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GOLDEN ROUTE TO CHAOS

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Abstract. *A strict mathematical description of appearance of a deterministic chaos in the set of orbits of the critical circle map* $\theta_{n+1} = \theta_n + \Omega - (1/2\pi)\sin(2\pi\theta_n) \mid \bmod 1$*, is considered. All relevant terms such as coupled oscillators, self-similar winding numbers, mode-locking, Farey sequences, Arnold tongues are thoroughly discussed. The locked-ratios of Fibonacci numbers that lie on the zigzag path on the Farey-tree approaching the golden mean* $\gamma = (\sqrt{5} - 1)/2$*, or its unitary complement, which is the stem of the expression "golden route to chaos". This pattern occurs in many dynamical systems such as negative resistance circuits, biochemical and chemical oscillators, thermofluid convections, lasers, cortical neural oscillators and so on.*

INTRODUCTION: COUPLED OSCILLATORS

Oscillators are everywhere around us. Many of them are connected by exchange of energy, in other words they are *coupled*. The Figure 1 shows some examples. Usually, one oscillator is active as a source of power (*driving* one), and the another is passive (*driven* one). The second half of XX century was recognized as a period of increasing interest for strange behavior of such "simple" physical systems.

The fact that oscillators as systems being thoroughly described by a set of well known equations still exhibit unpredictable dynamics was attractive enough to involve many extraordinary specialists from different fields of science: physics, mathematics, astronomy, chemistry, biology etc. One of the most astonishing phenomenon connected with oscillators is *chaos* – the state in which a dynamical system behaves unpredictable, following very complex patterns. It was early discovered that nonlinearity was responsible for chaotic motion regimes. Among the first mathematically correct explanations were those by Duffing (nonlinear spring driven oscillator and the corresponding famous equation [5]), Van der Pol (self-excited oscillator based on the electronic tube circuit [5]) and others. The underlying topology of coupled oscillators' dynamics was studied by Arnold, Zaslavskii, Hopf and others.

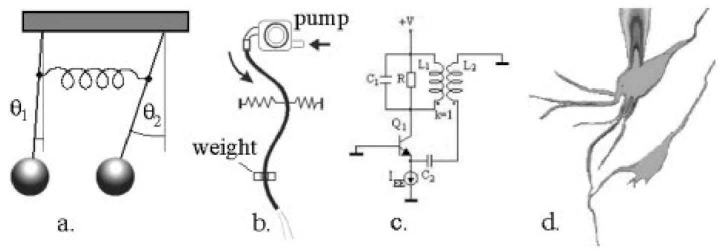

Figure 1. Coupled oscillators: a. Pendula; b. Hydrodynamics; c. Electronic; d. Neurals.

The driving force *A cos* ωt may cause *subharmonic* oscillations of the form B $\cos(\omega t/n + \varphi)$ playing an important role in pre-chaotic states of the system and its transition to chaos. The superposition of the two subharmonics gives,

$$x(t) = b_1 \cos ?_1 t + b_2 \cos ?_2 t\,, \tag{1}$$

where ω_1 is the driving–force frequency and ω_2 is the resonance frequency. If the ratio $\Omega = \omega_1/\omega_2$ is rational number, one speaks about *periodic* regime. If Ω is irrational, the oscillator is in *quasi–periodic* (or *aperiodic*) regime. The surface that "carries" the trajectory in the phase space is *torus* $T = T(b_1, b_2) = C(b_1)\times C(b_2)$, the Descartes product of two circles with radii b_1 and b_2 (Figure 2, left). Therefore, the *Zaslavskii map* (generalized torus to torus map [5]) is a very important tool in studying coupled oscillators' dynamics. In the case of periodic regime ($\Omega \in \mathbf{Q}$), the trajectory of the point $x(t)$ (orbit) makes a winding line on T known as *Hopf fibration* [4] (fig. 2, right). If $\Omega \notin \mathbf{Q}$, orbits densely cover the entire torus surface.

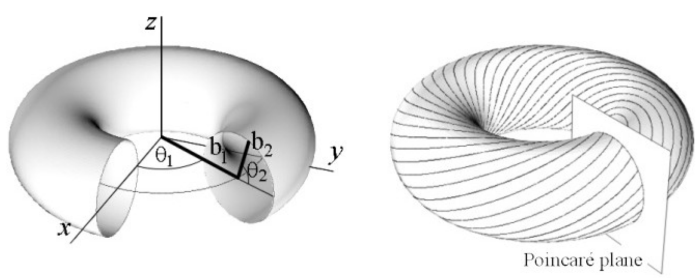

Figure 2. Torus $T = \{x = (b_1 + b_2 \cos\theta_2)\cos\theta_1,\ y = (b_1 + b_2\cos\theta_2)\sin\theta_1,\ z = b_2 \sin\theta_2\}$ and Hopf fibration on its surface for $\Omega = 21/34$ with the Poincaré plane.

From the point of view of complex and chaotic motion, it is sometimes enough to consider the *Poincaré sections* of T (see [1]), the cross section, with the *Poincaré plane*, i.e. the plane orthogonal to the first dimension of the torus (Figure 2, right).

MODE-LOCKING AND CIRCLE MAPS

One of the most interesting occurrences connected with quasiperiodicity of coupled oscillators is *mode-locking* (also: phase-locking, frequency-locking, synchronization). This was observed as early as in 17. Century by Duch physicist Christian Huygens. He noticed that two pendulum clocks, mechanically attached to a common structure become synchronized as the system evolves in time. Recently, many mode-locking models were found in diverse fields: solid-state physics, biological systems, chemistry, fluid mechanics etc.

If the frequencies ω_1 and ω_2 in (1) are commensurable, the Poincaré section of T consists of a finite number of discrete points, arranged in the vertices of a regular polygon inscribed in the circle $C(b_2)$. This is a periodic case, and $\Omega = \omega_1/\omega_2$ ισ rational. In the case of irrationality of Ω, the Poincaré section is the sequence of points covering the whole circle $C(b_2)$. This is the consequence of irregular structure of the sequence $\{\theta_\nu, n\in\mathbf{N}_0\}$, which is given by the iteration

$$\theta_{n+1} = \phi(\theta_n), \text{ where } \phi(\theta) = \theta + \Omega - \frac{K}{2\pi}\sin 2\pi\theta \,. \tag{2}$$

Here, $\Omega=\omega_1/\omega_2$ ισ the *frequency ratio* and K is a positive real constant called *coupling strength parameter* that controls nonlinearity. The mapping ϕ in (2) is known as the *standard circle map*. If it takes “mod 1” then ϕ clearly maps [0,1] into itself and therefore it maps circle to circle. In terms of equation (2), the mode-locking means that so called *winding number* (also *rotational number*) $\omega = \lim_{n\to\infty}(\theta_\nu - \theta_0)/n$ “locks” into rational ratios, with the curious tendency of having small denominators.

The mode locking can be observed even on the sky since planets are important coupled (non-dissipative) oscillators. For ex. the Moon’s spinning frequency relates to its revolution around Earth as 1:1. The same ratio for the planet Mercury is 2:3. It goes back to Gauss, who 1812 discovered that the orbit of the asteroid Pallas was locked to the orbital period of Jupiter in the precise ratio 7:18 ([6]). The planets are driven by gravitation, but neurons of the human brain also oscillate pairwise. In fact, they are electrically coupled oscillators [2]. Namely, hippocampal pyramidal and granule cells show small depolarization caused by attenuated dendritic action potentials. This causes electric coupling of two neurons (axonal coupling) which seems to be the mechanism for ultra-fast neuronal communication.

FAREY TREE

One of the most important mathematical theorems says that the set **Q** of rational numbers is countable. This means that the elements from **Q** may be “ordered”, i.e. put in the for of the sequence $\mathbf{r} = \{r_1, r_2, r_3, \ldots, r_n, \ldots\}$. There are infinitely many ways to construct the sequence **r**. One, connected with coupled oscillators, is known as *Farey tree*. It is the set **Q** arranged as a two-dimensional array (Figure 3).

Kappraff and Adamson [3] gave an effective algorithm for constructing the Farey tree. Here, this algorithm is formally encoded and an inverse algorithm is given as well.

<u>*Algorithm* 1.</u> (Direct algorithm $n \mapsto r_n$ ($\mathbf{N} \to \mathbf{Q}$)). Let $n \in \mathbf{N}$, $n = (b_0 b_1 ... b_m)_2 \Rightarrow (b_0 b_1 ... b_m b_m)_2 \Rightarrow \{\beta_0, \beta_1, ..., \beta_\mu\}$, where β_j = card{ $b_{\nu+1}, b_{\nu+2}, ..., b_{\nu+j}$ }, $b_\iota + b_{\iota+1} = 1 \wedge b_\iota b_{\iota+1} = 0$. Then,

$$r_n = \cfrac{1}{\beta_0 + \cfrac{1}{\beta_1 + \cfrac{1}{\cdots + \cfrac{1}{\beta_\mu}}}} \tag{3}$$

which usually shortens to $r_n = 1/\beta_0 + 1/\beta_1 + 1/...+ 1/\beta_\mu$, or to $r_n = [0; \beta_0, \beta_1, ..., \beta_\mu]$.

The Algorithm 1 gives r_1=1/2, r_2=2/3, r_3=1/3, r_4=3/4, r_5=3/5, r_6=2/5, r_7=1/4, r_8=4/5, r_9=5/7, r_{10}=5/8, r_{11}=4/7, r_{12}=3/7, r_{13}=3/8, r_{14}=2/7, r_{15}=1/5, r_{16}=5/6, r_{17}=7/9 etc. This sequence forms the Farey tree shown on Figure 3. Beginning with 1/2 and counting right to left, we can count all the rationals from [0, 1].

Two rationals p/q and r/s from Farey tree are called *adjacents*, if $|ps - qr| = 1$. A *Farey sum* (or *mediant*) of two adjacents is $(p/q) \oplus (r/s) = (p+r)/(q+s)$. Note that adjacent rationals from Farey tree always lie in adjacent levels and are south-east/north-west or south-west/north-east neighbors. Their mediant is the one-level-up element that is situated “between” them. For ex., 1/3 and 2/5 are adjacents and $(1/3) \oplus (2/5) = 3/8$. Also, $12/17 = (5/7) \oplus (7/10)$. The following inverse algorithm gives the position of a given rational number (from [0, 1]) in the hierarchy of Farey tree.

<u>*Algorithm* 2.</u> (Inverse algorithm $r = p/q \mapsto n$ ($\mathbf{Q} \to \mathbf{N}$)). Let $r \in \mathbf{Q}$ with continued fraction representation $r = [0; \beta_0, \beta_1, ..., \beta_\mu]$; Then, in binary representation, n = (1…1)(0…0)(1…1) … (1…1) where blocks of “ones” and “zeroes” contain $\beta_0, \beta_1, ..., \beta_{\mu-1}$ and $\beta_\mu - 1$ elements, respectively.

Application of this algorithm reveals that 6/7 is 32-nd member of the Farey tree hierarchy, while 1/7 is 63-rd. In fact, these two rationals are boundary elements of the fifth level of the Farey tree (see Fig. 3).

Inside the Farey tree, there is the special subsequence with indices 1, 2, 5, 10, 21, 42, 85, 170, 341, 682, 1365, 2730, … This is the sequence of Fibonacci numbers ratios F_{i+1}/F_{i+2}, for $i \in \mathbf{N}$,

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \frac{34}{55}, \frac{55}{89}, \frac{89}{144}, \frac{144}{233}, \frac{233}{377}, \ldots \tag{4}$$

which is marked on Figure 3 and linked by the bold line. Let us recall that the Fibonacci sequence is given by $F_0 = 0$, $F_1 = 1$, $F_{i+1} = F_i + F_{i-1}$, $i \in \mathbf{N}$. The leftmost and rightmost quotients in the basis of the tree are $0/1 = F_0/F_1$ and $1/1 = F_1/F_2$.

There is another algorithm ([6]) that allows calculating two immediate successors of an element r_n of Farey tree, and these are r_{2n} and r_{2n+1}. For ex. the immediate successors of $r_9 = 5/7$ are $r_{18} = 8/11$ and $r_{19} = 7/10$. The algorithm is based on a well known identity of continuous fractions $[0; a_1,\ldots,a_k] = [0; a_1,\ldots,a_k - 1,1]$.

<u>*Algorithm* 3.</u> (Immediate successors) Let r_n, $n \geq 1$, be any element of the Farey tree with the continued fraction expansion $r_n = [0; \beta_0, \beta_1, ..., \beta_\mu]$; Then,

$$r_{2n} = [0; \beta_0, \beta_1, ..., \beta_\mu + 1], \quad r_{2n+1} = [0; \beta_0, \beta_1, ..., \beta_\mu - 1, 2] \text{ for even } n;$$

$$r_{2n+1} = [0;\, \beta_0, \beta_1, ..., \beta_\mu + 1], \quad r_{2n} = [0;\, \beta_0, \beta_1, ..., \beta_\mu - 1, 2] \ \text{ for odd } \ n\,.$$

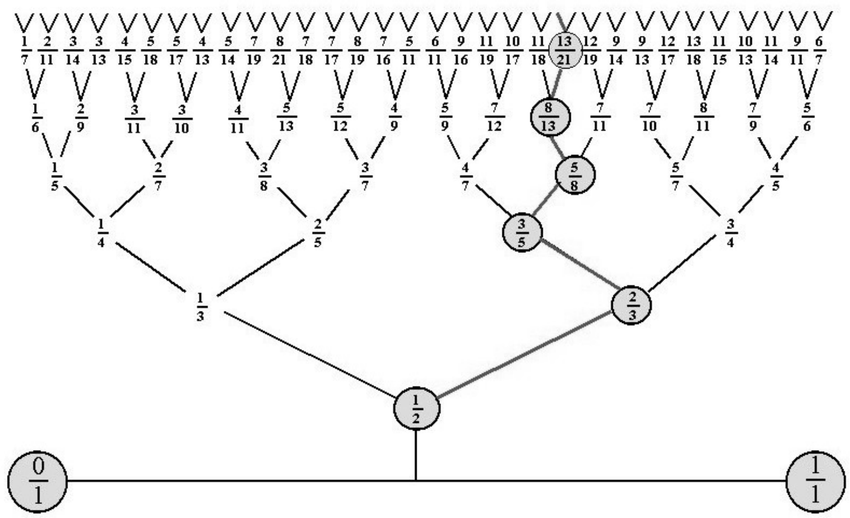

Figure 3. Farey tree.

The above example, $r_9 = 5/7$ gives the following continued fraction expansion 5/7 = [0; 1, 2, 2] = [0; 1, 2, 1, 1]. The last digits should be increased by one, so the first fraction yields [0; 1, 2, 3] = 7/10 and, by Algorithm 3 it is r_{19}, while the equivalent fraction gives [0; 1, 2, 1, 2] = 8/11 which is r_{18}. The 51252–nd element of the Farey tree is the fraction 327/769. The Algorithm 3 gives [0; 2, 2, 1, 5, 2, 1, 1, 3] as its continued fraction, so that [0;2, 2, 1, 5, 2, 1, 1, 4] = 418/983 (the 102504-th element) and [0; 2, 2, 1, 5, 2, 1, 1, 2, 2] = 563/1324 (the 102505-th element).

DEVIL'S STAIRCASE AND ARNOLD TONGUES

If the parameter K in (2) vanishes, two oscillators are uncoupled – no energy exchange exists. In this case, the winding number $\omega = \lim_{n\to\infty}(\theta_n - \theta_0)/n$ equals Ω. If the coupling strength increases, $0<K<1$, the oscillations may be periodic even though Ω is irrational. This is known as the *mode-locking* regime. In fact, the periodic or mode-locked motions persist for the whole finite-width interval of Ω. This means that the graph $\omega = \omega(\Omega)$ will have small plateaux. The "oldest" rational in Farey hierarchy, 1/2 corresponds to the widest plateau (Figure 4, left). Two symmetric, narrower plateaux correspond to the next two, 1/3 and 2/3, and so on.

The mediant gives the local hierarchy of the widths of the plateaux, determining the plateau with the greatest width between two plateaux characterized by adjacent rationals. The resulting graph is known as *devil's staircase* and has *fractal* structure, as close-ups in Figure 4 show. This graph is a line with constant segment at every rational.

Therefore, between two adjacent plateaux of periodicity p/q and r/s, there is smaller mediant plateau (p+r)/(q+s). Unlike the "classic" devil's staircase generated by the Cantor set that exhibit self-similar structure, this ones are *asymptotically self similar*. However, the precise scaling law is known only in the vicinity of the famous *Golden mean* $\gamma = (\sqrt{5} - 1)/2 = 0.61803\ldots$ The shape of devil's staircase depends on K. For K = 0 it becomes the straight line (all plateaux has zero width). By increasing K, the plateau widths increase as well. When K exceeds 1 (critically strong coupling), the plateaux overlap and the system starts behave chaotically.

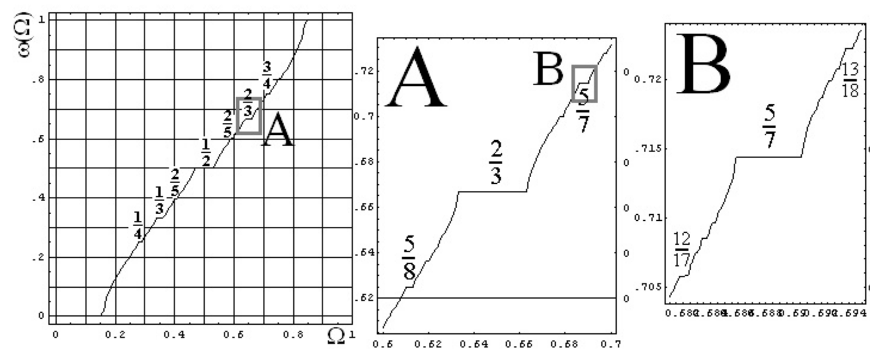

Figure 4. Devil's staircase for the circle map (K = 0.8) and close-ups.

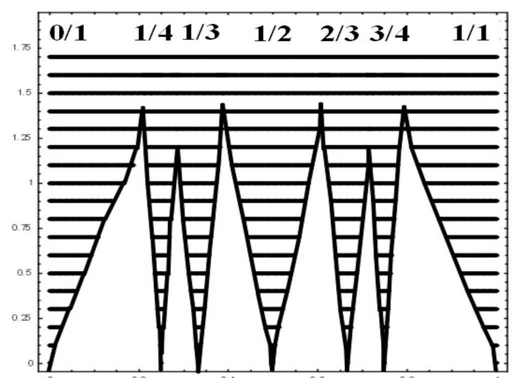

Figure 5. Arnold tongues for the critical circle map (K=1).

Increasing of plateaux is represented by the K versus Ω diagram that has the form of "tongues" (shaded area in Figure 5) known as *Arnold tongues*. This diagram shows the geometry of quasiperiodic scenario for transition to chaos. Since two parameters change, (K and Ω) we say that bifurcation sequence has codimension 2. Note that each Arnold tongue corresponds to one rational number. Between two adjacent rational tongues there is a narrower one corresponding to the Farey sum of the previous two.

If the system reaches critically strong coupling $K \to 1_-$, the chaotic regime become possible. Then, the circle map (2) is called *critical* and the coupled system "jumps" from one stable mode-locking regime to the next one in the hierarchy with the biggest possible denominator, i.e. to the mediant one. Since the "optimal" resonance is 1:1, it is the beginning of the route that may lead to chaos. The next mode-locking possibility is 1/2 which is the closest possibility and the first term in Farey tree. Next is mediant of the 1/1 and 1/2 which is 2/3. Then, mode-locking frequencies go to 3/5, 5/8, 8/13, etc, just following the sequence (4) and the bold zigzag line in Figure 3. In other words, the system follows the sequence of Fibonacci ratios F_i/F_{i+1} ($i \in \mathbf{N}$). Since $\lim_{i\to\infty} (F_i/F_{i+1}) = \gamma = (\sqrt{5} - 1)/2$ i.e. the Golden mean, this route is called *golden route to chaos*. Of course, by some circumstances, the system may start at the ratio 0/1 (the leftmost in Fig. 3). Then the route includes the following mode-locking states: 1/2, 1/3, 2/5, 3/8, 5/13, 8/21, 13/24 …, which is the sequence $\{F_i/F_{i+2},\ i \in \mathbf{N}\}$. This time, limit is γ^2 which is $1-\gamma = 0.381966...$

But, there are also other paths along the Farey tree. If a coupled system starts gliding to chaos, we say that the quasiperoidic route to chaos is on. The tendency of the frequency ratio Ω is to take rational values with as small denominators as possible. In this way, if it starts with a fraction which continued form is $\Omega_1 = [0; a_1,\dots, a_k]$ it continues to the ratio $[0; a_1,\dots, a_k, 1] = \Omega_2$, which is the one of immediate successors of Ω_1, then goes to the successor of Ω_2 which is $\Omega_3 = [0; a_1,\dots, a_k, 1, 1]$, then to $\Omega_4 = [0; a_1,\dots, a_k, 1, 1, 1]$ etc. It is easy to see (by Algorithm 3, for ex.) that the sequence $\{\Omega_i\}$ forms a zigzag path of fractions with small denominators. The route is as fast as the denominators are smaller. Therefore, the fastest route is this one starting in the first vertex of the Farey tree, [0; 2]= [0; 1, 1], and following the sequence [0; 1, 1], [0; 1, 1, 1], [0; 1, 1, 1, 1], [0; 1, 1, 1, 1, 1], … which is the Fibonacci ratios sequence $\{F_i/F_{i+1},\ i\in\mathbf{N}\}$. Thus, the "golden route" to chaos is the fastest one among all others. The opposite, slowest route goes along the harmonic sequence 1/2, 1/3, 1/4, 1/5, … 1/n , … which is represented by the rightmost branch of the Farey tree.

Note: The authors want to mention that Figures 2 – 5 and algorithms are created by their own software being developed in MATHEMATICA 4.1 environment.

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ZLATNI PUT U HAOS

Ljubiša M. Kocic i Liljana R. Stefanovska

U radu se razmatra pojava deterministickog haosa na skupu orbita kriticnog kružnog preslikavanja $\theta_{n+1} = \theta_n + \Omega - (1/2\pi)\sin(2\pi\theta_n)$ | mod 1, i daje se njegova stroga matematicka interpretacija. Detaljno se obraduju svi pridruženi fenomeni, kao sto su spregnuti oscilatori, samo-slicni zavojni brojevi, zakljucavanja faze, Fareyevi nizovi i Arnoldovi jezici. Fibonaccievi brojni odnosi zakljucavanja koji leže na cik-cak putanji Fareyevog drveta, i koji konvergiraju ka odnosu zlatnog preseka $\gamma = (\sqrt{5} - 1)/2$ ili ka njegovom jedinicnom komplementu, objašnjavaju naziv "zlatni put u haos". Ovakva šema prelaska u haos je karakteristicna za mnoge dinamicke sisteme, kao na primer za elektricna kola sa negativnom otpornošcu, za biohemijske i hemijske oscilatore, za provodjenje toplote fluidima, za lasere, za kortikalne neuro-oscilatore itd.