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Letter to Editor

THE VARIATIONAL PRINCIPLE FOR MONGE-AMPERE EQUATION BY THE SEMI-INVERSE METHOD

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Abstract. *By the semi-inverse method proposed Ji-Huan He, a variational principle is established for Monge-Ampere equation.*

Key words: *variational principle , Monge-Ampere , nonlinearity*

This paper studies the nonlinear Monge-Ampere equation in the form[1]:

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2+A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}-D=0\,, \tag{1}$$

where A,B,C, and D are constants.

Our aim is to search for, by the semi-inverse method [2,3], a variational principle whose stationary condition satisfies the above equation. To this end, we first consider the following linear partial differential equation:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}-D=0\,. \tag{2}$$

The Lagrangian of Eq. (2) can be found with ease:

$$L_1(u)=-\frac{1}{2}A\left(\frac{\partial u}{\partial x}\right)^2-\frac{1}{2}B\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}-\frac{1}{2}C\left(\frac{\partial u}{\partial y}\right)^2-Du\,. \tag{3}$$

To proceed, we regard the following Lagrangian

$$L_2(u)=au\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+bu\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\,, \tag{4}$$

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where a and b are constants.

The Euler equation for (4) is obviously

$$a\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+a\frac{\partial^2}{\partial x^2}(u\frac{\partial^2 u}{\partial y^2})+a\frac{\partial^2}{\partial y^2}(u\frac{\partial^2 u}{\partial x^2})+b(\frac{\partial^2 u}{\partial x\partial y})^2+2b\frac{\partial^2}{\partial x\partial y}(u\frac{\partial^2 u}{\partial x\partial y})=0\,. \quad (5)$$

By simple manipulation, Eq.(5) reduces to

$$3a\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+3b(\frac{\partial^2 u}{\partial x\partial y})^2+2(a+b)\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x\partial y^2}+2(a+b)u\frac{\partial^4 u}{\partial x^2\partial y^2}+2(a+b)\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x^2\partial y}=0\,. \quad (6)$$

If we set $a = 1/3$ and $b = -1/3$, then Eq.(6) becomes

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-(\frac{\partial^2 u}{\partial x\partial y})^2=0\,. \quad (7)$$

We, therefore, obtain the following variational principle

$$J(u)=\iint L dxdy\,, \quad (8)$$

where the Lagrangian , L, reads

$$L(u)=L_1(u)+L_2(u)=\frac{1}{3}u\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-\frac{1}{3}u(\frac{\partial^2 u}{\partial x\partial y})^2-\frac{1}{2}A(\frac{\partial u}{\partial x})^2-\frac{1}{2}B\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}-\frac{1}{2}C(\frac{\partial u}{\partial y})^2-Du\,. \quad (9)$$

It is easy to prove that the stationary condition of the obtained functional, Eq.(8), satisfies Eq.(1).

We illustrate hereby the effectiveness of the semi-inverse method, which is a powerful tool to the construction of variational formulations directly from the field equations.

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