exists a fuzzy set $u(t) \in E^n$ such that $[u(t)]^\beta = U_\beta(t),\ t \geq t_0$, and this completes the proof.

To find the connection between the solution $U_\beta(t)$, of (4.2) and the attainability set $A_\beta(U_0,t)$ of (2.5), we have the following result.

Theorem 4.2 *Let $F \in C[R_+ \times R^n \times I, K_c(R^n)]$ satisfy the assumptions of Theorem 3.4 for each $\beta \in I = [0,1]$ and assume that it is also quasiconcave in β as well. Then there exist a unique solution $U_\beta(t) = U_\beta(t,t_0,U_0)$ of (4.2) for $t \geq t_0$ and the attainable set $A_\beta(U_0,t)$ of the inclusion (2.5) such that*

$$A_\beta(U_0,t) \subset U_\beta(t,t_0,U_0), \quad t \geq t_0. \tag{4.3}$$

Proof: It is easy to verify that when $F(t,x;\beta)$ satisfies the assumptions required in Theorem 3.4. the desired conditions in Theorem 3.1 to 3.3 are also satisfied, in view of the monotone nondecreasing nature of the functions $g(t,w), q(t,w)$, the definition of D and the fact $H(t,A;\beta)$ is generated by $F(t,x;\beta)$. We have assumed conditions in terms of $H(t,A;\beta)$ since the set differential equations are treated as an independent subject. Thus for each $\beta \in I$, Theorem 3.4 yields the relation (4.3). Also, both $U_\beta(t)$ and $A_\beta(U_0,t)$ satisfy the assumptions of Theorem 3.5, because one can prove similarly the quasi-concavity of $A_\beta(U_0,t)$. Therefore, there exist fuzzy sets $u(t), v(t) \in E^n$ such that

$$[v(t)]^\beta = A_\beta(U_0,t) \text{ and } [u(t)]^\beta = U_\beta(t), \quad t \geq t_0.$$

The proof is complete.

We note that, in general, since $A_\beta(U_0, t)$ is only compact and not convex, only (4.3) holds. Equality in (4.3) is valid only in some special cases.

Recalling (2.9) in the example considered in Section 2, let us generate the set differential equation from F in (2.9), that is, from (4.1), we have

$$D_H U_\beta = H(t, U; \beta), \; U_\beta(t_0) = U_{0\beta} \in K_c(R).$$

This implies

$$D_H U_\beta = -U_\beta, \quad U_\beta(0) = U_{0\beta} \in K_c(R) \tag{4.4}$$

Since the values of the solution of (4.4) are interval functions, the equation (4.4) can be written as (suppressing the β),

$$[u_1', \; u_2'] = (-1)U = [-u_2, \; -u_1], \tag{4.5}$$

where $U = [u_1, \; u_2]$. The relation (4.5) is equivalent to the system of equations

$$\begin{aligned} u_1' &= -u_2, \qquad u_1(0) = u_{10}, \\ u_2' &= -u_1, \qquad u_2(0) = u_{20}, \end{aligned}$$

whose solution is

$$\left.\begin{aligned} u_1(t) &= \tfrac{1}{2}[u_{10} + u_{20}]e^{-t} + \tfrac{1}{2}[u_{10} - u_{20}]e^{t}, \\ u_2(t) &= \tfrac{1}{2}[u_{20} + u_{10}]e^{-t} + \tfrac{1}{2}[u_{20} - u_{10}]e^{t}, \quad t \geq 0. \end{aligned}\right\} \tag{4.6}$$

Given $U_0 \in K_c(R)$, if there exist $V_0, \; W_0 \in K_c(R)$ such that $U_0 = V_0 + W_0$, then the Hukuhara difference $U_0 - V_0 = W_0$. Let us choose

$$U_0 = [u_{10}, u_{20}], \;\; V_0 = \left[\frac{(u_{10} - u_{20})}{2}, \frac{(u_{20} - u_{10})}{2}\right],$$

so that

$$W_0 = \left[\frac{(u_{10} + u_{20})}{2}, \frac{(u_{20} + u_{10})}{2}\right].$$

Then it follows assuming $u_{10} \neq -u_{20}$, that

$$\begin{aligned} U(t, U_0) &= \left[\frac{-1}{2}(u_{20} - u_{10}), \; \frac{1}{2}(u_{20} - u_{10})\right] e^{t} \\ &+ \left[\frac{1}{2}(u_{10} + u_{20}), \; \frac{1}{2}(u_{10} + u_{20})\right] e^{-t}, \; t \geq 0. \\ U(t, V_0) &= \left[\frac{1}{2}(u_{10} - u_{20}), \; \frac{1}{2}(u_{20} - u_{10})\right] e^{t}, \text{ and} \\ U(t, W_0) &= \left[\frac{1}{2}(u_{10} + u_{20}), \; \frac{1}{2}(u_{20} + u_{10})\right] e^{-t}, \; t \geq 0. \end{aligned}$$

If on the other hand, $u_{10} = -u_{20}$, then it is enough to change the roles of U_0 and V_0. We note that for any general initial value U_0, the solution of SDE(4.4) contains both the desired and the undesired parts compared to the solution of the ODE(4.3). In order to isolate the desired part of the solution $U(t, U_0)$ of (4.4) which matches the solution of the ODE(4.3), we need to use the initial values satisfying the desired Hukuhara difference of the given two initial values. This is necessary only when the generated SDE from ODE has the property that no translate of $D_H U, \; U$ is included in the set $\{\theta\}$, which is the case in (4.4), when we write (4.4) as

$$D_H U + U = \{\theta\}.$$

If, on the other hand, we have the SDE as

$$D_H U = \lambda(t) U, \;\; U(0) = U_0, \tag{4.7}$$

which is generated by

$$u' = \lambda(t)u, \quad u(0) = u_0, \tag{4.8}$$

where $\lambda(t) > 0$ is a real valued function from $R_+ \to R_+$ such that $\lambda \in L^1(R_+)$, then we see that, with similar computation,

$$U(t, U_0) = U_0 \, exp \left[\int_0^t \lambda(s)ds \right], \quad t \geq 0,$$

for any $U_0 \in K_c(R)$. Hence we get the stability of the trivial solution of (4.7). In this case, we note that the solutions of both equations (4.7) and (4.8) match providing the same stability results. There is no necessity, therefore, to choose the initial values as before, since the undesirable part of the solution does not exist among solutions of (4.7). Consequently, it does not matter, whether we use the Hukuhara difference or not, we get the same conclusion. In order to be consistent and take care of all the situations, the stability results are formulated in terms of Hukuhara differences of the initial values.

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VEZA IZMEDJU SKUPA I FAZI DIFERENCIJALNIH JEDNAČINA

V. Lakshmikantham

Proučavanje fazi diferencijalnih jednačina (FDE) formira odgovarajuće okruženje praktičnih problema koji su prožeti neizvesnošću i nejasnoćama. Proteklih godina, teorija FDE je opširno proučavana u originalnoj formulaciji kao i u alternativnom okviru, što vodi ka običnom ubrajanju diferencijalnih višestrukih vrednosti. Nedavno se došlo do zaključka da uvodjenje proučavanja skupa diferencijalnih jednačina u metričkom prostoru ima nekoliko prednosti uz stvaranje prirodnog okruženja za proučavanje FDE. U ovom radu se predstavljaju zanimljivi rezultati u ovom pravcu uz neophodan materijal.

Ključne reči: *skup diferencijalnih jednačina, fazi diferencijalne jednačine*