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Invited Paper

A NEW UNIFIED CONCEPT OF STABILITY

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Abstract. *In the useful and rich field of stability theory for nonlinear systems, there have been many refinements, extensions and generalizations* [3, 4]. *Basically, stability concerns with comparing phase-space positions of solutions of perturbed and unperturbed equations, with classical Lyapunov stability being too stringent a requirement and orbital stability being too loose a demand. We define a new concept of stability that can unify theses two extreme cases (and possibly many other appropriate notions between these two) in terms of suitable topologies, following the idea of J. L. Massera* [6]. *Also, a further unification is achieved by using two measures* [5]. *In this unified frame work, we give sufficient conditions for these concepts to hold, via Lyapunov functions.*

Key words: *clock stability, Lyapunov stability, orbital stability*

1 INTRODUCTION

Consider the differential systems

$$x'(t) = F(t, x), \qquad x(t_0) = x_0, \tag{1.1}$$

$$y'(t) = f(t, y), \qquad y(\tau_0) = y_0, \tag{1.2}$$

where $F, f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$. Assume, for convenience, that the solutions $x(t, t_0, x_0)$, $y(t, \tau_0, y_0)$ of (1.1), (1.2) respectively exist and are unique for each (t_0, x_0) and (τ_0, y_0), $t \geq \tau_0 \geq t_0$. In classical Lyapunov Stability (LS) the phase-space positions of the perturbed and unperturbed solutions are compared at each t. i.e. $|x(t, t_0, x_0) - y(t, t_0, y_0)| < \epsilon$, for each $t \geq t_0$, whenever $|y_0 - x_0| < \delta(\epsilon)$. (We can consider (1.2) a perturbation of (1.1)). This requirement of closeness between the two solutions at every instant is quite restrictive from a physical point of view. The Orbital Stability (OS), on the other hand, compares the solutions over the entire time interval $[t_0, \infty)$, i.e.

$$\left(\inf_{s \in [t_0, \infty)} |x(s, t_0, x_0) - y(t, t_0, y_0)| \right) < \epsilon, \; t \geq t_0.$$

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In (OS), the two motions are compared at any two unrelated moments and we require only that the two trajectories be close to each other, in some sense.

The following motions in $\mathbb{R}^2$ illustrate the difference between these two notions. Let $0 < \epsilon < 1$ and consider the motions

(a) $x(t) = \cos t,\ y(t) = \sin t,$

(b) $x(t) = (1+\epsilon)\cos t,\ y(t) = (1+\epsilon)\sin t,$

(c) $x(t) = \cos(1+\epsilon)t,\ y(t) = \sin(1+\epsilon)t,$

(d) $x(t) = \cos 2t,\ y(t) = \sin 2t.$

It is easy to see that motion (b) is close to motion (a). However, many physicists consider motion (c) close to motion (a), though in the sense of Lyapunov, the distance between these two motions is 2. Motions (a), (d) are very different but they are close orbitally. These considerations suggest that a notion that can unify (LS) and (OS) may lead to concepts between these extreme cases which could have some physical significance.

The perturbation of a system can be realized when

(i) the dynamics changes i.e, F, f are different,

(ii) when initial position changes i.e, x_0 and y_0 are different, or

(iii) when starting times are different i.e. the solutions $x(t, t_0, x_0)$, and $y(t, \tau_0, y_0)$ are compared with $\tau_0 - t_0 = \eta > 0$.

In case (iii) (LS) can be modified as $|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)| < \epsilon$, for all $t \geq \tau_0$, provided $|y_0 - x_0| < \delta(\epsilon)$ and (OS) can be described by

$$\left(\inf_{s\in[\tau_0,\infty)} |y(t, \tau_0, y_0) - x(s - \eta, t_0, x_0)|\right) < \epsilon.$$

In literature, we consider $f = F + R$, with the perturbation term R satisfying suitable conditions in order to preserve the stability of the unperturbed motion.

2 A NEW CONCEPT OF STABILITY

Following up the idea of J.L. Massera [6] that the distance between the trajectories be measured, maintaining different time scales or "clock" with which time is measured along each motion, let us now define the new concepts of stability in terms of given topology of the function space.

Let E be the given space of all functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ each function $\sigma(t)$ representing a clock. We call $\sigma(t) = t$, the perfect clock. Let τ be any topology in E. Let $x(t, t_0, x_0)$ be any given solution of (1.1) and $y(t, \tau_0, y_0)$ be any solution of (1.2).

Definition 2.1 *The solution $x(t, t_0, x_0)$ is said to be*

(1) τ - stable, if given $\epsilon > 0$, t_0 $\tau_0 \in \mathbb{R}_+$ and a τ - neighborhood N of the perfect clock, there exists a $\delta = \delta(t_0, \tau_0, \epsilon) > 0$ such that for each y_0 with $|y_0 - x_0| < \delta$, there is a clock $\sigma \in N$ with $\sigma(\tau_0) = t_0$, satisfying

$$|y(t, \tau_0, y_0) - x(\sigma(t), t_0, x_0)| < \epsilon, \; t \geq t_0;$$

(2) τ - uniformly stable, if δ in (1) is independent of t_0, τ_0;

(3) τ - asymptotically stable if (1) holds and given $\epsilon > 0$, t_0, $\tau_0 \in \mathbb{R}_+$, there exists a $\delta_0 = \delta_0(t_0, \tau_0) > 0$, a τ - neighborhood N of the perfect clock, a $T = T(t_0, \tau_0, \epsilon) > 0$ and a clock $\sigma \in N$ such that for each y_0 with $|y_0 - x_0| < \delta_0$, $\sigma(\tau_0) = t_0$, we have

$$|y(t, \tau_0, y_0) - x(\sigma(t), t_0, x_0)| < \epsilon, \; \forall\, t \geq \tau_0 + T;$$

(4) τ - uniformly asymptotically stable if (2) holds and δ_0, T in (3) are independent of t_0, τ_0.

We note that a partial ordering of topologies of E induces a corresponding partial ordering of stability concepts. On the space E, we can consider the following topologies:

(τ_1) the discrete topology (where every set in E is open);

(τ_2) the chaotic topology (where only open sets are the empty set and entire clock space E);

(τ_3) the topology defined by the base

$$U_{\sigma_0,\epsilon} = \{\sigma, \sigma_0 \in E : \sup_{t \in [\tau_0, \infty)} |\sigma(t) - \sigma_0(t)| < \epsilon\}$$

with σ_0, a given clock, $\sigma, \sigma_0 \in C[\mathbb{R}_+, \mathbb{R}_+]$;

(τ_4) the topology defined by the base

$$U_{\sigma_0,\epsilon} = \{\sigma, \sigma_0 \in C^1[\mathbb{R}_+, \mathbb{R}_+] : |\sigma(\tau_0) - \sigma_0(t_0)| < \epsilon \text{ and } \sup_{t \in [\tau_0, \infty)} |\sigma'(t) - \sigma_0'(t)| < \epsilon\};$$

(τ_5) the topology of three open sets, the empty set, the entire clock space E and the set of all continuous increasing functions from $\mathbb{R}_+$ to $\mathbb{R}_+$.

It is easy to see that the topologies τ_3, τ_4, τ_5 lie between τ_1 and τ_2. The following remarks are in order:

(1) if $x(t, t_0, x_0)$ is the equilibrium position (trivial solution of (1.1)), then (OS) implies (LS).

(2) τ_1 - stability corresponds to (LS) if $\sigma(t) = t$ is the neighborhood consisting of only the perfect clock.

(3) τ_2 - stability corresponds to (OS) since

$$d\left(y(t,\tau_0,y_0), M(t_0,x_0)\right) = \inf_{s\in[\tau_0,\infty)} \left(|y(t,\tau_0,y_0) - x(s,t_0,x_0)|\right)$$

(with $M(t_0,x_0)$ being the entire motion $x([t_0,\infty),t_0,x_0))$, can be denoted by s_t, for each $t \geq t_0$ and designating the clock $\sigma(t)$ as s_t. This $\sigma \in E$ in τ_2 - topology and we obtain orbital stability of the motion $x(t,t_0,x_0)$ in terms of τ_2 - topology.

It can be shown [6] that τ_4 - stability implies τ_1 - stability if $|F(t,x(t,t_0,x_0))| \leq \bar{M},\ t \geq t_0$. For further details and examples, see [1, 2].

We know by the enormous volume of research on "stability" (various refinements and extensions) that is available in the literature, in order to unify most of the existing notions, in the current context, we need

(i) a comparison theorem for the new context,

(ii) the usage of two measures, $h_0,\ h$ (where h_0 is used to measure the change in initial values and h is used to measure the change in the solution), and

(iii) sufficient conditions in terms of Lyapunov function for $\tau_3,\ \tau_4,\ \tau_5$ - stabilities.

3 COMPARISON RESULTS

We need the following known results [3].

Theorem 3.1 *Let $g \in C[\mathbb{R}_+^3, \mathbb{R}]$, $g(t,u,v)$ be nondecreasing in v for each (t,u) and $r(t) = r(t,\tau_0,u_0)$ be the maximal solution of*

$$u' = g(t,u,u), \quad u(\tau_0) = u_0 \geq 0 \tag{3.1}$$

on $[\tau_0,\infty)$. Then the maximal solution $R(t) = R(t,\tau_0,u_0)$ of

$$u' = g(t,u,r(t)), \quad u(\tau_0) = u_0 \geq 0 \tag{3.2}$$

exist on $[\tau_0,\infty)$ and $r(t) \equiv R(t)$ on $[\tau_0,\infty)$.

Theorem 3.2 *Assume g is as in Theorem 3.1. Let $m \in C[\mathbb{R}_+,\mathbb{R}_+]$ satisfy*

$$D_-m(t) \leq g(t,m(t),v), \quad t \geq \tau_0. \tag{3.3}$$

Then, for all $v \leq r(t)$, we have

$$m(t) \leq r(t), \quad t \geq \tau_0. \tag{3.4}$$

To prove a comparison result in terms of Lyapunov function, let

$$\Omega = \{\sigma \in C^1[\mathbb{R}_+, \mathbb{R}_+] : \sigma(\tau_0) = t_0 \text{ and } w(t, \sigma, \sigma') \leq r(t), t \geq \tau_0\}$$

where $w \in C[\mathbb{R}_+^2 \times \mathbb{R}, \mathbb{R}_+]$, and $r(t)$ is the maximal solution of (3.1). For some $\sigma \in \Omega$, let $V(t, \sigma, x) \in C[\mathbb{R}_+^2 \times \mathbb{R}^n, \mathbb{R}_+]$ and define $D^+V(t, \sigma, y - x)$ as follows:
$D^+V(t, \sigma, y - x) \equiv$

$$\limsup_{h \to 0^+} \frac{1}{h} \left[V(t + h, \sigma(t + h), y - x + h(f(t, y) - F(\sigma(t), x)\sigma'(t))) - V(t, \sigma(t), y - x)\right]$$

Theorem 3.3 *Assume that for some $\sigma \in \Omega$, there exists a $V(t, \sigma, x) \in C[\mathbb{R}_+^2 \times \mathbb{R}^n, \mathbb{R}_+]$ which is locally Lipschitzian in x and satisfies*

$$D^+V(t, \sigma, y - x) \leq g(t, V(t, \sigma, y - x), w(t, \sigma, \sigma')),$$

where $g \in C[\mathbb{R}_+^3, \mathbb{R}]$ with $g(t, u, v)$ nondecreasing in v for each (t, u). Then

$$V(t, \sigma(t), y(t, \tau_0, y_0) - x(\sigma(t), t_0, x_0)) \leq r(t, \tau_0, u_0), \ \forall t \geq \tau_0,$$

provided $u_0 = V(t_0, \sigma(t_0), y_0 - x_0)$, where $y(t, \tau_0, y_0)$, $x(t, t_0, x_0)$ are solutions of (1.2), (1.1) respectively and $\sigma \in \Omega$.

Proof: Let $x(t, t_0, x_0)$, $y(t, \tau_0, y_0)$ be the solutions of (1.1), (1.2) through (t_0, x_0) and (τ_0, y_0) existing on $[t_0, \infty)$, $[\tau_0, \infty)$ respectively. With

$$m(t) = V(t, \sigma(t), y(t, \tau_0, y_0) - x(\sigma(t), t_0, x_0))$$

for some $\sigma \in \Omega$, it is easy to get the differential inequality

$$\begin{aligned} D^+m(t) &\leq g(t, m(t), w(t, \sigma, \sigma')), \ t \geq \tau_0 \\ &\leq g(t, m(t), r(t)), \ t \geq \tau_0, \end{aligned}$$

where $r(t) = r(t, \tau_0, u_0)$ is the maximal solution of (3.1). In view of Theorems 3.1 and 3.2, we obtain the estimate $m(t) \leq r(t)$, $t \geq \tau_0$, proving the comparison theorem.

4 SUFFICIENT CONDITIONS

In this section we give the sufficient conditions in terms of Lyapunov functions. Let $M(t_0, x_0) = M = x([t_0, \infty), t_0, x_0)$ and suppose it is closed.
Theorem 4.1 *Let $V \in C[\mathbb{R}_+ \times S(M, \rho), \mathbb{R}_+]$, $V(t, x)$ locally Lipschitzian in x and*

$$b(d(x, M)) \leq V(t, x) \leq a(d(x, M)),$$

a, b being standard $\mathcal{K}$ class functions [1, 2], and

$$D^+V(t,x) \le g(t, V(t,x)) \ on \ \mathbb{R}^+ \times S(M,\rho),$$

with $g(t,0) \equiv 0,\ g \in C[\mathbb{R}_+^2, \mathbb{R}]$. *Then the stability properties of the null solution of*

$$u' = g(t,u), \quad u(\tau_0) = u_0 \ge 0,$$

imply the corresponding τ_2 - stability (of the given solution $x(t,t_0,x_0)$) of (1.1).

For details, see [1, 2].

Theorem 4.2 *Let the assumptions of Theorem 3.3 hold. Suppose that*

(i) $b(|x|) \le V(t,\sigma,x) \le a(t,\sigma,|x|)$

(ii) $\tilde{d}(|t-\sigma|) \le w(t,\sigma,\sigma')$

with $b,\ \tilde{d} \in \mathcal{K},\ a(t,\sigma,\cdot) \in \mathcal{K}$. *Then the stability properties of the trivial solution of (3.1) imply the corresponding τ_3 - stability properties of (1.1) respectively.*

For details, see [1, 2].

Theorem 4.3 *Let assumptions of Theorem 4.2 hold. Assume that*

(i)* $b(|x|) \le V(t,\sigma,x) \le a_0(|x-y|) + a_1(|t-\sigma|)$,

$a_0, a_1, b \in \mathcal{K}$, *is satisfied in place of (i). Then the uniform stability properties of the trivial solution of (3.1) imply the corresponding τ_3 - uniform stability properties of (1.1) respectively.*

For details, see [1, 2].

Theorem 4.4 *Let assumptions of Theorem 3.3 hold and in addition to (i*) of Theorem 4.3, let* $\tilde{d}(|1-\sigma'(t)|) \le w(t,\sigma,\sigma')$, $\tilde{d} \in \mathcal{K}$. *Then the stability properties of the trivial solution of (3.1) imply the corresponding τ_4 - stability properties of (1.1) respectively.*

For details, see [1, 2].

Suitable choices for the comparison function g in Theorem 4.2 are

(1) $g(t,u,v) = -\alpha u + \lambda v,\ \lambda - \alpha = \beta > 0$. In this case, $r(t) = u_0 e^{-\beta(t-t_0)}$.

(2) $g(t,u,v) = \lambda(t) v,\ \lambda \in L^1[\mathbb{R}_+, \mathbb{R}_+]$. In this case, $r(t) = u_0 \, exp(\int_{t_0}^t \lambda(s)ds) \le u_0 e^{\tilde{N}}$, $\int_{t_0}^t \lambda(s)ds \le \tilde{N}$.

5 STABILITY CRITERIA IN TERMS OF TWO MEASURES

We need to use the following classes of functions in order to describe the current context.

Let

$$h_0,\ h \in \Gamma = \{z \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+],\ \inf_x z(t,x) = 0,\ \text{for each } t\},$$
$$\mathcal{L} = \{\tilde{\sigma} \in C[\mathbb{R}_+, \mathbb{R}_+] : \tilde{\sigma}(u) \text{ decreasing in } u \text{ and } \lim_{u\to\infty} \tilde{\sigma}(u) = 0\},$$
$$\mathcal{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a(u) \text{ non decreasing in } u,\ a(0) = 0\}.$$

Let E be the clock space of all functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ and $\sigma \in E$, with $\sigma(t) = t$ being the perfect clock. Let τ be any topology in E. We need $h_0, h \in \Gamma$ such that h_0 is uniformly finer than h, i.e., there exists a $\rho > 0$ and $\varphi \in \mathcal{K}$ such that $h_0(t,x) < \rho$ implies $h(t,x) \leq \varphi(h_0(t,x))$. We can now define τ - stability in terms of two measures.

Definition 5.1 *The systems (1.1) and (1.2) are $(h_0,\ h;\ \tau)$ - stable if given $\epsilon > 0$, $\tau_0, t_0 \in \mathbb{R}_+$, a τ - neighborhood N of the perfect clock, there exists a $\delta = \delta(t_0, \tau_0, \epsilon)$ such that for each y_0 with $h_0(t_0, y_0 - x_0) < \delta$, there is a clock $\sigma \in N$ with $\sigma(\tau_0) = t_0$ satisfying*

$$h(t, y(t, \tau_0, y_0) - x(\sigma(t), t_0, x_0)) < \epsilon, \text{ for all } t \geq \tau_0.$$

Definition 5.2 *The systems (1.1) and (1.2) are $(h_0,\ h;\ \tau)$ - uniformly stable if δ in the above definition is independent of $t_0,\ \tau_0$.*

Other definitions can be formulated similarly. In order to see the greater unification achieved by using two measures (see [5]), we make the following choices for $h_0,\ h$:

(1) $h_0(t,y) = h(t,y) = |y - x(t, t_0, x_0)|$. This gives the τ - stability of the solution $x(t, t_0, x_0)$;

(2) $h(t,y) = |y - x(t, t_0, x_0)|_s$, $1 \leq s \leq n$, and $h_0(t,y) = |y - x(t, t_0, x_0)|$. This gives the τ - partial stability of the solution $x(t, t_0, x_0)$;

(3) $h_0(t,y) = h(t,y) = d(y, M)$, $M \subset \mathbb{R}^n$; This gives the τ - stability of the invariant set M.

(4) $h_0(t,y) = h(t,y) = d(y, C)$, gives the orbital stability of the closed orbit C (periodic solution);

(5) $h(t,y) = d(y, B)$, $h_0(t,y) = d(y, A)$, where $A \subset B \subset \mathbb{R}^n$, B being conditionally invariant with respect to A, gives the stability of the conditionally invariant set B;

(6) $h_0(t,y) = h(t,y) = |y| + \ell(t)$, $\ell \in \mathcal{L}$ gives the stability of the asymptotically selfinvariant set $\{0\}$.

(7) $h_0(t,y) = h(t,y) = |y - x(t, t_0, x_0)| + \ell(t)$, $\ell \in \mathcal{L}$ gives the τ - eventual stability of the solution $x(t, t_0, x_0)$.

We shall now give a typical result that provides sufficient conditions for $(h_0, h; \tau)$ - stability in τ_4 - topology.

Theorem 5.1 *Assume that for some $\sigma \in \Omega$ (see Theorem 3.3), there exists a Lyapunov function $V(t, \sigma, x)$ such that*

(i) $V(t, \sigma, x)$ is locally Lipschitzian in x, $V \in C[\mathbb{R}_+^2 \times \mathbb{R}^n, \mathbb{R}_+]$;

(ii) $V(t, \sigma, x)$ is h - positive definite and h_0 - decrescent i.e. there exists $b \in \mathcal{K}$, such that for some $\rho > 0$, $b(h(t,x)) \leq V(t, \sigma, x)$ whenever $h(t,x) < \rho$, and there exists a $a(t, s, \cdot) \in \mathcal{K}$ such that for some $\rho > 0$,

$$V(t, \sigma(t), x) \leq a(t, \sigma(t), h_0(t,x)) \text{ whenever } h_0(t,x) < \rho;$$

(iii) $D^+V(t, \sigma(t), y - x) \leq g(t, V(t, \sigma(t), y - x), w(t, \sigma(t), \sigma'(t)))$, where g is as in Theorem 3.3;

(iv) $d(|t - \sigma(t)|) \leq w(t, \sigma(t), \sigma'(t))$, $d \in \mathcal{K}$;

Then the stability properties of the trivial solution of $u' = g(t, u)$ imply the corresponding $(h_0, h; \tau_4)$ - stability properties of the systems (1.1), (1.2)

Proof: We shall prove $(h_0, h; \tau_4)$ stability. Let $x(t, t_0, x_0)$ be the given solution of (1.1). Since V is h - positive definite, there exists a $\lambda > 0$ and a $b \in \mathcal{K}$ satisfying

$$b(h(t,x)) \leq V(t, \sigma, x), \ (t,x) \in S(h, \lambda), \tag{5.1}$$

where $S(h, \lambda) = [(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : h(t,x) < \lambda]$.

Let $0 < \epsilon < \lambda$ and $t_0, \tau_0 \in \mathbb{R}_+$ be given. Suppose that the trivial solution of (3.1) is stable. Then, given $b(\epsilon) > 0$, $\tau_0 \in \mathbb{R}_+$, there exists a $\delta_1 = \delta_1(\tau_0, \epsilon)$ such that

$$u_0 < \delta_1 \text{ implies } u(t, \tau_0, u_0) < b(\epsilon), \ t \geq \tau_0, \tag{5.2}$$

where $u(t, \tau_0, u_0)$ is any solution of (3.1). Choose $u_0 = V(\tau_0, \sigma(\tau_0), y_0 - x_0)$. Since V is h_0 - decrescent and h_0 is uniformly finer than h, there exists a $\lambda_0 > 0$ and $a(t, s, \cdot) \in \mathcal{K}$, such that

$$h_0(\tau_0, y_0 - x_0) \leq \lambda_0 \text{ and } V(\tau_0, \sigma(\tau_0), y_0 - x_0) \leq a(\tau_0, \sigma(\tau_0), h_0(\tau_0, y_0 - x_0)). \tag{5.3}$$

It then follows that

$$\begin{aligned} b(h(\tau_0, y_0 - x_0)) &\leq V(\tau_0, \sigma(\tau_0), y_0 - x_0) \\ &\leq a(\tau_0, \sigma(\tau_0), h_0(\tau_0, y_0 - x_0)). \end{aligned} \tag{5.4}$$

Choose $\delta = \delta(t_0, \tau_0, \epsilon)$ such that $\delta \in (0, \lambda_0]$ and $\eta = \eta(\epsilon) > 0$, satisfying

$$a(\tau_0, \sigma(\tau_0), \delta) < \delta_1, \ \eta = d^{-1}(b(\epsilon)). \tag{5.5}$$

Let $h_0(\tau_0, y_0 - x_0) < \delta$. Then (5.4) shows that $h(\tau_0, y_0 - x_0) < \epsilon$, since $\delta_1 < b(\epsilon)$. Also, using assumption (iv) we get

$$\begin{aligned} d(|t - \sigma(t)|) &\leq W(t, \sigma(t), \sigma'(t)) \leq r(t, \tau_0, u_0) \\ &\leq r(t, \tau_0, \delta_1) < b(\epsilon). \end{aligned} \tag{5.6}$$

It follows that $|t - \sigma(t)| < \eta$ and therefore $\sigma \in N$. We claim that whenever $h_0(\tau_0, y_0 - x_0) < \delta$ and $\sigma \in N$, one obtains that

$$h(t, y(t, \tau_0, y_0) - x(\sigma(t), t_0, x_0)) < \epsilon, \ t \geq \tau_0.$$

If not, there exists a solution $y(t, \tau_0, y_0)$ and $t_1 > \tau_0$ such that

$$h(t_1, y(t_1, \tau_0, y_0) - x(\sigma(t_1), t_0, x_0)) = \epsilon \tag{5.7}$$

and

$$h(t, y(t, \tau_0, y_0) - x(\sigma(t), t_0, x_0)) < \epsilon, \ \tau_0 \leq t < t_1.$$

We then get from (5.1), (5.2) and (5.7),

$$\begin{aligned} b(\epsilon) &= b(h(t_1, y(t_1, \tau_0, y_0) - x(\sigma(t_1), t_0, x_0))) \\ &\leq V(t_1, \sigma(t_1), y(t_1, \tau_0, y_0) - x(\sigma(t_1), t_0, x_0))) \\ &\leq r(t_1, \tau_0, u_0) < r(t_1, \tau_0, \delta_1) < b(\epsilon), \end{aligned}$$

a contradiction which proves $(h_0, h; \tau_4)$ - stability.

Based on the proof, it is not difficult to construct the proofs of other $(h_0, h; \tau_4)$ - stability properties [4, 5]. We do not repeat the rest of the proof.

If we wish to prove $(h_0, h; \tau_5)$ - stability properties, we only need to change the condition (iv) in Theorem 5.1 to

$$d(|1 - \sigma'(t)|) < w(t, \sigma(t), \sigma'(t))$$

and follow apppropriate modifications in the proofs.

REFERENCES

1. Lakshmikantham, V. (2000). *Criteria for a new concept of stability,* J. Korean Math. Soc., 37. 657 -664.
2. Lakshmikantham, V. and Leela, S., (2002). *A new concept unifying Lyapunov and orbital stabilities,* Comm. Appl. Analysis, 6, 289-296.
3. Lakshmikantham, V. and Leela, S., (1969). *Differential and integral inequalities, Vol I,* Academic Press.
4. Lakshmikantham, V., Leela, S. and Martynyuk, A. A., (1989). *Stability analysis of nonlinear systems,* Marcel Dekker, New York.
5. Lakshmikantham, V. and X. Liu., (1993). *Stability analysis in terms of two measures,* World Scientific.
6. Massera, J. L. (1964). *The meaning of stability,* Bol. Fac. Ingen. Agrimens. Montevideo 8. 405-429.

NOVI UNIFICIRANI KONCEPT STABILNOSTI

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U korisnoj i bogatoj oblasti teorije stabilnosti nonlinearnih sistema, dogodila su se mnoga prečišćavanja, proširenja i generalizacije [3, 4]. U osnovi stabilnost se odnosi na uporedjivanje položaja faznog prostora rešenja poremećenih i neporemećenih jednačina sa klasičnom Ljapinovom stabilnošću koja postavlja isuviše strog zahtev i orbitalne stabilnosti, koja postavlja isuviše slab zahtev. Novi koncept stabilnosti se definiše tako da može da objedinjuje ova dva ekstremna slučaja (i verovatno mnoge druge odgovarajuće pomove i pojave izmedju ovih dvaju) na osnovu odgovarajuće topologije prateći ideju J. L. Masera [6]. Dalja unifikacija se takodje ostvaruje koristeći dve mere [5]. U ovom unificiranom okviru, ukazujemo na potrebne uslove za održivost ovih koncepata preko Ljapinovih funkcija.

Ključne reči: *stabilnost časovnika, Ljapinova stabilnost, orbitalna stabilnost.*