FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol.4, N° 16, 2004, pp. 169 - 171

Letter to Editor

THE VARIATIONAL PRINCIPLE FOR YANG-MILLS EQUATION BY THE SEMI-INVERSE METHOD

UDC 517.9(045)

Hong-Mei Liu

College of Science, Donghua University, P.O. Box 471, 1882 Yan'an Xilu Road, Shanghai 200051, China

Abstract. *A variational principle is obtained for Yang-Mills equation by Ji-Huan He's semi-inverse method, which is proved to be a powerful mathematical tool to the search for a variational formulation directly from the field equations.*

Key words: *variational principle, Yang-Mills equation, nonlinearity*

In this paper we will study the coupled Yang-Mills equations in the forms [1]:

$$\beta^2 y'' - e^2\rho^2 z^2 y = 0\,, \tag{1}$$

$$\beta^2 z'' - e^2\alpha^2 y^2 z = 0\,, \tag{2}$$

where α, β, *e*, and ρ are constants.

In order to search for a variational formulation for the above coupled equations, we apply Ji-Huan He's semi-inverse method[3,4]. To this end, we construct the following trial-functional with an unknown function *F*:

$$J(y,z) = \int \left\{ \frac{1}{2}\beta^2 y'^2 + \frac{1}{2}e^2\rho^2 z^2 y^2 + F \right\} dx\,, \tag{3}$$

where *F* is an unknown function of z and/or its derivatives.

There exist many alternative approaches to constructing trial-functionals, illustrating examples can be found in details in Refs. [5,6,7]. The advantage of the above trial-functional lies on the fact that the stationary condition with respect to *y* results in Eq. (1).

Calculating the first order variation of the above functional, Eq. (3), with respect to *z* yields the following Euler equation:

Received November 20, 2003

$$e^2\rho^2 zy^2 + \frac{\delta F}{\delta z} = 0\,, \tag{4}$$

where $\frac{\delta F}{\delta z}$ is called variational derivative with respect to z, which is defined as

$$\frac{\delta F}{\delta z} = \frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\frac{\partial F}{\partial z_x} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial z_{xx}} - \cdots\,.$$

We search for such an F so that Eq.(4) turns out to be one of the field equations, Eq. (2), so we set

$$\frac{\delta F}{\delta z} = -e^2\rho^2 zy^2 = -\frac{\rho^2\beta^2}{\alpha^2}z''\,, \tag{5}$$

from which we can immediately identify F in the form:

$$F = \frac{1}{2}\frac{\rho^2\beta^2}{\alpha^2}z'^2\,. \tag{6}$$

Finally we obtain the following variational principle:

$$J(y,z) = \int\left\{\frac{1}{2}\beta^2 y'^2 + \frac{1}{2}\frac{\rho^2\beta^2}{\alpha^2}z'^2 + \frac{1}{2}e^2\rho^2 z^2 y^2\right\}dx \tag{7}$$

[Proof] Assuming that the values of y and z are prescribed on the boundary, i.e., $\delta y = \delta z = 0$, and calculating variation of the above obtained functional, Eq.(7), we have

$$\begin{aligned}
\delta J(y,z) &= \delta\int\left\{\frac{1}{2}\beta^2 y'^2 + \frac{1}{2}\frac{\rho^2\beta^2}{\alpha^2}z'^2 + \frac{1}{2}e^2\rho^2 z^2 y^2\right\}dx \\
&= \int\left\{\beta^2 y'\delta y' + \frac{\rho^2\beta^2}{\alpha^2}z'\delta z' + e^2\rho^2 z^2 y\delta y + e^2\rho^2 zy^2\delta z\right\}dx \\
&= \int\left\{-\beta^2 y''\delta y - \frac{\rho^2\beta^2}{\alpha^2}z''\delta z + e^2\rho^2 z^2 y\delta y + e^2\rho^2 zy^2\delta z\right\}dx \\
&= \int\left\{-(\beta^2 y'' - e^2\rho^2 z^2 y)\delta y - (\frac{\rho^2\beta^2}{\alpha^2}z'' - e^2\rho^2 zy^2)\delta z\right\}dx = 0.
\end{aligned}$$

In view of the arbitrary of δy and δz, we obtain the following Euler equations:

$$\beta^2 y'' - e^2\rho^2 z^2 y = 0\,, \tag{8}$$

$$\frac{\rho^2\beta^2}{\alpha^2}z'' - e^2\rho^2 zy^2 = 0\,. \tag{9}$$

It is obvious that the system of Eqs. (8) and (9) is equivalent to the original system of Eqs. (1) and (2).

From the obtained functional, Eq. (7), we can obtain the following Hamiltonian:

$$\frac{1}{2}\beta^2 y'^2 + \frac{1}{2}\frac{\rho^2\beta^2}{\alpha^2} z'^2 - \frac{1}{2}e^2\rho^2 z^2 y^2 = H\ , \tag{10}$$

which is an invariant.

To conclude, we find the semi-inverse method proposed by He is a powerful mathematical tool to the construction of variational formulations for physical problems.

REFERENCES

1. Shi-Kuo Liu, Shi-Da Liu. Nonlinear equations in physics, Peking University press, 2000, Beijing
2. He, J.H., Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics. *Int.. J. Turbo & Jet-Engines*, **14**(1), 1997, 23~28
3. He JH. *Generalized Variational Principles in Fluids*, Science & Culture Publishing House of China, Hongkong, China, 2003 (in Chinese)
4. He, J.H. Coupled variational principles of piezoelectricity, *Int. J. Engineering Sciences* , 39(3),2000, 323-341
5. He, J.H. A Classical Variational Model for Micropolar Elastodynamics, *International J. of Nonlinear Sciences and Numerical Simulation,* 1(2), 2000: 133-138
6. He,J.H. Variational theory for linear magneto-electro-elasticity , *International Journal of Nonlinear Science and Numerical Simulation* , 2(4), 2001, 309-316
7. Hao.T.H. Application of the Lagrange Multiplier Method the Semi-Inverse Method to the Search for Generalized Variational Principle in Quantum Mechanics, *International Journal of Nonlinear Science and Numerical Simulation,* 4(3), 2003, 311-312