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ON GENERALIZATION OF MULTIVARIABLE HARMONIC POLYNOMIALS

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Abstract. *In this paper is presented a class of multivariable homogeneous orthogonal polynomials, obtained as linear combination of classical generalized Laguerre polynomials. Using them, the generalized harmonic polynomials are defined. It is proven that multivariable harmonic polynomials are particular case of generalized harmonic polynomials.*

Key words: *Multivariable harmonic polynomials, Multivariable hypergeometric polinomials, Gauss hypergeometric polynomials, Lauricella functions; Laguerre polynomials, Multivariable Appell polynomials; Jacobi shifting polynomials.*

1. INTRODUCTION AND PRELIMINARIES

The polynomials in r variables satisfying Laplace partial equation

$$\Delta F(x_1,\dots,x_r)=0\,, \tag{1}$$

where $\Delta=\dfrac{\partial^2}{\partial x_1}+\dots+\dfrac{\partial^2}{\partial x_r}$ $\quad(r\in N\setminus\{1\})$ are harmonic. They are used in mechanics, electromagnetic, theory of oscillations, and especially in construction of cubature formulas.

In paper [13] the notes of cubature formulas are common roots of linear combinations of basic orthogonal polynomials $V_{n-i,i}(x,y)$ $(n=4,6;\ i=0,1,\dots,n)$ for unit circle and weight function equal to one [14, p.169, table 83] and monic Hermite polynomials $H_{n-i,i}(x,y):=H_{n-i}(x)H_i(y)$ for the whole plane and weight function $(x,y)\to w(x,y):=e^{-x2-y2}$.

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The orthogonal polynomials of fourth degree

$$\begin{aligned} e &:= V_{4,0} - 6V_{2,2} + V_{0,4} = x^4 - 6x^2y^2 + y^4, \\ f &:= H_{4,0} - 6H_{2,2} + H_{0,4} = x^4 - 6x^2y^2 + y^4, \\ g &:= V_{3,1} - V_{1,3} = x^3y - xy^3, \\ h &:= H_{3,1} - H_{1,3} = x^3y - xy^3 \end{aligned} \tag{2}$$

and the orthogonal polynomials of degree six

$$\begin{aligned} e_1 &:= V_{6,0} - 15V_{4,2} + 15V_{2,4} - V_{0,6} = x^6 - 15x^4y^2 + 15x^2y^4 - y^6, \\ f_1 &:= H_{6,0} - 15H_{4,2} + 15H_{2,4} - H_{0,6} = x^6 - 15x^4y^2 + 15x^2y^4 - y^6, \\ g_1 &:= V_{5,1} - \frac{10}{3}V_{3,3} + V_{1,5} = x^5y - \frac{10}{3}x^3y^3 + xy^5, \\ h_1 &:= H_{5,1} - \frac{10}{3}H_{3,3} + H_{1,5} = x^5y - \frac{10}{3}x^3y^3 + xy^5 \end{aligned} \tag{3}$$

are harmonic polynomials.

The first author of this paper came to the idea to form linear combinations of the orthogonal polynomials of arbitrary degrees n, which also gave harmonic polynomials of the degree n, used in construction of Gauss's cubature formulas exact for all polynomials of the degree $2n$-1 [2].

By construction of cubature formulas with weight function $(x,y)\rightarrow w(a,b;x,y) := |x|^a |y|^b \varphi(x^2+y^2)$ the polynomials with parameters a and b, which for $a = b = 0$ reduce to harmonic polynomials, are necessary.

In paper [5] the generalization of harmonic polynomials in two variables is realized by considering linear combination of monic generalized Hermite polynomials $H_n(a,t)$, orthogonal on interval $(-\infty,+\infty)$with weight function $|t|^a e^{-t^2}$ and defined as [15]

$$H_n(a,t) := t^\delta L_l^{(\frac{a-1}{2}+\delta)}(t^2) \tag{4}$$

$$\left(l = \left[\frac{n}{2}\right]; \delta = n - 2l = \begin{cases} 0; & \text{n even} \\ 1; & \text{n odd} \end{cases} \right),$$

where $L_l^{(a-1)}(t) := \sum_{i=0}^{l} (-1)^{l+i}(a+i)_{l-i} t^i$ are monic generalized Laguerre polynomials, orthogonal on $[0,+\infty)$ with weight function $t^{a-1}e^{-t}$, where $[k]$ denotes the greatest integer in $k \in R$ and $(a)_n := a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol (precisely, rising factorial n-th power of a), with general definition

$$(a)_{n,d} := \begin{cases} 1, & (n = 0; a \neq 0) \\ a(a+d)...(a+(n-1)d), & (d \geq 0; n \in N) \end{cases} \tag{5}$$

for which it holds $(a)_n = (a)_{n,1}$.

Also note that

$$a_{n,d} = d^n \left(\frac{a}{d}\right)_n . \tag{6}$$

By forming a linear combination of form

$$\sum_{l=0}^{s} (-1)^l \binom{s}{l} (a+1+2\delta_1+2s-2l)_{l,2}(b+1+2\delta_2+2l)_{s-l,2} H_{2s-2l+\delta_1,2l+\delta_2}(a,b;x,y),$$

where is

$$\begin{aligned} &H_{2s-2l+\delta_1,2l+\delta_2}(a,b;x,y) = H_{2s-2l+\delta_1}(a;x)H_{2l+\delta_2}(b;y) \\ &:= \sum_{j=0}^{s-l}\sum_{k=0}^{l} (-1)^{s+j+k} \binom{s-l}{j}\binom{l}{k} 2^{j+k-s}(a+1+2j+2\delta_1)_{s-l-j,2}(b+1+2k+2\delta_2)_{l-k,2}\, x^{2j+\delta_1} y^{2k+\delta_2} \quad (7) \\ &(s \in N_0; \delta_1, \delta_2 \in \{0,1\}; a,b \in C^+; x,y \in C). \end{aligned}$$

one can get the generalized harmonic polynomials [5].

The linear combination of monic generalized Laguerre polynomials, $\sum_{i=0}^{s} (-1)^i \binom{s}{i} (a+s-i)_i (b+i)_{s-i}$ $L_{s-i}^{(a-1)}(x) L_i^{(b-1)}(y)$, gives the polynomials [4]

$$Z_s(a,b;x,y) := \sum_{l=0}^{s} (-1)^l \binom{s}{l} (a+s-l)_l (b+l)_{s-l}\, x^{s-l} y^l , \tag{8}$$

$$(s \in N_0; a,b \in C^+; x,y \in C) .$$

The generalized harmonic polynomials are defined using Z polynomials in the following way

$$H_{G2-\delta_1,2-\delta_2}^{(2s+\delta)}(a,b;x,y) := 2^s x^{\delta_1} y^{\delta_2} Z_s\left(\frac{a+2\delta_1+1}{2}, \frac{b+2\delta_2+1}{2}; x^2, y^2\right) , \tag{9}$$

$$(\delta_1, \delta_2 \in \{0,1\}; \delta := \delta_1 + \delta_2) .$$

The first index is polynomial degree and subscripts indicate parity (even - (2), odd - (1)) of degree of variables *x* and *y*, respectively.

In paper [7] is presented algorithm and described program package for symbolic generation of basic orthogonal polynomials in two variables $E_{n-i,i}^{(\alpha,\beta)}(a,b;x,y)$, ,($n = 0,1,...;$ $i = 0,1,...,n$) over the triangle $T_2 := \{(x,y)|x + y \le 1; x,y \ge 0\}$, with weight function $(x,y) \to w(\alpha,\beta,a,b;x,y) := x^{a-1}y^{b-1}(x + y)^{\beta}(1 - x - y)^{\alpha}$, $(a,b > 0; \alpha,\beta > -1)$.

The basic orthogonal polynomials in two variables are related to homogeneous orthogonal polynomials $Z_S(a,b;x,y)$ and Jacobi shifted orthogonal polynomials of variable $t := x + y$ with the following equality

$$\| d_{s+1,j+1} \| \| E_{n-i,i}^{(\alpha,\beta)}(a,b;x,y) \| = \| Z_s(a,b;x,y) P_{n-s}^{(\alpha,\beta+2s+a+b-1)}(x+y) \| , \tag{10}$$

where the general element of quadratic matrix of order n+1 is of form

$$d_{s+1,j+1}=\sum_{i=0}^{s}(-1)^i\binom{s}{i}\binom{n-s}{j-i}(a+s-i)_i(b+i)_{s-i}\ , \qquad (11)$$

$$(s=0,1,...,n;\ j=0,1,...,n).$$

$\| E_{n-i,i}^{(\alpha,\beta)}(a,b;x,y)\|$ and $\| Z_s(a,b;x,y)P_{n-s}^{(\alpha,\beta+2s+a+b-1)}(x+y)\|$ are column vectors of order $n+1$.

In equality (10) $P_{n-s}^{(\alpha,\beta+2s+a+b-1)}(t)$ are shifted Jakobi polynomials of degree $n-s$ for weight function $t^{\beta+2s+a+b-1}(1-t)^{\alpha}$ on interval $[0,1]$.

The equality (10) can be also used for obtaining other basic orthogonal polynomials for first quadrant and other axisymetric areas. For example, in order to get basic orthogonal polynomials $L^{\alpha}_{n-i,i}(a,b;x,y)$ for first quadrant $D_2 := \{(x,y)|\ 0 \le x < +\infty,\ 0 \le y < +\infty\}$ and weight function $(x,y) \to w(\alpha,a,b;x,y) := x^{a-1}y^{b-1}(x+y)^{\alpha}e^{-x-y}$, $(a,b > 0;\ \alpha \ge 0)$, it is necessary to replace in equality (10) the shifted Jakobi polynomials by monic generalized Laguerre polynomials $L_{n-s}^{(\alpha+2s+a+b-1)}(x+y)$.

The definition of Z polynomials in three variables and corresponding generalized harmonic polynomials have been given in [8]. We will be dealing with them in section 3.

In paper [9] Z polynomials in three variables and corresponding generalized harmonic polynomials have been used for construction of cubature formulas of degree up to 11.

All those resaults are milestons to generalization of multivariable harmonic polynomials. The starting point for obtaining of r-dimensional Z and generalized harmonic polynomials are the results given in [1,10,11,12].

2. Multivariable Z polynomials and harmonic polynomials

At construction of cubature formulas for central symmetric areas of r-dimensional Euclid space and weight functions of form $(x_1,...,x_r)\to w(a_1,...,a_r;\ x_1,...,x_r) := |x_1|^{a_1}...\ |x_r|^{a_r}\varphi(x_1^2+...+x_r^2)$, the generalization of harmonic polynomials with parameters a_j, $(j = 1,2,...,r)$ is necessary. Those polynomials are denoted as generalized harmonic polynomials (H_G). Define at first a new class of homogeneous polynomials in r variables as $Z^{(s,r;t_1,...,t_{r-2})}(a_1,...,a_r;\ x_1,...,x_r)$, where s ($s \in N_0$) is degree of polynomial, r is number of parameters and arguments, t_i $(i = 1,...,r-2)$ are orders of polynomials, a_j,x_j $(j = 1,...,r)$ are positive parameters and real arguments, respectively.

The definition of polynomials is

$$Z^{(s,r;t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)$$
$$:=\sum_{l_1=0}^{s-t_1}\sum_{l_2=0}^{t_1-t_2}\cdots\sum_{l_{r-2}=0}^{t_{r-3}-t_{r-2}}\sum_{l_{r-1}=0}^{t_{r-2}}(-1)^{l_1+l_2+...+l_{r-1}}\binom{s-t_1}{l_1}\binom{t_1-t_2}{l_2}\cdots\binom{t_{r-3}-t_{r-2}}{l_{r-2}}\binom{t_{r-2}}{l_{r-1}}$$
$$(a_1+s-t_1-l_1)_{l_1}(a_2+t_1-t_2-l_2)_{l_2}\cdots(a_{r-1}+t_{r-2}-l_{r-1})_{l_{r-1}}(a_r+l_1+l_2+...+l_{r-1})_{s-l_1-l_2-...-l_{r-1}} \qquad (12)$$
$$x_1^{s-t_1-l_1}x_2^{t_1-t_2-l_2}\cdots x_{r-1}^{t_{r-2}-l_{r-1}}x_r^{l_1+...+l_{r-1}},$$
$$(s\in N_0;\ t_1=0,1,\ldots,s;\ t_i=0,1,..,t_{i-1};\ i=2,\ldots,r-2).$$

The whole number of Z polynomials of degree s is $h(s,r)=\binom{s+r-2}{r-2}$.

Z polynomials have the following features:

- Z polynomials are part of right hand side of generalized multinomial formula

$$\sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\cdots\sum_{n_{r-2}=0}^{n_{r-1}}\sum_{n_{r-1}=0}^{n_r}\sum_{l_1=0}^{s-t_1}\sum_{l_2=0}^{t_1-t_2}\cdots\sum_{l_{r-2}=0}^{t_{r-3}-t_{r-2}}\sum_{l_{r-1}=0}^{t_{r-2}}(-1)^{l_1+l_2+\ldots+l_{r-1}}\binom{s-t_1}{l_1}\binom{t_1-t_2}{l_2}\cdots\binom{t_{r-3}-t_{r-2}}{l_{r-2}}\binom{t_{r-2}}{l_{r-1}}$$
$$\binom{n-s}{n_1-t_1-l_1}\binom{n_1-t_1-l_1}{n_2-t_2-l_1-l_2}\binom{n_2-t_2-l_1-l_2}{n_3-t_3-l_1-l_2-l_3}\cdots\binom{n_{r-3}-t_{r-3}-l_1-l_2-\ldots-l_{r-3}}{n_{r-2}-t_{r-2}-l_1-l_2-\ldots-l_{r-3}-l_{r-2}}$$
$$\binom{n_{r-2}-t_{r-2}-l_1-l_2-\ldots-l_{r-3}-l_{r-2}}{n_{r-1}-l_1-l_2-\ldots-l_{r-2}-l_{r-1}}(a_1+s-t_1-l_1)_{l_1}(a_2+t_1-t_2-l_2)_{l_2}\ldots(a_{r-1}+t_{r-2}-l_{r-1})_{l_{r-1}} \tag{13}$$
$$(a_r+l_1+l_2+\ldots+l_{r-1})_{s-l_1-l_2-\ldots-l_{r-1}}x_1^{n-n_1}x_2^{n_1-n_2}\ldots x_{r-1}^{n_{r-2}-n_{r-1}}x_r^{n_{r-1}}$$
$$=(x_1+\ldots+x_r)^{n-s}Z^{(s,r;t_1,\ldots,t_{r-2})}(a_1,\ldots,a_r;x_1,\ldots,x_r),$$

which for $s=0$ becomes classical multinomial formula;

- Z polynomials are part of right hand side of functional relation

$$\sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\cdots\sum_{n_{r-2}=0}^{n_{r-1}}\sum_{n_{r-1}=0}^{n_r}\sum_{l_1=0}^{s-t_1}\sum_{l_2=0}^{t_1-t_2}\cdots\sum_{l_{r-2}=0}^{t_{r-3}-t_{r-2}}\sum_{l_{r-1}=0}^{t_{r-2}}(-1)^{l_1+l_2+\ldots+l_{r-1}}\binom{s-t_1}{l_1}\binom{t_1-t_2}{l_2}\cdots\binom{t_{r-3}-t_{r-2}}{l_{r-2}}\binom{t_{r-2}}{l_{r-1}}$$
$$\binom{n-s}{n_1-t_1-l_1}\binom{n_1-t_1-l_1}{n_2-t_2-l_1-l_2}\binom{n_2-t_2-l_1-l_2}{n_3-t_3-l_1-l_2-l_3}\cdots\binom{n_{r-3}-t_{r-3}-l_1-l_2-\ldots-l_{r-3}}{n_{r-2}-t_{r-2}-l_1-l_2-\ldots-l_{r-3}-l_{r-2}}$$
$$\binom{n_{r-2}-t_{r-2}-l_1-l_2-\ldots-l_{r-3}-l_{r-2}}{n_{r-1}-l_1-l_2-\ldots-l_{r-2}-l_{r-1}}(a_1+s-t_1-l_1)_{l_1}(a_2+t_1-t_2-l_2)_{l_2}\ldots(a_{r-1}+t_{r-2}-l_{r-1})_{l_{r-1}} \tag{14}$$
$$(a_r+l_1+l_2+\ldots+l_{r-1})_{s-l_1-l_2-\ldots-l_{r-1}}G_{n-n_1,n_1-n_2,\ldots,n_{r-2}-n_{r-1},n_{r-1}}(\lambda,a_1,\ldots,a_r;x_1,\ldots,x_r)$$
$$=G_{n-s}(\lambda+s,a_1+\ldots+a_r+2s;x_1+\ldots+x_r)Z^{(s,r;t_1,\ldots,t_{r-2})}(a_1,\ldots,a_r;x_1,\ldots,x_r),$$

where monic Lauricella's polynomial $G_{n-n_1,n_1-n_2,\ldots,n_{r-2}-n_{r-1},n_{r-1}}(\lambda,a_1,\ldots,a_r;x_1,\ldots,x_r)$ is defined as

$$G_{n-n_1,n_1-n_2,\ldots,n_{r-2}-n_{r-1},n_{r-1}}(\lambda,a_1,\ldots,a_r;x_1,\ldots,x_r)$$
$$=(-1)^n\frac{(a_1)_{n-n_1}(a_2)_{n_1-n_2}\ldots(a_{r-1})_{n_{r-2}-n_{r-1}}(a_r)_{n_{r-1}}}{(\lambda)_n}$$
$$F_A^{(r)}[\lambda,-n+n_1,-n_1+n_2,\ldots,-n_{r-2}+n_{r-1},-n_{r-1};a_1,\ldots,a_r;x_1,\ldots,x_r] \tag{15}$$
$$:=\sum_{i_1=0}^{n-n_1}\sum_{i_2=0}^{n_1-n_2}\cdots\sum_{i_{r-1}=0}^{n_{r-2}-n_{r-1}}\sum_{i_r=0}^{n_{r-1}}(-1)^{n+i_1+\ldots+i_r}\binom{n-n_1}{i_1}\binom{n_1-n_2}{i_2}\cdots\binom{n_{r-2}-n_{r-1}}{i_{r-1}}\binom{n_{r-1}}{i_r}$$
$$\frac{(a_1+i_1)_{n-n_1-i_1}(a_2+i_2)_{n_1-n_2-i_2}\ldots(a_{r-1}+i_{r-1})_{n_{r-2}-n_{r-1}-i_{r-1}}(a_r+i_r)_{n_{r-1}-i_r}}{(\lambda+i_1+\ldots+i_r)_{n-i_1-\ldots-i_r}}x_1^{i_1}\ldots x_r^{i_r}$$

and monic Gauss's hypergeometric polynomial $G_{n-s}(\lambda+s,a_1+\ldots+a_r+2s;x_1+\ldots+x_r)$ as

$$G_{n-s}(\lambda+s,a_1+...+a_r+2s;x_1+...+x_r)$$
$$=(-1)^{n-s}\frac{(a_1+...+a_r+2s)_{n-s}}{(\lambda+s)_{n-s}}\,{}_2F_1(\lambda+s,-n+s;a_1+...+a_r+2s;x_1+...+x_r) \quad (16)$$
$$:=\sum_{i=0}^{n-s}(-1)^{n-s+i}\binom{n}{i}\frac{(a_1+...+a_r+2s+i)_{n-s-i}}{(\lambda+s+i)_{n-s-i}}(x_1+...+x_r)^i.$$

Relation (14) for $s = 0$ reduces to functional relation [12, p.359, Eq.(4.8)]

$$\sum_{n_1+...+n_r=n}\binom{n}{n_1,...,n_r}(a_1)_{n_1}...(a_r)_{n_r}F_A^{(r)}[\lambda,-n_1,...,-n_r;a_1,...,a_r;x_1,...,x_r]$$
$$=(A)_n\,{}_2F_1(-n,\lambda;A;x_1+...+x_r) \quad (17)$$
$$(A:=a_1+...+a_r;n,n_j\in N_0;A,a_j\notin Z_0^-;j=1,...,r).$$

- Z polynomials can be obtained as linear combinations of classical monic Laguerre's polynomials $L_n^{(a-1)}(t)$, what follows from the next functional relation

$$\sum_{l_1=0}^{s-t_1}\sum_{l_2=0}^{t_1-t_2}...\sum_{l_{r-2}=0}^{t_{r-3}-t_{r-2}}\sum_{l_{r-1}=0}^{t_{r-2}}(-1)^{l_1+l_2+...+l_{r-1}}\binom{s-t_1}{l_1}\binom{t_1-t_2}{l_2}...\binom{t_{r-3}-t_{r-2}}{l_{r-2}}\binom{t_{r-2}}{l_{r-1}}$$
$$(a_1+s-t_1-l_1)_{l_1}(a_2+t_1-t_2-l_2)_{l_2}...(a_{r-1}+t_{r-2}-l_{r-1})_{l_{r-1}}(a_r+l_1+l_2+...+l_{r-1})_{s-l_1-l_2-...-l_{r-1}} \quad (18)$$
$$L_{s-t_1-l_1}^{(a_1-1)}(x_1)L_{t_1-l_2}^{(a_2-1)}(x_2)...L_{t_{r-2}-l_{r-1}}^{(a_{r-1}-1)}(x_{r-1})L_{l_1+...+l_{r-1}}^{(a_r-1)}(x_r)=Z^{(s,r:t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)$$

  which can be proven by mathematical induction;

- Polynomials $Z^{(s,r:t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)$ are in connection with Lauricella hypergeometric function $F_B^{(r)}[a_1,...,a_r,b_1,...,b_r;c;x_1,...,x_r]$ by relation

$$Z^{(s,r:t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)=$$
$$(a_r)_s x_1^{s-t_1}x_2^{t_1-t_2}...x_{r-1}^{t_{r-2}}F_B^{(r)}[-s+t_1,-t_1+t_2,...,-t_{r-3}+t_{r-2},-a_1-s+t_1+1,-a_2-t_1+t_2+1,...,$$
$$-a_{r-1}-t_{r-2}+1;a_r;-\frac{x_r}{x_1},...,-\frac{x_r}{x_{r-1}}] \quad (19)$$
$$:=(a_r)_s x_1^{s-t_1}x_2^{t_1-t_2}...x_{r-1}^{t_{r-2}}\sum_{l_1=0}^{s-t_1}\sum_{l_2=0}^{t_1-t_2}....\sum_{l_{r-2}=0}^{t_{r-3}-t_{r-2}}\sum_{l_{r-1}=0}^{t_{r-2}}(-s+t_1)_{l_1}(-t_1+t_2)_{l_2}...(-t_{r-3}+t_{r-2})_{l_{r-2}}$$
$$(-t_{r-2})_{l_{r-1}}\frac{(-a_1-s_1+t_1)_{l_1}(-a_2-t_1+t_2+1)_{l_2}...(-a_{r-1}-t_{r-2}+1)_{l_{r-1}}}{(a_r)_{l_1+...+l_{r-1}}}\frac{\left(-\frac{x_r}{x_1}\right)^{l_1}}{l_1!}...\frac{\left(-\frac{x_r}{x_{r-1}}\right)^{l_{r-1}}}{l_{r-1}!}$$

- Using $Z^{(s,r:t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)$ polynomials, the generalized harmonic polynomials of r variables can be defined as follows

$$H_{G\,2-\delta_1,...,2-\delta_r}^{(2s+\delta,r:t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)$$
$$:=2^s x_1^{\delta_1}....x_r^{\delta_r}Z^{(s,r:t_1,...,t_{r-2})}(\frac{a_1+2\delta_1+1}{2},...,\frac{a_r+2\delta_r+1}{2};x_1^2,...,x_r^2) \quad (20)$$
$$(s\in N_0;t_1=0,1,\ldots,s,t_i=0,1,..,t_{i-1};i=2,\ldots,r-2;\delta_j\in\{0,1\};j=1,...,r;\delta:=\delta_1+...+\delta_r).$$

In case when all degree of variables are even, (20) reduces to

$$H_{G2,...,2}^{(2s,r:t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)$$
$$:= 2^s Z^{(s,r:t_1,...,t_{r-2})}(\frac{a_1+1}{2},...,\frac{a_r+1}{2};x_1{}^2,...,x_r{}^2).$$

The second case appears when degree of variable x_1 is odd and the rest of degrees are even

$$H_{G1,2,...,2}^{(2s+1,r:t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)$$
$$:= 2^s x_1 Z^{(s,r:t_1,...,t_{r-2})}(\frac{a_1+3}{2},\frac{a_2+1}{2},...,\frac{a_r+1}{2};x_1{}^2,...,x_r{}^2).$$

At least, 2^r-th case appears when all degrees of variable x_i are odd

$$H_{G1,...,1}^{(2s+r,r:t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)$$
$$:= 2^s x_1...x_r Z^{(s,r:t_1,...,t_{r-2})}(\frac{a_1+3}{2},...,\frac{a_r+3}{2};x_1{}^2,...,x_r{}^2).$$

Using identity (6) for $d = 2$ the generalized harmonic polynomials can be written in developed form:

$$H_{G2-\delta_1,...,2-\delta_r}^{(2s+\delta,r:t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)$$
$$:= \sum_{l_1=0}^{s-t_1}\sum_{l_2=0}^{t_1-t_2}...\sum_{l_{r-2}=0}^{t_{r-3}-t_{r-2}}\sum_{l_{r-1}=0}^{t_{r-2}}(-1)^{l_1+l_2+...+l_{r-1}}\binom{s-t_1}{l_1}\binom{t_1-t_2}{l_2}..\binom{t_{r-3}-t_{r-2}}{l_{r-2}}\binom{t_{r-2}}{l_{r-1}}$$
$$(a_1+2\delta_1+1+2s-2t_1-2l_1)_{l_1,2}(a_2+2\delta_2+1+2t_1-2t_2-2l_2)_{l_2,2}\cdots \quad (21)$$
$$(a_{r-1}+2\delta_{r-1}+1+2t_{r-21}-2l_{r-1})_{lr-1,2}(a_r+2\delta_r+1+2l_1+...+2l_{r-1})_{s-l_1-...-l_{r-1},2}$$
$$x_1{}^{2s-2t_1-2l_1+\delta_1}x_2{}^{2t_1-2t_2-2l_2+\delta_2}...x_{r-1}{}^{2t_{r-2}-2l_{r-1}+\delta_{r-1}}+x_r{}^{2l_1+...+2l_{r-1}+\delta_r}.$$

The harmonic polynomials in r variables are defined by following general formula [1, p.84]

$$H_{2-\delta_1,...,2-\delta_r}^{(2s+\delta,r:t_1,...,t_{r-2})}(a_1,...,a_r;x_1,...,x_r)$$
$$:= \sum_{l_1=0}^{s-t_1}\sum_{l_2=0}^{t_1-t_2}...\sum_{l_{r-2}=0}^{t_{r-3}-t_{r-2}}\sum_{l_{r-1}=0}^{t_{r-2}}(-1)^{l_1+l_2+...+l_{r-1}}\binom{l_1+...+l_{r-1}}{l_1,...,l_{r-1}}$$
$$\binom{2s+\delta}{2s-2t_1-2l_1+\delta_1,2t_1-2t_2-2l_2+\delta_2,...,2t_{r-2}-2l_{r-1}+\delta_{r-1},2l_1+...+2l_{r-1}+\delta_r} \quad (22)$$
$$x_1{}^{2s-2t_1-2l_1+\delta_1}x_2{}^{2t_1-2t_2-2l_2+\delta_2}...x_{r-1}{}^{2t_{r-2}-2l_{r-1}+\delta_{r-1}}+x_r{}^{2l_1+...+2l_{r-1}+\delta_r}$$
$$(s\in N_0; t_1=0,1,\ldots,s, \mathrm{t_i}=0,1,..,\mathrm{t_{i-1}}; \mathrm{i}=2,\ldots,\mathrm{r}-2; \delta_j\in\{0,1\}; j=1,...,r; \delta:=\delta_1+...+\delta_r),$$

where is $\binom{n_1+...+n_r}{n_1,...,n_r} := \frac{(n_1+...+n_r)!}{n_1!...n_r!}$.

Involving the normalization factor of form

$$K_{2-\delta_1,\dots,2-\delta_r} = \frac{2^{2s} s!(2s+\delta_r+1)_{\delta-\delta_r}}{(2s-2t_1+\delta_1)!(2t_1-2t_2+\delta_2)!\dots(2t_{r-3}-2t_{r-2}+\delta_{r-2})!(2t_{r-2}+\delta_{r-1})!},$$

harmonic polynomials in r variables can be obtained using polynomials $Z^{(s,r:t_1,\dots,t_{r-2})}(\delta_1+\frac{1}{2},\dots,\delta_r+\frac{1}{2};x_1{}^2,\dots,x_r{}^2)$, what follows from the next theorem.

Theorem 1. Harmonic polynomials in r variables $H^{(2s+\delta,r:t_1,\dots,t_{r-2})}_{2-\delta_1,\dots,2-\delta_r}(x_1,\dots,x_r)$ and polynomials $Z^{(s,r:t_1,\dots,t_{r-2})}(\delta_1+\frac{1}{2},\dots,\delta_r+\frac{1}{2};x_1{}^2,\dots,x_r{}^2)$ are connected with the identity

$$H^{(2s+\delta,r:t_1,\dots,t_{r-2})}_{2-\delta_1,\dots,2-\delta_r}(x_1,\dots,x_r) = K_{2-\delta_1,\dots,2-\delta_r}\, x_1{}^{\delta_1}\dots x_r{}^{\delta_r} Z^{(s,r:t_1,\dots,t_{r-2})}(\delta_1+\frac{1}{2},\dots,\delta_r+\frac{1}{2};x_1{}^2,\dots,x_r{}^2). \quad (23)$$

Proof. From $Z^{(s,r:t_1,\dots,t_{r-2})}(\delta_1+\frac{1}{2},\dots,\delta_r+\frac{1}{2};x_1{}^2,\dots,x_r{}^2)$ polynomial definition, the right hand side in Theorem can be written in the developed form

$$\begin{aligned}
&\frac{2^{2s} s!(2s+\delta_r+1)_{\delta-\delta_r}}{(2s-2t_1+\delta_1)!(2t_1-2t_2+\delta_2)!\dots(2t_{r-3}-2t_{r-2}+\delta_{r-2})!(2t_{r-2}+\delta_{r-1})!}\\
&\sum_{l_1=0}^{s-t_1}\sum_{l_2=0}^{t_1-t_2}\dots\sum_{l_{r-2}=0}^{t_{r-3}-t_{r-2}}\sum_{l_{r-1}=0}^{t_{r-2}}(-1)^{l_1+l_2+\dots+l_{r-1}}\binom{s-t_1}{l_1}\binom{t_1-t_2}{l_2}\dots\binom{t_{r-3}-t_{r-2}}{l_{r-2}}\binom{t_{r-2}}{l_{r-1}}\\
&(s-t_1-l_1+\delta_1+\tfrac{1}{2})_{l_1}(t_1-t_2-l_2+\delta_2+\tfrac{1}{2})_{l_2}\dots(t_{r-2}-l_{r-1}+\delta_{r-1}+\tfrac{1}{2})_{lr-1}\\
&(l_1+\dots+l_{r-1}+\delta_{r3}+\tfrac{1}{2})_{s-l_1-\dots-l_{r-1}}\, x_1{}^{2s-2t_1-2l_1+\delta_1} x_2{}^{2t_1-2t_2-2l_2+\delta_2}\dots x_{r-1}{}^{2t_{r-2}-2l_{r-1}+\delta_{r-1}} x_r{}^{2l_1+\dots+2l_{r-1}+\delta_r}
\end{aligned} \quad (24)$$

Using formulas

$$(m+d+\frac{1}{2})_n = \frac{(2n+2m+d)!m!}{(m+n)!(2m+d)!2^{2n}} \text{ and } \binom{m}{i} = \frac{m!}{(m-i)!i!} (d \in \{0,1\}) \quad (25)$$

and after obvious simplifications and using identity $(2s+\delta_r)!(2s+\delta_r+1)_{\delta-\delta_r} = (2s+\delta)!$, the previous relation reduces to the right hand side of (22). □

Involving parameters $\delta_i (i = 1,\dots,r)$ values from the set $\{0,1\}$, one can obtain 2^s different harmonic polynomials.

3. TRIVARIABLE Z POLYNOMIALS AND HARMONIC POLYNOMIALS

We will especially deal with Z polynomials, generalized harmonic H_G and harmonic H polynomials of three variables, because this case is the most interesting for appliance. The case of two variable was enquired in papers [2,3,4,5,6].

For $r = 3$, $t_1 = t$, $a_1 = a$, $a_2 = b$, $a_3 = c$, $x_1 = x$, $x_2 = y$, $x_3 = z$, the following relations and formulas hold:

$$Z^{(s,3:t)}(a,b,c;x,y,z) := \sum_{l_1=0}^{s-t}\sum_{l_2=0}^{t}(-1)^{l_1+l_2}\binom{s-t}{l_1}\binom{t}{l_2}(a+s-t-l_1)_{l_1}(b+t-l_2)_{l_2} (c+l_1+l_2)_{s-l_1-l_2}x^{s-t-l_1}y^{t-l_2}z^{l_1+l_2}, \tag{26}$$

$$(s\in N_0; t=0,1,\dots,s; a,b,c\in C^+; x,y,z\in C).$$

The overall number of Z polynomials of degree s is s+1.

Z polynomials have the following features:

- Z polynomials are part of the right hand side of generalized trinomial formula

$$\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{l_1=0}^{s-t}\sum_{l_2=0}^{t}(-1)^{l_1+l_2}\binom{s-t}{l_1}\binom{t}{l_2}\binom{n-s}{i-l_1-t}\binom{i-l_1-t}{j-l_1-l_2} (a+s-t-l_1)_{l_1}(b+t-l_2)_{l_2}(c+l_1+l_2)_{s-l_1-l_2}x^{n-i}y^{i-j}z^j = (x+y+z)^{n-s}Z^{(s,3:t)}(a,b,c;x,y,z), \tag{27}$$

which for s=0 becomes classical trinomial formula;

- Z polynomials are part of the right hand side of functional relation

$$\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{l_1=0}^{s-t}\sum_{l_2=0}^{t}(-1)^{l_1+l_2}\binom{s-t}{l_1}\binom{t}{l_2}\binom{n-s}{i-l_1-t}\binom{i-l_1-t}{j-l_1-l_2} (a+s-t-l_1)_{l_1}(b+t-l_2)_{l_2}(c+l_1+l_2)_{s-l_1-l_2} G_{n-i,i-j,j}(\lambda,a,b,c;x,y,z) = G_{n-s}(\lambda+s,a+b+c+2s;x+y+z)Z^{(s,3:t)}(a,b,c;x,y,z) \tag{28}$$

$$(n=N_0; s=0,1,\dots,n; t=0,1,\dots,s; \lambda,a,b,c\in C^+; x,y,z\in C),$$

where monic Lauricella's polynomials $G_{n-i,i-j,j}(\lambda,a,b,c;x,y,z)$ are defined as

$$G_{n-i,i-j,j}(\lambda,a,b,c;x,y,z) = (-1)^n\frac{(a)_{n-i}(b)_{i-j}(c)_j}{(\lambda)_n}F_A^{(3)}[\lambda,-n+i,-i+j,-j;a,b,c;x,y,z] := \sum_{i_1=0}^{n-i}\sum_{i_2=0}^{i-j}\sum_{i_3=0}^{j}(-1)^{n+i_1+i_2+i_3}\binom{n-i}{i_1}\binom{i-j}{i_2}\binom{j}{i_3} \frac{(a+i_1)_{n-i-i_1}(b+i_2)_{i-j-i_2}(c+i_3)_{j-i_3}}{(\lambda+i_1+i_2+i_3)_{n-i_1-i_2-i_3}}x^{i_1}y^{i_2}z^{i_3}, \tag{29}$$

and monic Gauss hypergeometric polynomials $G_{n-s}(\lambda+s,a+b+c+2s;x+y+z)$ as

$$G_{n-s}(\lambda+s,a+b+c+2s;x+y+z) = (-1)^{n-s}\frac{(a+b+c+2s)_{n-s}}{(\lambda+s)_{n-s}}{}_2F_1(\lambda+s;-n+s;a+b+c+2s;x+y+z) := \sum_{i=0}^{n-s}(-1)^{n-s+i}\binom{n-s}{i}\frac{(a+b+c+2s+i)_{n-s-i}}{(\lambda+s+i)_{n-s-i}}(x+y+z)^i; \tag{30}$$

- Z polynomials can be obtained as linear combinations of classical monic Laguerre's polynomials $L_n^{(a-1)}(t)$, what follows from the next functional relation

$$\sum_{l_1=0}^{s-t}\sum_{l_2=0}^{t}(-1)^{l_1+l_2}\binom{s-t}{l_1}\binom{t}{l_2}(a+s-t-l_1)_{l_1}(b+t-l_2)_{l_2}$$
$$(c+l_1+l_2)_{s-l_1-l_2}L_{s-t-l_1}^{(a-1)}(x)L_{t-l_2}^{(b-1)}(y)L_{l_1+l_2}^{(c-1)}(z) \tag{31}$$
$$=Z^{(s,3:t)}(a,b,c;x,y,z),$$

  which can be proven by mathematical induction;

- Z polynomials are in connection with Appell's hypergeometric function $F_3(a_1,a_2,b_1,b_2;c;x;y)$ by relation

$$Z^{(s,3:t)}(a,b,c;x,y,z)$$
$$=(c)_s x^{s-t}y^t F_3(-s+t,-t,-a-s+t+1,-b-t+1;c;-\frac{z}{x},-\frac{z}{y})$$
$$:=(c)_s x^{s-t}y^t\sum_{l_1=0}^{s-t}\sum_{l_2=0}^{t}\frac{(-s+t)_{l_1}(-t)_{l_2}(-a-s+t+1)_{l_1}(-b-t+1)_{l_2}}{(c)_{l_1+l_2}} \tag{32}$$
$$\frac{(-\frac{z}{x})^{l_1}}{l_1!}\frac{(-\frac{z}{y})^{l_2}}{l_2!}.$$

Using $Z^{(s,3:t)}(a,b,c;x,y,z)$ polynomials, the generalized harmonic polynomials of three variables can be defined as follows.

$$H_{G2-\delta_1,2-\delta_2,2-\delta_3}^{(2s+\delta,3:t)}(a,b,c;x,y,z):=$$
$$2^s x^{\delta_1}y^{\delta_2}z^{\delta_3}Z^{(s,3:t)}(\frac{a+2\delta_1+1}{2},\frac{b+2\delta_2+1}{2},\frac{c+2\delta_3+1}{2};x^2,y^2,z^2) \tag{33}$$
$$(s\in N_0;t=0,1,...,s;\delta_1,\delta_2,\delta_3\in\{0,1\};\delta:=\delta_1+\delta_2+\delta_3).$$

The first index is polynomial degree, *t* is polynomial order, and subscripts indicate parity (even - (2), odd - (1)) of degree of variables *x*, *y*, *z*, respectively.

Using identity (6) for $d = 2$, the generalized harmonic polynomials can be written as follows.

$$H_{G2-\delta_1,2-\delta_2,2-\delta_3}^{(2s+\delta,3:t)}(a,b,c;x,y,z)$$
$$:=\sum_{l_1=0}^{s-t}\sum_{l_2=0}^{t}(-1)^{l_1+l_2}\binom{s-t}{l_1}\binom{t}{l_2}(a+2\delta_1+1+2s-2t-2l_1)_{l_1,2} \tag{34}$$
$$(b+2\delta_2+1+2s+2t-2l_2)_{l_2,2}(c+2\delta_3+1+2l_1+2l_2)_{s-l_1-l_2,2}$$
$$x^{2s-2t-2l_1+\delta_1}y^{2t-2l_2+\delta_2}z^{2l_1+2l_2+\delta_3}.$$

The harmonic polynomials of three variables can be defined using following formula [1, p.14]

$$H^{(2s+\delta,3:t)}_{2-\delta_1,2-\delta_2,2-\delta_3}(a,b,c;x,y,z) := \sum_{l_1=0}^{s-t}\sum_{l_2=0}^{t}(-1)^{l_1+l_2}\binom{l_1+l_2}{l_1,l_2}\binom{2s+\delta}{2s-2t-2l_1+\delta_1,2t-2l_2+\delta_2,2l_1+2l_2+\delta_3} x^{2s-2t-2l_1+\delta_1}y^{2t-2l_2+\delta_2}z^{2l_1+2l_2+\delta_3} \tag{35}$$

$(s\in N_0; t=0,1,...,s; \delta_1,\delta_2,\delta_3\in\{0,1\}; \delta:=\delta_1+\delta_2+\delta_3)$.

Involving the normalization factor of form $K_{2-\delta_1,2-\delta_2,2-\delta_3}=\dfrac{2^{2s}s!(2s+\delta_3+1)_{\delta_1+\delta_2}}{(2s-2t+\delta_1)!(2t+\delta_2)!}$, harmonic polynomials of three variable can be obtained using polynomials $Z^{(s,3:t)}(a,b,c;x,y,z)$. It follows from the next.

Theorem 2. Harmonic polynomials of three variable $H^{(2s+\delta,3:t)}_{2-\delta_1,2-\delta_2,2-\delta_3}(a,b,c;x,y,z)$ and polynomials $Z^{(s,3:t)}(\delta_1+\frac{1}{2},\delta_2+\frac{1}{2},\delta_3+\frac{1}{2};x^2,y^2,z^2)$ are connected with the identity

$$H^{(2s+\delta,3:t)}_{2-\delta_1,2-\delta_2,2-\delta_3}(a,b,c;x,y,z) = K_{2-\delta_1,2-\delta_2,2-\delta_3}x^{\delta_1}y^{\delta_2}z^{\delta_3}Z^{(s+1,3:t)}(\delta_1+\frac{1}{2},\delta_2+\frac{1}{2},\delta_3+\frac{1}{2};x^2,y^2,z^2). \tag{36}$$

Proof. Using definition of polynomials $Z^{(s,3:t)}(\delta_1+\frac{1}{2},\delta_2+\frac{1}{2},\delta_3+\frac{1}{2};x^2,y^2,z^2)$, the right hand side in Theorem can be written in the developed form

$$\frac{2^{2s}s!(2s+\delta_3+1)_{\delta_1+\delta_2}}{(2s-2t+\delta_1)!(2t+\delta_2)!}\sum_{l_1=0}^{s-t}\sum_{l_2=0}^{t}(-1)^{l_1+l_2}\binom{s-t}{l_1}\binom{t}{l_2} (s-t-l_1+\delta_1+\frac{1}{2})_{l_1}(t-l_2+\delta_2+\frac{1}{2})_{l_2}(l_1+l_2+\delta_3+\frac{1}{2})_{s-l_1-l_2} x^{2s-2t-2l_1+\delta_1}y^{2t-2l_2+\delta_2}z^{2l_1+2l_2+\delta_3}. \tag{37}$$

Using formulas (25) after obvious simplifications and using identity

$$(2s+\delta)!=(2s+\delta_3)!(2s+\delta_3+1)_{\delta_1+\delta_2}, \quad (\delta:=\delta_1+\delta_2+\delta_3) \tag{38}$$

the previous relation becomes

$$\sum_{l_1=0}^{s-t}\sum_{l_2=0}^{t}(-1)^{l_1+l_2}\frac{(l_1+l_2)!}{l_1!l_2!}\frac{(2s+\delta)!}{(2s-2t-2l_1+\delta_1)!(2t-2l_2+\delta_2)!(2l_1+2l_2+\delta_3)!} x^{2s-2t-2l_1+\delta_1}y^{2t-2l_2+\delta_2}z^{2l_1+2l_2+\delta_3}. \tag{39}$$

After taking usual prefixes for polynomial coefficients, the polynomials $H_{2-\delta_1,2-\delta_2,2-\delta_3}^{(2s+\delta,3:t)}(a,b,c;x,y,z)$ are obtained, and the Theorem is proven. □

Involving values from the set {0,1} to parameters δ_1, δ_2, δ_3, one can obtain eight different harmonic polynomials.

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O GENERALIZACIJI HARMONIČNIH POLINOMA VIŠE PROMENLJIVIH

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U ovom radu je predstavljena jedna klasa homogenih ortogonalnih polinoma više promenljivih koja se dobija kao linearna kombinacija klasičnih generalisanih Laguerreovih polinoma. Pomoću njih su definisani generalisani harmonični polinomi. Dokazano je da su harmonični polinomi više promenljivih partikularni slučajevi generalisanih harmoničnih polinoma.