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EIGENAMPLITUDE VECTORS AND FUNCTIONS EXTENDED ORTHOGONALITY OF SMALL OSCILLATIONS MIXED SYSTEMS OF THE COUPLED DISCRETE AND CONTINUOUS SUBSYSTEMS

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Abstract. *In this paper, by using general examples of mixed systems of the coupled discrete subsystem of rigid bodies and continuous subsystem, the extended orthogonality expression of own amplitude vectors and of own amplitude functions of small oscillations are derived. By using examples, the analogy between frequency equations of some classes of these systems is identified.*

Key words: *Mixed systems, coupled subsystems, rigid bodies, continuous, elastic body, eigen amplitude vectors*

1. INTRODUCTION

In many classical university books like Theory of Oscillations (see Ref. [1]) we can find many examples of classical tasks of frequency equations of discrete or continuous system oscillations, which are excited with initial perturbations of equilibrium natural state. During a the long time period, as a professor of Elastodynamics and Theory of Oscillations at the Faculty of Mechanical Engineering, I was the author of many original examine tasks and corresponding solutions of these tasks. In the teaching process I must show the students rational explanations of the same solutions, and properties of oscillatory processes of the system dynamics. By introducing some of my assistants to the teaching process I discuss the possibilities for different solutions of the equations of oscillatory systems dynamics and small transformations of the examine tasks definitions to compose new tasks but with the same solution philosophy. Nowdays, by using computer tools such as MathCad, Mathematica, Math Lab, a new powerful possibilities for the visualization of oscillatory processes in dynamical systems applied in engineering practice are very useful for the university teaching of the theory of oscillations as a

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accompanying tools to the analytical method and pure mathematical explanations. Using these WEB and MathCad information tools the examine tasks of Elastodynamics and The Theory of Oscillations in the teaching process and studies at the Faculty of Mechanical Engineering are presented at *www.masfak.ni.ac.yu/Elastodynamika* (see Ref. [2]). Some of these original tasks with the solutions are published in the three books (see Refs. [3], [4] and [5]).

The current research in the theory of discrete dynamical system oscillations is directed to the nonlinear phenomena (See Ref. [6]), as well as to the nonstationary processes, and also to the stochastic-like and chaotic-like processes in purely deterministic dynamical systems and conditions. In the theory of the oscillations of the continuous systems also nonlinear phenomena and damage and fracture structure of dynamical systems are the topics of the same premier journals and international scientific meetings and conferences (See Proceedings of ENOC Copenhagen 1999, Moscow 2002; ICNM Shanghai 1998, 2002.; Control Oscillations and Chaos COC 2000 Saint Petersburg, ...Issues of Journal Applied Mechanics Reviews and Refferativniy Zhurnal Mechanika Moscow.....). The pure elastic system is presently not in the focus of researchers (see Ref. [7]).

New materials in engineering systems are inspirations of many researchers for new constitutive relations discoveries in mathematical sense and for investigations on the dynamics of these constructions. In some published papers of the author the dynamics of discrete systems of the material particles which are constrained by the standard hereditary, rheological, or creep light elements (see Ref. [8], [9], [10], [11] and [12]) are investigated. These papers are inspired by the papers of Goroshko O. A. et. all. (see Ref. [13]).

In the monograph [14] by Goroshko O. A. and Hedrih (Stevanović) K. an analytical dynamics of the discrete hereditary systems, and corresponding solutions are first published as an integral theory of this kind of systems.

As a new material in active systems the piezoceramics is used. In the papers [15] and [16] piezoceramics behaviour in the vibrations regimes are presented as results of the analytical, numerical and experimental investigations of the vibrations frequency spectra. These results are important for investigations of active structure oscillations and control of oscillations.

In the paper [17] and [18] longitudinal hereditary vibrations and creep vibrations of a fractional derivative rheological rod with variable cross section are examined. Partial differential equation and particular solutions for the case of natural creep longitudinal vibrations of the rod of creep material of a fractional derivative order is accomplished. For the case of natural creep vibrations, eigenfunction and time-function, for different examples of boundary conditions are determined. Different boundary conditions are analysed and the series of eigenvalues and natural circular frequencies of longitudinal creep vibrations, as well as tables of these values are completed. Using the MathCad software a graphical representation of the time-function is presented.

Papers [19], [20], [21] and [22], consider the problem of transversal oscillations of a bar, which is free or under the action of the length-wise random forces.

In the paper [22] the problem of the transversal oscillations of two layer straight bar under the action of the length-wise random forces is considered. The excitation process is a bounded noise excitation. It is assumed, that the layers of the bar were made of creep continuously non homogenous material and that the corresponding modulus of elasticity and creep fractional derivative order constitutive relation of the each layer are continuous function of the length coordinate and the thickness coordinates. The equation of the transversal creep vibrations of a fractional derivative order constitutive relation beam are examined. Partial fractional-differential equation and particular solutions for the case of natural creep vibrations of the beam of creep material of a fractional derivative order constitutive relation in the case of the influence of rotation inertia is derived. For the case of natural creep vibrations, eigenfunction and time-function, for different examples of boundary conditions, are determined.

The paper [23] presents the discrete continuum method showing examples of homogenous discrete systems with the limited number of degrees of freedom. These systems are in the form of homogenous chains and nets in the space and plain. Material points of these nets and chains are tied by elastic, standard hereditary or creep elements. By introducing the trigonometric method for studying the properties and the equations of dynamics of discrete homogenous continuums author sets up the discrete continuum method for the study of dynamics of chain systems with hereditary or creeping connections. This systems dynamics is described by a system of integro-differential equations or differential equations with fractional derivatives. A light standard creep element is defined by a constitutive relation of stress-strain state, for the creation of which fractional order derivatives were used.

In the paper [23] we can follow keywords: discrete continuum, discrete hereditary system, discrete homogenous chain, discrete homogenous material net, elastic element, standard hereditary light element, standard creep light element, integro-differential relation, fractional derivatives order, Jules-Lissajous figure, trigonometric method, small vibrations. We can see an interaction between notions of words discrete continuum and continuous or discrete systems. It was inspiration for me to return my focus and pay attention to the mixed systems of the coupled discrete subsystem of rigid bodies and continuous subsystem and to compose characteristic – frequency equations of the small oscillations of these systems.

Papers [24] and [25] are also directed to the examination of the classical knowledge of continuous and discrete systems in order to make some new conclusions. Papers [26] and [27] give visualizations of oscillatory processes in classical oscillatory models of real systems and give new illumination of the properties of these systems variety of oscillatory processes.

This work is one new contribution to the knowledge of the mixed systems of the coupled discrete subsystem of rigid bodies and continuous subsystem to compose characteristic – frequency equations of the small oscillations.

We can conclude that new computer tools with power computer possibilities directed philosophy of considerations of real systems dynamics by using discretization of continuum as the way and method for solutions of problems, and by using many iterations continualizations of solutions. Discretizations and continualizations in the process of solutions and analysis of dynamical processes are opposite directions and good method for proving calculations and conclusions.

In accordance with narrow specializations of researchers we can not find more examples which considered mixed systems consisting of coupled discrete and continuous systems. And not very often there are some analytical results. In the époque of the large numerical experiments over the dynamical systems I think that it is very important to make some new classical examples of the frequency equations useful for the teaching process in the Theory of vibrations.

2. MODEL OF MIXED SYSTEMS OF THE COUPLED DISCRETE SUBSYSTEM OF RIGID BODIES AND CONTINUOUS SUBSYSTEM

Let consider two subsystems: one elastic rod, the axis of which is straight, as a continuous system solid deformable body with the following parameters: $\mathbf{E}$, ρ, ℓ, $\mathbf{A}$, and with two rigid weights at free ends with masses m_p and m_0 (see Figure 1.); This rod is constrained by the spring with the stiffness c_0 and coupled with discrete systems with n degree of freedom. For example, this discrete subsystem is a chain system of the n material particles with masses m_i, $i = 1,2,3,...n$, translatory movable along the line parallel to the axis of the rod; these masses are connected by the springs with stiffnesses c_i, $i = 1,2,3,...n$, We consider the relations between the longitudinal vibrations of the elastic rod and the free oscillations of the chain material particles system. Let's determine the frequency equations of the defined mixed system of the coupled discrete subsystem of rigid bodies and the continuous subsystem.

2.1. Differential Equation of the Longitudinal Oscillations of the Elastic Rod and Boundary Conditions

In accordance with the notations in the Figure 1. we can see that $u(x,t)$ represents the longitudinal displacement of the rod's cross section at the distance x measured from the left end of the rod in the axis direction at the time t. Partial differential equation of the longitudinal oscillations is:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c_e^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

where is: $c_e^2 = \dfrac{\mathbf{E}}{\rho}$.

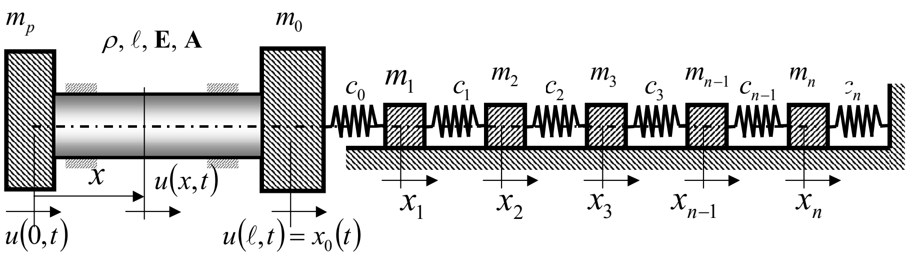

Fig. 1. Small oscillations of the mixed system of the coupled discrete and continuous subsystems Longitudinal oscillations of the beam with multi body chain with changeable numbers of material particles.

Solution of the equation (1) is in the following form:

$$u(x,t) = \mathbf{X}(x)\mathbf{T}(t) \tag{2}$$

where are:

$$\mathbf{X}(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \tag{3}$$

$$\mathbf{T}(t) = A \cos \omega t + B \sin \omega t \tag{4}$$

Using the boundary conditions of the subsystem of the longitudinal rod's oscillations (se Ref. [1]) as well as the compatibility conditions of the displacements and forces as interactions of the coupled subsystems we can write:

$$\left[m_P \frac{\partial^2 u(x,t)}{\partial t^2} \right]\Bigg|_{x=0} = \left[\mathbf{EA} \frac{\partial u(x,t)}{\partial x} \right]\Bigg|_{x=0} \tag{5}$$

$$\left[m_P \frac{\partial^2 u(x,t)}{\partial t^2} \right]\Bigg|_{x=\ell} = \left[-\mathbf{EA} \frac{\partial u(x,t)}{\partial x} \right]\Bigg|_{x=\ell} + \mathbf{F}(t) \tag{6}$$

$$u(\ell,t) = x_0(t)$$

Lets introduce the following notations:

$$\mu_P = \frac{m_P}{\rho \mathbf{A} \ell} \quad \mu_o = \frac{m_0}{\rho \mathbf{A} \ell} \quad \xi = \lambda \ell \quad \omega^2 = \lambda^2 \frac{\mathbf{E}}{\rho} = \frac{\xi^2}{\ell^2} \frac{\mathbf{E}}{\rho} = \xi^2 \omega_0^2 \quad \omega_0^2 = \frac{\mathbf{E}}{\rho \ell^2} \quad \kappa = \frac{c_0}{c_e} \tag{7}$$

$$\frac{d\mathbf{X}(x)}{dx}\Bigg|_{x=\ell} = \lambda \frac{d\tilde{\mathbf{X}}(\xi)}{d\xi} = \frac{\xi}{\ell} \frac{d\tilde{\mathbf{X}}(\xi)}{d\xi}; \quad \tilde{\mathbf{X}}(\xi) = C_1 \cos \xi + C_2 \sin \xi; \quad c_e = \frac{\mathbf{EA}}{\ell} \quad u_0 = \frac{m_0 \omega_0^2}{c_0}.$$

Introducing the proposed solutions (2) into boundary conditions and the conditions of the compatibility for displacement and forces we can write:

$$\mu_P \xi^2 \tilde{\mathbf{X}}(0) + \xi \tilde{\mathbf{X}}'(0) = 0 \tag{8}$$

$$(\mu_0 \xi^2 - \kappa) \tilde{\mathbf{X}}(\xi) - \xi \tilde{\mathbf{X}}'(\xi) + \kappa A_1 = 0 \tag{9}$$

Using is the relation: $\tilde{\mathbf{X}}(\xi) = C_1 \cos \xi + C_2 \sin \xi$ and the corresponding derivative with respect to the argument ξ: $\tilde{\mathbf{X}}'(\xi) = -C_1 \sin \xi + C_2 \cos \xi$, from the previous equations we can obtain:

$$\mu_P \xi^2 C_1 + \xi C_2 = 0 \tag{10}$$

$$C_1[(\mu_0 \xi^2 - \kappa)\cos \xi + \xi \sin \xi] + C_2[(\mu_0 \xi^2 - \kappa)\sin \xi - \xi \cos \xi] = -\kappa A_1$$

Determinant of the previous algebraic system of equations with respect to C_1, C_2 is:

$$\Delta(\xi) = \begin{vmatrix} \mu_P \xi^2 & \xi \\ [(\mu_0\xi^2-\kappa)\cos\xi + \xi\sin\xi] & [(\mu_0\xi^2-\kappa)\sin\xi - \xi\cos\xi] \end{vmatrix}$$

$$\Delta(\xi) = \xi\{\mu_P\xi[(\mu_0\xi^2-\kappa)\sin\xi - \xi\cos\xi] - [(\mu_0\xi^2-\kappa)\cos\xi + \xi\sin\xi]\} \tag{11}$$

and these coefficients can express in the following way:

$$C_1 = \frac{\kappa A_1 \xi}{\xi\{\mu_P\xi[(\mu_0\xi^2-\kappa)\sin\xi - \xi\cos\xi] - [(\mu_0\xi^2-\kappa)\cos\xi + \xi\sin\xi]\}} \tag{12}$$

$$C_2 = -\frac{\kappa \mu_P \xi^2 A_1}{\xi\{\mu_P\xi[(\mu_0\xi^2-\kappa)\sin\xi - \xi\cos\xi] - [(\mu_0\xi^2-\kappa)\cos\xi + \xi\sin\xi]\}} \tag{13}$$

2.2. Differential Equations for a Discrete System of Material Particles with Boundary Condition

Now, we consider a subsystem of discrete material particles with n degrees of freedom and we choose n generalized coordinates x_i, $i = 1,2,3,...,n$, and the corresponding matrix **A** of the inertia coefficients and the matrix **C** of the quasi-elastic coefficients:

$$\begin{matrix} \mathbf{A} = (a_{ij}), \; i = 1,2,3,...,n; \; j = 1,2,3,...,n \\ \mathbf{C} = (c_{ij}), \; i = 1,2,3,...,n; \; j = 1,2,3,...,n \end{matrix} ; \tag{14}$$

The system of differential equations of the discrete subsystem with boundary condition is:

$$\mathbf{A}\{\ddot{x}\} + \mathbf{C}\{x\} = -c_0(x_1 - x_0)\mathbf{I}_0\{I\} \tag{15}$$

where is:

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{Bmatrix} \qquad \mathbf{I}_0 = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} \qquad \{I\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

The solution of the previous system (16) is assumed in the following form:

$$\{x\} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{n-1} \\ A_n \end{Bmatrix} \mathbf{T}(t) = \{A\}\mathbf{T}(t) \qquad \{\ddot{x}\} = -\omega^2\{A\}\mathbf{T}(t) = -\xi^2\omega_0^2\{A\}\mathbf{T}(t) \tag{17}$$

2.3. Frequency Equation of the Coupled Longitudinal Oscillations of the Elastic Rod and Discrete System of the Material Particles (First approach)

Taking into consideration that:

$$u(\ell,t) = x_0(t) = \mathbf{X}(\ell)\mathbf{T}(t) = (C_1\cos\xi + C_2\sin\xi)\mathbf{T}(t) \tag{18}$$

and that: $\frac{1}{c_0}\mathbf{C} = \overline{\mathbf{C}}$ $\quad \frac{1}{m_0}\mathbf{A} = \overline{\mathbf{A}}$ form the system of differential equations in matrix form (15) we can obtain the following matrix equation:

$$(\overline{\mathbf{C}} - \xi^2 u_0\overline{\mathbf{A}} + \mathbf{I}_0)\{A\} - (C_1\cos\xi + C_2\sin\xi)\{I\} = \{0\} \tag{19.1}$$

and using the boundary conditions (10) of deformable elastic rod ends we can write the following matrix equation:

$$\begin{pmatrix} \mu_P\xi^2 & \xi \\ [(\mu_0\xi^2-\kappa)\cos\xi + \xi\sin\xi] & [(\mu_0\xi^2-\kappa)\sin\xi - \xi\cos\xi] \end{pmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} + kA_1\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{19.2}$$

In the scalar form these two matrix equation are the system of $n+2$ algebraic homogeneous equations with unknown amplitudes A_k, $k = 1,2,...,n$ and C_1 and C_2:

$$\sum_{j=1}^{n}(\overline{c}_{1j} - \xi^2 u_0\overline{a}_{1j})A_j + A_1 - C_1\cos\xi - C_2\sin\xi = 0$$

$$\sum_{j=1}^{n}(\overline{c}_{1j} - \xi^2 u_0\overline{a}_{1j})A_j = 0\,,\; i = 2,3,4,....,n \tag{19.3}$$

$$\mu_P\xi^2 C_1 + \xi C_2 = 0$$

$$\kappa A_1 + C_1[(\mu_0\xi^2-\kappa)\cos\xi + \xi\sin\xi] + C_2[(\mu_0\xi^2-\kappa)\sin\xi - \xi\cos\xi] = 0$$

The previous two matrix equations are $n+2$ algebra homogeneous equations and for nontrivial solutions it is necessary that determinant of this system is equal to zero. From this condition we can obtain the following characteristic frequency transcendent equation:

$$\Delta(\xi) = \begin{vmatrix} \overline{c}_{11}+1-\xi^2u_0\overline{a}_{11} & \overline{c}_{21}-\xi^2u_0\overline{a}_{21} & : & \overline{c}_{n1}-\xi^2u_0\overline{a}_{n1} & -\cos\xi & -\sin\xi \\ \overline{c}_{12}-\xi^2u_0\overline{a}_{12} & \overline{c}_{22}-\xi^2u_0\overline{a}_{22} & : & \overline{c}_{n2}-\xi^2u_0\overline{a}_{n2} & 0 & 0 \\ : & : & : & : & 0 & 0 \\ \overline{c}_{1n}-\xi^2u_0\overline{a}_{1n} & \overline{c}_{2n}-\xi^2u_0\overline{a}_{2n} & : & \overline{c}_{nn}-\xi^2u_0\overline{a}_{nn} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_P\xi^2 & \xi \\ k & 0 & 0 & 0 & [(\mu_0\xi^2-\kappa)\cos\xi+\xi\sin\xi] & [(\mu_0\xi^2-\kappa)\sin\xi-\xi\cos\xi] \end{vmatrix} = 0 \tag{20}$$

This is the first main result of this consideration of the mixed system of the coupled subsystems free oscillations. We can see that this equation consists of four submatrices: two main submatrices: one part is the expression of the frequency equations of the discrete system oscillations, and the second of the deformable body frequency equation, and two submatrices as the matrices of coupling of main matrices. The series of roots of this characteristic equation (20) *are characteristic number* ξ_s, $s = 1,2,3,..., \infty$.

Unknown amplitudes A_k, $k=1,2,3,...,n$ and C_1 and C_2 correspond to the eigen characteristic number ξ_s we obtain from the relations:

$$\frac{A_k^{(s)}}{\mathbf{K}_{n+2,k}^{(s)}}=\frac{C_1^{(s)}}{\mathbf{K}_{n+2,n+1}^{(s)}}=\frac{C_2^{(s)}}{\mathbf{K}_{n+2,n+2}^{(s)}}=\overline{C}_s \quad s=1,2,3,...,\infty \tag{20}$$

where $\mathbf{K}_{n+2,k}^{(s)}$ is the co-factor of the system determinant element (20) of the order $n+2$ and k is the column for the eigen characteristic number ξ_s. Then, for the amplitudes A_k, $k=1,2,...,n$ and C_1 and C_2 we obtain:

$$\begin{aligned} A_k^{(s)} &= \mathbf{K}_{n+2,k}^{(s)}C_s \qquad s=1,2,3,...,\infty \\ C_1^{(s)} &= \mathbf{K}_{n+2,n+1}^{(s)}C_s \\ C_2^{(s)} &= \mathbf{K}_{n+2,n+2}^{(s)}C_s \end{aligned} \tag{21}$$

Extended solutions are in the following forms:

$$u(x,t)=\sum_{s=1}^{\infty}\mathbf{X}_{(\mathbf{s})}(x)\mathbf{T}_{\mathbf{s}}(t)=\sum_{s=1}^{\infty}\left(\mathbf{K}_{n+2,n+1}^{(s)}\cos\frac{\xi_s}{\ell}x+\mathbf{K}_{n+2,n+2}^{(s)}\sin\frac{\xi_s}{\ell}x\right)C_s\mathbf{T}_{\mathbf{s}}(t) \tag{22}$$

$$u(\ell,t)=\sum_{s=1}^{\infty}\mathbf{X}_{(\mathbf{s})}(\ell)\mathbf{T}_{\mathbf{s}}(t)=\sum_{s=1}^{\infty}(\mathbf{K}_{n+2,n+1}^{(s)}\cos\xi_s+\mathbf{K}_{n+2,n+2}^{(s)}\sin\xi_s)C_s\mathbf{T}_{\mathbf{s}}(t) \tag{22*}$$

$$x_k(t)=\sum_{s=1}^{\infty}A_k^{(s)}\mathbf{T}_{\mathbf{s}}(t)=\sum_{s=1}^{\infty}\mathbf{K}_{n+2,k}^{(s)}C_s\mathbf{T}_{\mathbf{s}}(t)\ , \quad k=1,2,...,n \tag{23}$$

2.4. Frequency Equation of the Coupled Longitudinal Oscillations of the Elastic Rod and Discrete System of the Material Particles (Second Approach)

Using the solutions (12) and (13) for C_1 and C_2, and taking into account that:

$$u(\ell,t)=x_0(t)=\mathbf{X}(\ell)\mathbf{T}(t)=(C_1\cos\xi+C_2\sin\xi)\mathbf{T}(t)=\frac{\kappa A_1\xi}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\mathbf{T}(t) \tag{24}$$

and that is $\frac{1}{c_0}\mathbf{C}=\overline{\mathbf{C}}$ and $\frac{1}{m_0}\mathbf{A}=\overline{\mathbf{A}}$ form the system of differential equations in the matrix form (15) we can obtain the following matrix equation:

$$\left(\overline{\mathbf{C}}-\xi^2u_0\overline{\mathbf{A}}+\left[1-\frac{\kappa\xi}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\right]\mathbf{I}_0\right)\{A\}=\{0\} \tag{25}$$

The previous matrix equations are the algebraic homogeneous equations and for nontrivial solutions it is necessary that determinant of this system is equal to zero. From this condition we can obtain the following characteristic frequency transcendent equation:

$$\left|\overline{\mathbf{C}} - \xi^2 u_0 \overline{\mathbf{A}} + \left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\right]\mathbf{I}_0\right| = 0 \tag{26}$$

This is the second main result of this consideration of the mixed system of the coupled subsystems free oscillations. We can see that this equation consists of two parts: one part is expression of the frequency equations of the discrete system oscillations, and the second part is the expression of the deformable body frequency equation connected by one member with previous.

2.5. Special Cases of the Frequency Equation of the Coupled Longitudinal Oscillations of the Elastic Rod and Discrete System of the Material Particles

Derived frequency equation (20) of the free oscillations of the mixed system of the coupled subsystems can be written in the developed form in the following way:

$$\begin{vmatrix} k_{11} - \xi^2 u_0 \mu_{11} + \left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\right] & k_{12} - \xi^2 u_0 \mu_{12} & k_{13} - \xi^2 u_0 \mu_{13} & k_{1n} - \xi^2 u_0 \mu_{1n} \\ k_{21} - \xi^2 u_0 \mu_{21} & k_{22} - \xi^2 u_0 \mu_{22} & k_{23} - \xi^2 u_0 \mu_{23} & k_{2n} - \xi^2 u_0 \mu_{2n} \\ k_{31} - \xi^2 u_0 \mu_{31} & k_{32} - \xi^2 u_0 \mu_{32} & k_{33} - \xi^2 u_0 \mu_{33} & k_{3n} - \xi^2 u_0 \mu_{3n} \\ & & & \\ k_{n1} - \xi^2 u_0 \mu_{n1} & k_{2n} - \xi^2 u_0 \mu_{2n} & k_{3n} - \xi^2 u_0 \mu_{3n} & k_{nn} - \xi^2 u_0 \mu_{nn1} \end{vmatrix} = 0 \tag{26*}$$

For the case of the coupled elastic rod longitudinal oscillations and chain discrete material particles system oscillations, the previous frequency equations take the following form:

$$\begin{vmatrix} k_1 - \xi^2 u_0 \mu_1 + \left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\right] & -k_1 & & & \\ -k_1 & k_1 + k_2 - \xi^2 u_0 \mu_2 & -k_2 & & \\ & -k_2 & k_2 + k_3 - \xi^2 u_0 \mu_3 & \vdots & \\ & & \cdots & \cdots & \\ & & & & -k_{n-1} \\ & & & \vdots & k_{n-1} + k_n - \xi^2 u_0 \mu_n \end{vmatrix} = 0$$

For the case that elastic rod is connected with one material particle with two springs we obtain:

$$k_1 - \xi^2 u_0 \mu_1 + \left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\right] = 0$$

and taking into account expression (11) we can write the following:

$$\xi(k_1 + 1 - \xi^2 u_0 \mu_1)\{\mu_P \xi[(\mu_0 \xi^2 - \kappa)\sin\xi - \xi\cos\xi] -$$
$$-[(\mu_0 \xi^2 - \kappa)\cos\xi + \xi\sin\xi]\} - \kappa\xi(\cos\xi - \mu_P \xi \sin\xi) = 0$$

For the case that one end of the rod is fixed – the case of the cantilever rod, in the previous frequency equation we can introduce $\mu_p \to \infty$, and then we obtain:

$$tg\xi = \frac{\xi(k_1 + 1 - \xi^2 u_0 \mu_1)}{\{\xi^2[\mu_0(k_1+1) + \kappa u_0 \mu_1 - \mu_0 \xi^2 u_0 \mu_1] - k_1 \kappa\}}$$

For the case of a free material particle and connected by one spring for rod we can write:

$$\xi tg\xi = \frac{(1 - \xi^2 u_0 \mu_1)}{(\mu_0 + \kappa u_0 \mu_1 - \mu_0 \xi^2 u_0 \mu_1)} \tag{27}$$

For the two material particle connected for rod as a chain, we obtain:

$$\left\{k_1 - \xi^2 u_0 \mu_1 + \left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi \sin\xi)\right]\right\}(k_1 + k_2 - \xi^2 u_0 \mu_2) - k_1^2 = 0$$

where:

$$\Delta(\xi) = \xi\{\mu_P \xi[(\mu_0 \xi^2 - \kappa)\sin\xi - \xi\cos\xi] - [(\mu_0\xi^2 - \kappa)\cos\xi + \xi\sin\xi]\}$$

For the three material particles chain, the frequency equation is:

$$\begin{vmatrix} k_1 - \xi^2 u_0 \mu_1 + \left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P \xi\sin\xi)\right] & -k_1 & \\ -k_1 & k_1 + k_2 - \xi^2 u_0 \mu_2 & -k_2 \\ & -k_2 & k_2 + k_3 - \xi^2 u_0 \mu_3 \end{vmatrix} = 0 \tag{28}$$

For the case that we have discrete material particles homogeneous chain, the frequency equation obtains the following form:

$$\begin{vmatrix} 1 - \xi^2 \tilde{u}_0 + \frac{1}{k_1}\left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right] & -1 & & & & \\ -1 & 2 - \xi^2\tilde{u}_0 & -1 & & & \\ & -1 & 2 - \xi^2 \tilde{u}_0 & \vdots & & \\ & & \cdots & \cdots & & \\ & & & & 2 - \xi^2\tilde{u}_0 & -1 \\ & & & & -1 \vdots & 2 - \xi^2 \tilde{u}_0 \end{vmatrix} = 0$$

For the special case of three material particles homogeneous chain, the frequency equation obtain the following form:

$$[(2 - \xi^2\tilde{u}_0)^2 - 1]\left\{1 - \xi^2\tilde{u}_0 + \frac{1}{k_1}\left[1 - \frac{\kappa\xi}{\Delta(\xi)}(\cos\xi - \mu_P\xi\sin\xi)\right]\right\} - (2 - \xi^2\tilde{u}) = 0 \tag{29}$$

3. EIGEN AMPLITUDE VECTORS ORHOGONALITY OF SMALL OSCILLATIONS MIXED SYSTEMS OF THE COUPLED DISCRETE AND CONTINUOUS SUBSYSTEMS

Using the main results of the previous consideration of the mixed system of the coupled subsystems free oscillations frequency equation (20) and solving the equation, it is possible to find series of the characteristic own numbers ξ_s,$s = 1,2,3,...,\infty$, as a series of the frequency equation roots. Then using (12) and (13) we can express the characteristic coefficients $C_1^{(s)}$ and $C_2^{(s)}$ in the following form:

$$C_1^{(s)} = \frac{\kappa A_1^{(s)}\xi_s}{\Delta(\xi_s)} = \tilde{C}_1^{(s)}(\xi_s)A_1^{(s)} \qquad \tilde{C}_1^{(s)}(\xi) = \frac{\kappa\xi_s}{\Delta(\xi_s)} \tag{12*}$$

$$C_2^{(s)} = -\frac{\kappa\mu_P\xi_s^2 A_1^{(s)}}{\Delta(\xi_s)} = -\tilde{C}_2^{(s)}(\xi_s)A_1^{(s)} = -\mu_P\xi\tilde{C}_1^{(s)}(\xi_s)A_1^{(s)} \qquad s = 1,2,3...,\infty \tag{13*}$$

$$\Delta(\xi_s) = \xi_s\{\mu_P\xi_s[(\mu_0\xi_s^2 - \kappa)\sin\xi_s - \xi_s\cos\xi_s] - [(\mu_0\xi_s^2 - \kappa)\cos\xi_s + \xi_s\sin\xi_s]\}$$

Then, characteristic amplitude functions are in the following form:

$$\mathbf{X}_{(\mathbf{s})}(x) = \tilde{C}_1^{(s)}(\xi_s)A_1^{(s)}\left[\cos\frac{\xi_s}{\ell}x - \mu_P\xi_s\sin\frac{\xi_s}{\ell}x\right] = A_1^{(s)}\tilde{\mathbf{X}}_{(\mathbf{s})}(x)$$

and

$$\mathbf{X}_{(\mathbf{s})}(\ell) = \tilde{C}_1^{(s)}(\xi_s)A_1^{(s)}[\cos\xi_s - \mu_P\xi_s\sin\xi_s] = A_1^{(s)}\tilde{\tilde{\mathbf{X}}}_{(s)}(\xi_s) \tag{30}$$

The extended solution is:

$$u(x,t) = \sum_{s=1}^{\infty}\mathbf{X}_{(\mathbf{s})}(x)\mathbf{T}_{\mathbf{s}}(t) = \sum_{s=1}^{\infty}A_1^{(s)}\tilde{\mathbf{X}}_{(\mathbf{s})}(x)\mathbf{T}_{\mathbf{s}}(t) = \sum_{s=1}^{\infty}\mathbf{K}_{n1}^{(s)}\tilde{\mathbf{X}}_{(\mathbf{s})}(x)C_s\mathbf{T}_{\mathbf{s}}(t) \tag{31}$$

$$u(\ell,t) = \sum_{s=1}^{\infty}\mathbf{X}_{(\mathbf{s})}(\ell)\mathbf{T}_{\mathbf{s}}(t) = \sum_{s=1}^{\infty}A_1^{(s)}\tilde{\mathbf{X}}_{(\mathbf{s})}(\ell)\mathbf{T}_{\mathbf{s}}(t) = \sum_{s=1}^{\infty}\mathbf{K}_{n1}^{(s)}\tilde{\tilde{\mathbf{X}}}_{(\mathbf{s})}(\xi_s)C_s\mathbf{T}_{\mathbf{s}}(t) \tag{31*}$$

$$x_k(t) = \sum_{s=1}^{\infty}A_k^{(s)}\mathbf{T}_{\mathbf{s}}(t) = \sum_{s=1}^{\infty}\mathbf{K}_{nk}^{(s)}C_s\mathbf{T}_{\mathbf{s}}(t)\,, \quad k = 1,2,...,n \tag{32}$$

where

$$\frac{A_k^{(s)}}{\mathbf{K}_{nk}^{(s)}} = C_s \qquad s = 1,2,3,....,\infty$$

where $\mathbf{K}_{n,k}^{(s)}$ is the co-factor of the system determinant element of the order n and k is the column for each eigen characteristic number ξ_s, $s = 1,2,3,...,\infty$.

Using the two differential equations:

$$\mathbf{X}''_{(\mathbf{s})}(x) + \lambda_s^2\mathbf{X}_{(s)}(x) = 0$$

$$\mathbf{X}''_{(r)}(x) + \lambda_r^2\mathbf{X}_{(r)}(x) = 0 \tag{33}$$

which must satisfy two different arbitrary eigen amplitude functions $\mathbf{X}_{(s)}(x)$ and $\mathbf{X}_{(r)}(x)$ for different eigen characteristic numbers λ_s and λ_r when $s \neq r$, and multiplying with $\mathbf{X}_{(s)}(x)dx$ and $-\mathbf{X}_{(r)}(x)dx$, then integrating along x from the left to the right end of the rod, from zero to ℓ, after summarizing we obtain the following expression:

$$(\lambda_s^2-\lambda_r^2)\int_0^\ell \mathbf{X}_{(s)}(x)\mathbf{X}_{(r)}(x)dx=\int_0^\ell[\mathbf{X}_{(s)}(x)\mathbf{X}''_{(r)}(x)-\mathbf{X}''_{(s)}(x)\mathbf{X}_{(r)}(x)]dx$$

$$(\lambda_s^2-\lambda_r^2)\int_0^\ell \mathbf{X}_{(s)}(x)\mathbf{X}_{(r)}(x)dx=\mathbf{X}_{(s)}(\ell)\mathbf{X}'_{(r)}(\ell)-\mathbf{X}'_{(s)}(\ell)\mathbf{X}_{(r)}(\ell)+\mathbf{X}'_{(s)}(0)\mathbf{X}_{(r)}(0)-\mathbf{X}_{(s)}(0)\mathbf{X}'_{(r)}(0)$$

$$\begin{aligned}\frac{1}{\ell^2}(\xi_s^2-\xi_r^2)\left\{\int_0^\ell \mathbf{X}_{(s)}(x)\mathbf{X}_{(r)}(x)dx+\mu_0\ell\mathbf{X}_{(s)}(\ell)\mathbf{X}_{(r)}(\ell)-\mu_P\ell\mathbf{X}_{(s)}(0)\mathbf{X}_{(r)}(0)\right\}=\\=\frac{k}{\ell}A_1^{(s)}A_1^{(r)}[\tilde{\tilde{\mathbf{X}}}_{(s)}(\xi_s)-\tilde{\tilde{\mathbf{X}}}_{(r)}(\xi_r)]\end{aligned}\tag{34}$$

Using the matrix equation in the form (19)

$$\left(\overline{\mathbf{C}}-\xi^2u_0\overline{\mathbf{A}}+\left[1-\frac{\kappa\xi}{\Delta(\xi)}(\cos\xi-\mu_P\xi\sin\xi)\right]\mathbf{I}_0\right)\{A\}=\{0\}$$

and introducing the notation (30) we can write:

$$(\overline{\mathbf{C}}-\xi^2u_0\overline{\mathbf{A}}+[1-\tilde{\tilde{\mathbf{X}}}(\xi)]\mathbf{I}_0)\{A\}=\{0\}$$

This equation must satisfy two different arbitrary eigen amplitude vectors $\{A^{(s)}\}$ and $\{A^{(r)}\}$ for different eigen characteristic numbers λ_s and λ_r when $s\neq r$, and multiplying with $(A^{(r)})$ and $-(A^{(s)})$, then after summarizing we obtain the following expression:

$$\begin{aligned}(\overline{\mathbf{C}}-\xi_s^2u_0\overline{\mathbf{A}}+[1-\tilde{\tilde{\mathbf{X}}}_{(s)}(\xi_s)]\,\mathbf{I}_0)\{A^{(s)}\}=\{0\}\\(\overline{\mathbf{C}}-\xi_r^2u_0\overline{\mathbf{A}}+[1-\tilde{\tilde{\mathbf{X}}}_{(r)}(\xi_r)]\mathbf{I}_0)\{A^{(r)}\}=\{0\}\end{aligned}\tag{35}$$

$$\begin{aligned}(A^{(r)})\overline{\mathbf{C}}\{A^{(s)}\}-\xi_s^2u_0(A^{(r)})\overline{\mathbf{A}}\{A^{(s)}\}+[1-\tilde{\tilde{\mathbf{X}}}_{(s)}(\xi_s)](A^{(r)})\mathbf{I}_0\{A^{(s)}\}=\{0\}\\(A^{(s)})\overline{\mathbf{C}}\{A^{(r)}\}-\xi_r^2u_0(A^{(s)})\overline{\mathbf{A}}\{A^{(r)}\}+[1-\tilde{\tilde{\mathbf{X}}}_{(r)}(\xi_r)](A^{(s)})\mathbf{I}_0\{A^{(r)}\}=\{0\}\end{aligned}$$

$$(\xi_r^2-\xi_s^2)u_0(A^{(r)})\overline{\mathbf{A}}\{A^{(s)}\}+[\tilde{\tilde{\mathbf{X}}}_{(r)}(\xi_r)-\tilde{\tilde{\mathbf{X}}}_{(s)}(\xi_s)](A^{(r)})\mathbf{I}_0\{A^{(s)}\}=\{0\}\tag{36}$$

Comparing the expressions (34) and (36) we can write the following:

$$\frac{1}{\ell^2}(\xi_s^2-\xi_r^2)\left\{\int_0^\ell \mathbf{X}_{(s)}(x)\mathbf{X}_{(r)}(x)dx+\mu_0\ell\mathbf{X}_{(s)}(\ell)\mathbf{X}_{(r)}(\ell)-\mu_P\ell\mathbf{X}_{(s)}(0)\mathbf{X}_{(r)}(0)+k\ell u_0(A^{(r)})\overline{\mathbf{A}}\{A^{(s)}\}\right\}=0$$

The characteristic eigen numbers ξ_s and ξ_r are different for $s\neq r$ and the expressions in the brackets are equal to zero, and then the extended orthogonality condition of the eigen amplitude vectors and eigen amplitude functions is in the following form:

$$ku_0(A^{(r)})\overline{\mathbf{A}}\{A^{(s)}\}+\frac{1}{\ell}\int_0^\ell \mathbf{X}_{(s)}(x)\mathbf{X}_{(r)}(x)dx+\mu_0\mathbf{X}_{(s)}(\ell)\mathbf{X}_{(r)}(\ell)-\mu_P\mathbf{X}_{(s)}(0)\mathbf{X}_{(r)}(0)=0\tag{37}$$

for $s\neq r$

The second form of the extended orthogonality condition of the eigen amplitude vectors and the eigen amplitude functions is in the following form:

$$(A^{(r)})\overline{\mathbf{C}}\{A^{(s)}\}+\xi_s^2 u_0\mathbf{EA}\left[\int_0^{\ell}\mathbf{X}_{(s)}(x)\mathbf{X}_{(r)}(x)dx+\mu_0\mathbf{X}_{(s)}(\ell)\mathbf{X}_{(r)}(\ell)-\mu_P\mathbf{X}_{(s)}(0)\mathbf{X}_{(r)}(0)\right]+ \tag{38}$$

$$+[1-\tilde{\tilde{\mathbf{X}}}_{(\mathbf{s})}(\xi_s)](A^{(r)})\mathbf{I}_0\{A^{(s)}\}=0$$

$$\text{for } s\neq r$$

The orthogonality conditions are:

$$\tilde{a}_{sr}=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\mathbf{K}_{ni}^{(s)}\mathbf{K}_{nj}^{(r)}+$$

$$+\mathbf{K}_{n1}^{(s)}\mathbf{K}_{n1}^{(r)}\left[\rho\mathbf{A}\int_0^{\ell}\tilde{\mathbf{X}}_{(s)}(x)\tilde{\mathbf{X}}_{(r)}(x)dx+m_P\ell\tilde{\mathbf{X}}_{(s)}(0)\tilde{\mathbf{X}}_{(r)}(0)-m_0\ell\tilde{\mathbf{X}}_{(s)}(\ell)\tilde{\mathbf{X}}_{(r)}(\ell)\right]=\begin{cases}\tilde{a}_{ss} & s=r\\ 0 & s\neq r\end{cases}$$

$$\tilde{c}_{sr}=\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}\mathbf{K}_{ni}^{(s)}\mathbf{K}_{ni}^{(r)}+$$

$$+\mathbf{K}_{n1}^{(s)}\mathbf{K}_{n1}^{(r)}\left[\mathbf{EA}\ell\int_0^{\ell}\tilde{\mathbf{X}}''_{(s)}(x)\tilde{\mathbf{X}}_{(r)}(x)dx+\mathbf{EA}\ell\tilde{\mathbf{X}}'_{(s)}(0)\tilde{\mathbf{X}}_{(r)}(0)-\mathbf{EA}\ell\tilde{\mathbf{X}}'_{(s)}(\ell)\tilde{\mathbf{X}}_{(r)}(\ell)+c_0\ell(1-\tilde{\mathbf{X}}_{(s)}(\ell))(1-\tilde{\mathbf{X}}_{(r)}(\ell))\right]$$

$$=\begin{cases}\tilde{c}_{ss} & s=r\\ 0 & s\neq r\end{cases}$$

$$\tilde{c}_{sr}=\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}\mathbf{K}_{ni}^{(s)}\mathbf{K}_{ni}^{(r)}-\mathbf{K}_{n1}^{(s)}\mathbf{K}_{n1}^{(r)}\left[\mathbf{EA}\ell\int_0^{\ell}\tilde{\mathbf{X}}'_{(s)}(x)\tilde{\mathbf{X}}'_{(r)}(x)dx-c_0\ell(1-\tilde{\mathbf{X}}_{(s)}(\ell))(1-\tilde{\mathbf{X}}_{(r)}(\ell))\right]=\begin{cases}\tilde{c}_{ss} & s=r\\ 0 & s\neq r\end{cases}$$

4. Analogy between of the Frequency Equation of the Coupled Longitudinal Oscillations of the Elastic Rod and Discrete System of the Material Particles and Coupled Torsion Oscillations of the Elastic Rod and Corresponding Discrete System of the Material Particles

Using the analogy (see Ref. [27]) between the two systems, and specially between the longitudinal and torsional oscillations of the elastic rod with the circle cross section we can use the previous analytical results for determining the frequency equation of the coupled small oscillations of the mixed systems presented in Figure 2. and 3.

In a general case a mixed system consists of the two subsystems: one elastic rod-shaft, whose axis is straight, with parameters: $\mathbf{G},\rho,\ell,\mathbf{A},\mathbf{I}_0$, and with two rigid discs at free ends with mass inertia moment with respect to the shaft axis: $\mathbf{J}_P$ and $\mathbf{J}_0$. This rod-shaft is constrained by the torsion spring with the stiffness c_0 and coupled with the discrete systems with n degrees of freedom. For example, this discrete subsystem is a mechanism in the form of a chain system of the n material particles (or rigid bodies) with generalized masses m_i, $i = 1,2,3,...,n$, torsion (rotation) movable along the circle line coaxial to the axis of the rod-shaft; these masses are connected by torsion springs with stiffnesses c_i, $i = 1,2,3,...,n$. We consider relations between the torsional vibrations of the elastic rod-shaft and the free oscillations of the generalized chain material particles system-mechanism (examples in Figure 2. and 3.). Let us determine the frequency equations of the defined mixed system of the coupled discrete subsystem of rigid bodies and continuous subsystem.

Using the analogy

$$\theta(x,t) \Rightarrow u(x,t) \quad \mathbf{J}_P \Rightarrow m_P \quad \mathbf{J}_P \Rightarrow m_P \quad x_i \Rightarrow \theta_i$$

and taking into account that:

$$\mu_P = \frac{\mathbf{J}_P}{\rho \mathbf{I}_0 \ell} \quad \mu_o = \frac{\mathbf{J}_0}{\rho \mathbf{I}_0 \ell} \quad \xi = \lambda \ell \quad \omega^2 = \lambda^2 \frac{\mathbf{G}}{\rho} = \frac{\xi^2}{\ell^2}\frac{\mathbf{G}}{\rho} = \xi^2 \omega_0^2 \quad \omega_0^2 = \frac{\mathbf{E}}{\rho \ell^2}$$

$$c_t = \frac{\mathbf{G}\mathbf{I}_0}{\ell} \quad \kappa = \frac{c_0}{c_e} \quad u_0 = \frac{\mathbf{J}_0 \omega_0^2}{c_0}. \tag{39}$$

we can write the frequency equation of the considered mixed system with torsional oscillations of the shaft and coupled mechanisms (Figure 2.) in the same form as the frequency equation (20).

Extended system solutions are:

$$\theta(x,t) = \sum_{s=1}^{\infty} \mathbf{X}_{(\mathbf{s})}(x)\mathbf{T}_{\mathbf{s}}(t) = \sum_{s=1}^{\infty} A_1^{(s)} \tilde{\mathbf{X}}_{(\mathbf{s})}(x)\mathbf{T}_{\mathbf{s}}(t) = \sum_{s=1}^{\infty} \mathbf{K}_{n1}^{(s)} \tilde{\mathbf{X}}_{(\mathbf{s})}(x) C_s \mathbf{T}_{\mathbf{s}}(t) \tag{40}$$

$$\theta(\ell,t) = \sum_{s=1}^{\infty} \mathbf{X}_{(\mathbf{s})}(\ell)\mathbf{T}_{\mathbf{s}}(t) = \sum_{s=1}^{\infty} A_1^{(s)} \tilde{\mathbf{X}}_{(\mathbf{s})}(\ell)\mathbf{T}_{\mathbf{s}}(t) = \sum_{s=1}^{\infty} \mathbf{K}_{n1}^{(s)} \tilde{\tilde{\mathbf{X}}}_{(\mathbf{s})}(\xi_s) C_s \mathbf{T}_{\mathbf{s}}(t) \tag{40*}$$

$$\theta_k(t) = \sum_{s=1}^{\infty} A_k^{(s)} \mathbf{T}_{\mathbf{s}}(t) = \sum_{s=1}^{\infty} \mathbf{K}_{nk}^{(s)} C_s \mathbf{T}_{\mathbf{s}}(t) \;, \quad k = 1,2,...,n$$

For the special case of the mixed system presented in Figure 3. with cantilever shaft, and corresponding mechanisms in accordance with the representation in Figure 3. b*, or 3.c* or 3.d* we can write the following frequency equation for some of the above examples:

$$\mu \xi tg \xi = \frac{9\xi^2 - k^2}{56\xi^2 - 9k^2} \tag{41}$$

When ξ is small we can consider that tg $\xi \approx \xi$ and for the approximation of the previous frequency equation we can write:

$$\mu \xi^2 (56\xi^2 - 9k^2) - 9\xi^2 + k^2 \approx 0 \Rightarrow \quad 56\xi^4 - 9\xi^2 \left(k^2 + \frac{1}{\mu}\right) + \frac{k^2}{\mu} = 0$$

and the minimal value of the eigen values of small oscillations of the mixed system is:

$$\xi_{1,2}^2 = \frac{9\left(k^2 + \frac{1}{\mu}\right) \mp \sqrt{81\left(k^2 + \frac{1}{\mu}\right)^2 - 4 * 56 \frac{k^2}{\mu}}}{112}.$$

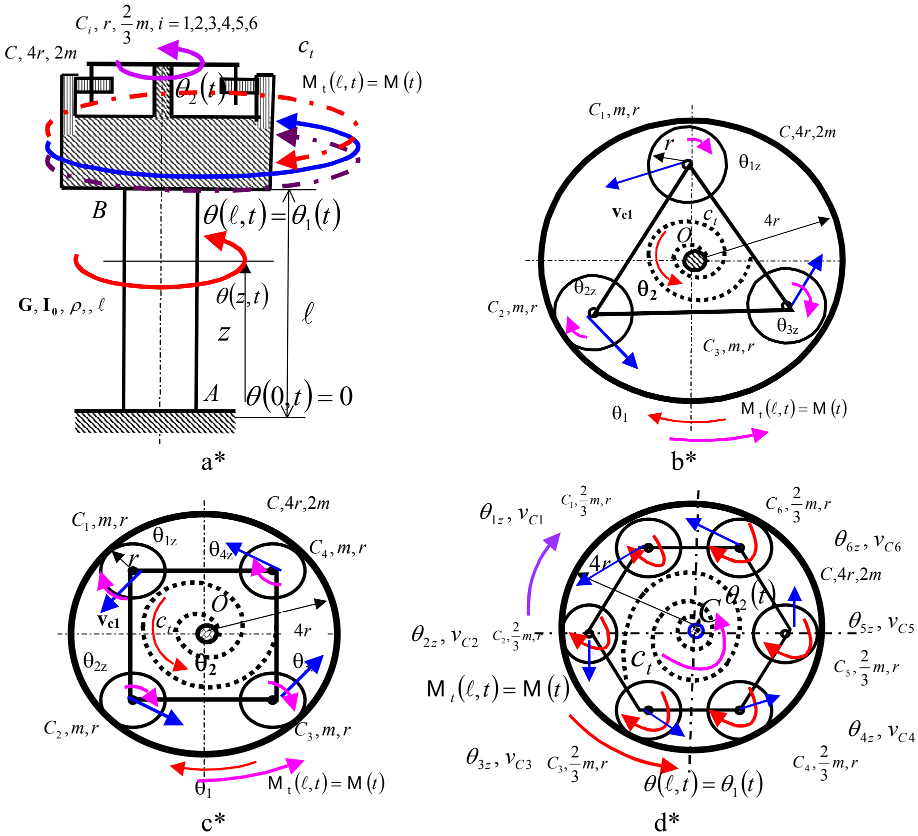

Fig. 2. Small oscillations of the mixed system of the coupled discrete and continuous subsystems. Torsion oscillations of the cantilever shaft with multi body mechanisms with changeable numbers of discs.

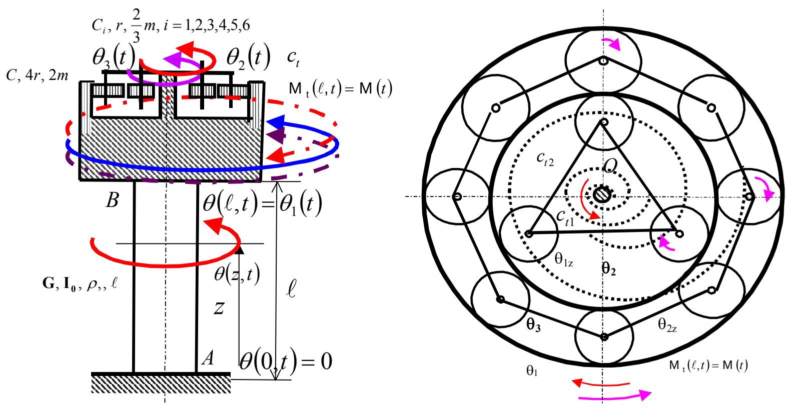

Fig. 3. Small oscillations of the mixed system of the coupled discrete and continuous subsystems. Torsion oscillations of the cantilever shaft with multi body mechanisms with two chain of the changeable numbers of discs.

6. CONCLUDING REMARKS

From the obtained analytical and numerical results for the natural longitudinal vibrations of an elastic rod coupled with material particle discrete system, it can be seen that the connections are convenient for changing characteristic function depending on the discrete system material parameters, and that fundamental eigen-function depending on space coordinate is dependent on the boundary conditions and the geometrical properties of coupled discrete system.

Using general examples of the mixed systems of the coupled discrete subsystem of rigid bodies and continuous subsystem, the extended orthogonality expression for the eigen amplitude vectors and the eigen amplitude functions of small oscillations are derived. We can see that this extended orthogonality condition for the eigen amplitude vectors and the eigen amplitude functions of small oscillations contain three parts in the expression: one term which corresponds to the discrete subsystem $ku_0(A^{(r)})\overline{\mathbf{A}}\{A^{(s)}\}$, *one term correspond to the continuous subsystem* $\frac{1}{\ell}\int_0^\ell \mathbf{X}_{(s)}(x)\mathbf{X}_{(r)}(x)dx$, *and two terms correspond to the boundary condition of the deformable body-subsystem* $\mu_0\mathbf{X}_{(s)}(\ell)\mathbf{X}_{(r)}(\ell)-\mu_P\mathbf{X}_{(s)}(0)\mathbf{X}_{(r)}(0)$.

We can also see that this extended orthogonality condition of the eigen amplitude vectors and of the eigen amplitude functions of small oscillations, in the second form, contain four parts in the expression: one term which correspond to the discrete subsystem $(A^{(r)})\overline{\mathbf{C}}(A^{(s)})$, *one term corresponds to the continuous subsystem* $\xi_s^2 u_0\mathbf{EA}\left[\int_0^\ell \mathbf{X}_{(s)}(x)\mathbf{X}_{(r)}(x)dx\right]$, *two terms correspond to the boundary condition of the deformable body-subsystem* $\xi_s^2 u_0\mathbf{EA}[\mu_0\mathbf{X}_{(s)}(\ell)\mathbf{X}_{(r)}(\ell)-\mu_P\mathbf{X}_{(s)}(0)\mathbf{X}_{(r)}(0)]$, *and last part* $[1-\tilde{\tilde{\mathbf{X}}}_{(\mathbf{s})}(\xi_s)](A^{(r)})\mathbf{I}_0\{A^{(s)}\}$ *correspond to the coupling of the discrete subsystem and the continuous subsystem.*

In this paper we returned to classical, but new problems of the theory of oscillations, coupled elastic bodies and systems of discrete material points using selected examples and at the same time we determined the corresponding orthogonality conditions to the system. Results [30] of a numerical experiment are shown on the frequency function graphs that consist out of members that express the influence of discrete systems on the frequency equations over the potential functions and terms that express the influence of deformable bodies and which contain transcendent functions themselves. From the graph in the paper [30]we can see the visualizations of the perturbations of frequency spectra of the eigen circular frequencies, deformable bodies oscillations or vice versa. Similar disturbances can be seen on the frequency spectra of a discrete system but with opposite effects. We can see "the continualization off the frequency spectra of discrete system" on the graph of the frequency functions. At the same time we can interpret these results as discretization of the part of frequency spectra of the continuous system as a result of coupling with a discrete system. What should also be stated here is the analogy used between these mixed systems with coupled subsystems, continuous and discrete when it is possible to establish a direct analogy between the longitudinal and torsional oscillations of a deformable body with the annular cross-sections. That enabled an analytical *analysis to be conducted for one type of the system and the results to be used on another type. And at the end, it should be stressed again that the goal of this paper was the solution of a classical but very concurrent task since the literature contains a very small number of examples of such a task. Methodology of the continuum discretization and of the continualization of a discrete system meet at border cases of the study of properties of real systems.*

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PROŠIRENI USLOV ORTOGONALNOSTI SOPSTVENIH AMPLITUDNIH VEKTORA I FUNKCIJA MALIH OSCILACIJA KOMPLEKSNIH SISTEMA SPREGNUTIH DISKRETNIH I KONTINUALNIH PODSISTEMA

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O frekventnim jednačinama malih oscilacija kompleksnih sistema spregnutih diskretnih i kontinualnih podsistema održano je predavanje na Seminaru Matematičkog instituta SANU, a rad je oštampan u 33 tomu časopisa "Mehanika tvrdogo tela" internacionalnog zbornika radova Instituta primenjene matematike i mehanike Nacionalne akademije nauka Ukrajine. Ovim radom se nastavlja prikazivanje rezultata istraživanja malih oscilacija kompleksnih sistema spregnutih diskretnih i kontinualnih podsistema, kroz izvođenje i dokaz proširenog uslova otogonalnosti sopstvenih amplitudnih vektora i sopstvenih amplitudnih funkcija.

Korišćenjem izvedenog uslova proširene ortogonalnosti sopstvenih amplitudnih vektora i funkcija sistema izvode se izrazi za sopstvene i prinudne oscilacije takvih sistema.

Ključne reči: *mešoviti sistemi, spregnuti podsistemi, kruta tela, kontinualni sistem, elastično telo, vektori sopstvenih amplituda, ortogonalnost.*