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FORCED DUFFING OSCILLATOR WITH SLIGHT VISCOUS DAMPING AND HARDENING NON-LINEARITY

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Abstract. *The response of a one-degree of freedom system with cubic nonlinearities to a principal resonance is investigated. The modified homotopy perturbation method (MHPM) is used to determine the equations that describe the second-order approximate periodic solutions of the system. The stability of these solutions is determined using Floquet theory.*

1. INTRODUCTION

A large number of studies have been dedicated to the Duffing oscillator with hardening non-linearity [e.g.1-10]. The interest in this system lies in the variety of physical phenomena that it models, such as the rolling motion of a ship, and the fact that it is isomorphic with other systems of importance in physics and engineering (e.g. Josephson junction oscillator and Foucault pendulum). Particularly interesting is the response of the Duffing oscillator to a harmonic excitation in the presence of viscous damping, which has been found to exhibit, among other features, hysteretic and chaotic behaviors. Thus, we consider this latter system governed by a non-dimensional differential equation of the form

$$\ddot{u} + \omega^2 u + 2\varepsilon\mu\dot{u} + \varepsilon\alpha u^3 = \varepsilon k \cos\Omega\, \mathrm{t} \qquad (1)$$

where ε is a small parameter, ω, μ, α, k and Ω are positive constant parameters. Primary resonance (i.e. $\Omega\approx\omega$) are considered in the next section. To determine the dependence of $u(t)$ on the parameters $\omega(\Omega)$,μ,α,k and ε we develop an approximate second-order solution using MHPM. The stability of this solution is then determined using Floquet theory.

2. PROBLEM FORMULATION

We have been considering systems governed by equations having the form

$$\ddot{u} + \omega^2 u = F(\Omega\, t, u, \dot{u}) \qquad (2)$$

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where, in general F is a nonlinear analytical function, with the period T in the first variable. For the equation (2), we construct a one-parameter family of equations

$$\frac{\partial^2 U(t,p)}{\partial t^2}+\Lambda(p)U(t,p)=pF\left(\Omega t, U(t,p), \frac{\partial U(t,p)}{\partial t}\right) \tag{3}$$

where $p\in[0,1]$ is an embedding parameter, and $U(t,p)$ is an analytical function of both t and p. At p=0, we have obviously $U(t,0)=u_0(t)$ and $u_0(t)$ is an initial approximation of Eq. (2) which not necessarily satisfies the boundary conditions. At p=1, Eq. (3) is exactly the same as Eq. (2), respectively, so that $U(t,1)=u(t)$ and u(t) is exactly the solution that we want to know. As the embedding parameter p varies from zero to one, $U(t,p)$ varies continuously from u_0(t) and $\Lambda(p)$ varies from $\Lambda(0)$ to $\Lambda(1)=\omega^2$.

Suppose that $U(t,p)$ and $\Lambda(p)$ have derivatives with respect to the embedding variable p evaluated at p=0:

$$\left.\frac{\partial^j U(t,p)}{\partial p^j}\right|_{p=0}=u_o^{[j]}(t);\left.\frac{\partial^j \Lambda(p)}{\partial p^j}\right|_{p=0}=\Lambda_0^{[j]}, j\ge 1 \tag{4}$$

By Taylor's formula, we have:

$$U(t,p)=u_0(t)+\sum_{j\ge 1}\frac{u_0^{[j]}(t)}{j!}p^j \tag{5}$$

and

$$\Lambda(p)=\Lambda(0)+\sum_{j\ge 1}\frac{\Lambda_0^{[j]}}{j!}p^j \tag{6}$$

Setting p = 1, we obtain:

$$u(t)=u_0(t)+\sum_{j\ge 1}\frac{u_0^{[j]}(t)}{j!} \tag{7}$$

and

$$\omega^2=\Lambda(0)+\sum_{j\ge 1}\frac{\Lambda_0^{[j]}}{j!} \tag{8}$$

provided that the radii of convergence of series (5) and (6) are not less than 1. Note that (7) gives a relation between the initial approximation u_0(t) and solution $u(t)$; meanwhile, (8) provides a link between the initial approximation $\Lambda(0)$ and the square of the frequency ω. The key of the problem becomes how to solve derivatives $u_0^{[j]}(t)$ and $\Lambda_0^{[j]}$ ($j\ge 1$). For this purpose, we must first of all give equations governing u_0(t). For the case of principal resonance ($\Omega\approx\omega$), setting p=0 into (3), we obtain equation

$$\ddot{u}_0(t)+\Lambda(0)u_0(t)=0, \Lambda(0)=\Omega^2 \tag{9}$$

Differentiating Eq. (3) with respect to p and setting p=0, we have:

$$\ddot{u}_0^{[1]}(t)+\Lambda(0)u_0^{[1]}(t)=F(\Omega t, u_0(t), \dot{u}_0(t))-\Lambda_0^{[1]}u_0 t \tag{10}$$

where u_0(t) is given by Eq. (9) and avoiding the secular term, we obtained $\Lambda_0^{[1]}$ and the relationship between the constants of integration from Eq. (9). In the same way, we can obtain all of the j-th order deformation equations governing $u_0^{[j]}(t)$ (j≥2) which are similar in form to Eq(10) except the inhomogeneous terms. For example for $j = 2$, we obtain the second-order equation:

$$\ddot{u}_0^{[2]}(t) + \Lambda(0)u_0^{[2]}(t) = 2F^{[1]}(\Omega t, u_0(t), \dot{u}_0(t)) - 2\Lambda_0^{[1]}u_0^{[1]}(t) - \Lambda_0^{[2]}u_0(t) \tag{11}$$

where

$$F^{[1]}(\Omega t, u_0, \dot{u}_0) = \left.\frac{dF(\Omega t, u, \dot{u})}{dp}\right|_{p=0} = \left.\frac{dF(\Omega t, u, \dot{u})}{\partial u}\right|_{p=0} u_0^{[1]}(t) + \left.\frac{dF(\Omega t, u, \dot{u})}{\partial u}\right|_{p=0} \dot{u}_0^{[1]}(t) \tag{12}$$

with $u_0^{[1]}(t)$ given by Eq. (10) and $\Lambda_0^{[2]}$ can be determined avoiding the secular term in Eq. (10).

Another case to be conceived is that if there is a real parameter ε (small) such as $F(\Omega t, u, \dot{u}) = \varepsilon f(\Omega t, u, \dot{u})$. Eq. (2) becomes:

$$\ddot{u}(t) + \omega^2 u(t) = \varepsilon f(\Omega t, u, \dot{u}) \tag{13}$$

With the notations:

$$u_0^{[j]}(t) = \varepsilon^j u_0^{(j)}(t); \Lambda_0^{[j]} = \varepsilon^j \Lambda_0^{(j)}, j \geq 1 \tag{14}$$

Eqs. (7), (8), (9), (10) and (11) are respectively

$$u(t) = u_0(t) + \sum_{j\geq 1} \frac{\varepsilon^j u_0^{(j)}(t)}{j!} \tag{15}$$

$$\omega^2 = \Lambda(0) + \sum_{j\geq 1} \frac{\varepsilon^j \Lambda_0^{(j)}}{j!} \tag{16}$$

$$\ddot{u}_0(t) + \Lambda(0)u_0(t) = 0 \tag{17}$$

$$\ddot{u}_0^{(1)}(t) + \Lambda(0)u_0^{(1)}(t) = f(\Omega t, u_0(t), \dot{u}_0(t)) - \Lambda_0^{(1)}u_0(t) \tag{18}$$

$$\ddot{u}_0^{(2)}(t) + \Lambda(0)u_0^{(2)}(t) = 2f^{(1)}(\Omega t, u_0(t), \dot{u}_0(t)) - 2\Lambda_0^{(1)}u_0^{(1)}(t) - \Lambda_0^{(2)}u_0(t) \tag{19}$$

where

$$f^{(1)}(\Omega t, u_0, \dot{u}_0) = \left.\frac{df(\Omega t, u_0, \dot{u}_0)}{dp}\right|_{p=0} = \left.\frac{\partial f}{\partial u}\right|_{p=0} u_0^{(1)}(t) + \left.\frac{\partial f(\Omega t, u, \dot{u})}{\partial \dot{u}}\right|_{p=0} \dot{u}_0^{(1)}(t) \tag{20}$$

3. PERIODIC SOLUTIONS OF EQ. (1)

To determine second-order uniform periodic solutions of Eq. (1) we use MHPM and therefore Eqs (13), (17), (18) and (19). In this case, the function $f(\Omega t, u, \dot{u})$ becomes:

$$f(\Omega t, u, \dot{u}) = -\alpha u^3 - 2\mu\dot{u} + k\cos\Omega t \tag{21}$$

Eq. (17) can be written as:

$$\ddot{u}_0(t) + \Omega^2 u_0(t) = 0 \tag{22}$$

The solutions of Eq. (22) become:

$$u_0(t) = A\cos\Omega t + B\sin\Omega t \tag{23}$$

where A and B are real unknown constants.

Substituting Eqs. (21) and (23) into Eq. (18) yields:

$$\begin{aligned}\ddot{u}_0^{(1)} + \Omega^2 u_0^{(1)} &= (k - 2\mu\Omega B - \frac{3}{4}\alpha A^3 - \frac{3}{4}\alpha AB^2 - A\Lambda_0^{(1)}\cos\Omega t + \\ &+ (2\mu\Omega A - \frac{3}{4}\alpha B^3 - \frac{3}{4}\alpha A^2 B - B\Lambda_0^{(1)})\sin\Omega t + \\ &+ (\frac{3}{4}\alpha AB^2 - \frac{1}{4}A^3)\cos 3\Omega t + (\frac{1}{4}\alpha B^2 - \frac{3}{4}\alpha A^2 B)\sin 3\Omega t\end{aligned} \tag{24}$$

The conditions for the elimination of secular terms in Eq. (24) are:

$$\frac{3}{4}\alpha A(A^2 + B^2) + A\Lambda_0^{(1)} + 2\mu\Omega B = k \tag{25}$$

$$\frac{3}{4}\alpha B(A^2 + B^2) + B\Lambda_0^{(1)} - 2\mu\Omega A = 0 \tag{26}$$

Now, into (25) and (26) we put:

$$A = r\sin\varphi, \quad B = r\cos\varphi, \quad r, \varphi \in \Re \tag{27}$$

and we obtain:

$$\begin{gathered} r = \frac{k\cos\varphi}{2\mu\Omega}; A = \frac{k\sin 2\varphi}{4\mu\Omega}; B = \frac{k(1+\cos 2\varphi)}{4\mu\Omega}; \\ \Lambda_0^{(1)} = 2\mu\Omega tg\varphi - \frac{3\alpha k^2\cos^2\varphi}{16\mu^2\Omega^2}; \cos\varphi \neq 0; \varphi \in \Re \end{gathered} \tag{28}$$

Substituting Eqs. (28) in Eq. (23) we obtain the first-order solution:

$$u_0(t) = \frac{k\cos\varphi}{2\mu\Omega}\sin(\Omega t + \varphi); \cos\varphi \neq 0; \varphi \in \Re \tag{29}$$

The solution of Eq. (24) can be expressed as

$$u_0^{(1)}(t) = C\cos\Omega t + D\sin\Omega t + \frac{\alpha A(A^2 - 3B^2)}{32\Omega^2}\cos 3\Omega t + \frac{\alpha B(3A^2 - B^2)}{32\Omega^2}\sin 3\Omega t \tag{30}$$

where C and D are real unknown constants.

Substituting Eqs. (23) and (30) into Eq. (19) yields:

$$\begin{aligned} u_0^{(2)}(t)+\Omega^2 u_0^{(2)}(t) &= [-2\mu\Omega D-2\Lambda_0^{(1)}C-\Lambda_0^{(2)}A-\frac{3\alpha^2 A(A^2-B^2)(A^2-3B^2)}{128\Omega^2}- \\ &-\frac{3\alpha^2 AB^2(3A^2-B^2)}{64\Omega^2}-\frac{3}{2}\alpha C(A^2+B^2)-\frac{3}{4}\alpha C(A^2-B^2)-\frac{3}{2}\alpha ABD]\cos\Omega t+ \\ &+[2\mu\Omega C-2\Lambda_0^{(1)}D-\Lambda_0^{(2)}B+\frac{3\alpha^2 A^2 B(A^2-3B^2)}{64\Omega^2}-\frac{3\alpha^2 B(A^2-B^2)(3A^2-B^2)}{128\Omega^2}- \\ &-\frac{3}{2}\alpha CAB-\frac{3}{2}\alpha B(A^2+B^2)+\frac{3}{4}\alpha D(A^2-B^2)]\sin\Omega t+N.S.T. \end{aligned} \quad (31)$$

where N.S.T. stands for terms that do not produce secular terms.

Avoiding the presence of secular terms needs:

$$[\frac{3}{4}\alpha(3A^2+B^2)+2\Lambda_0^{(1)}]C+\frac{3}{2}\alpha AB+2\mu\Omega]D+\Lambda_0^{(2)}A+\frac{3\alpha^2 A(A^2+B^2)^2}{128\Omega^2}=0 \quad (32)$$

$$[\frac{3}{2}\alpha AB-2\mu\Omega]C+\frac{3}{4}\alpha(A^2+3B^2)+2\Lambda_0^{(1)}]D+\Lambda_0^{(2)}B+\frac{3\alpha^2 B(A^2+B^2)^2}{128\Omega^2}=0 \quad (33)$$

This set of equations can be solved and we obtain:

$$C=0; D=0; \Lambda_0^{(2)}=-\frac{3\alpha^2(A^2+B^2)^2}{128\Omega^2} \quad (34)$$

Substituting Eqs. (28) and (34) in Eq. (30) we obtain

$$u_0^{(1)}=-\frac{\alpha k^3\cos^3\varphi}{256\mu^3\Omega^5}\sin(3\Omega t+3\varphi) \quad (35)$$

Substituting Eqs. (29) and (35) into Eq. (15), we find that the second-order approximation to the solution of Eq. (1) for the primary resonant case ($\Omega\approx\omega$) is

$$u(t)=\frac{k\cos\varphi}{2\mu\Omega}\sin(\Omega t+\varphi)-\varepsilon\frac{\alpha k^3\cos^3\varphi}{256\mu^3\Omega^5}\sin(3\Omega t+3\varphi)+0(\varepsilon^2) \quad (36)$$

where $\cos\varphi\neq 0, \varphi\in\Re$.

From Eqs. (28), (34) and (16) we obtain:

$$\omega^2=\Omega^2+\varepsilon(2\mu\Omega tg\varphi-\frac{3\alpha k^2\cos^2\varphi}{16\mu^2\Omega^2})-\varepsilon^2\frac{3\alpha^2 k^4\cos^4\varphi}{2048\mu^4\Omega^6}+0(\varepsilon^3); \cos\varphi\neq 0; \varphi\in\Re \quad (37)$$

Note that in Eq. (37), tg φ remains finite for $\Omega\approx\omega$.

4. ORBITAL STABILITY

To ascertain the stability of the periodic orbits given by Eq. (36), we examine the time evolution of the orbit after the application of an infinitesimal arbitrary disturbance $\xi(t)$ in the form

$$x(t) = u(t) + \xi(t) \tag{38}$$

The stability of $u(t)$ then depends on whether $\xi(t)$ grows or decays with t. Substituting Eqs. (38) and (36) into Eq. (1) and keeping linear terms in $\xi(t)$, we obtain

$$\ddot{\xi} + \omega^2\xi = -2\varepsilon\mu\dot{\xi} - \varepsilon\frac{3\alpha k^2\cos^2\varphi}{8\mu^2\Omega^2}[1-\cos(2\Omega t + 2\varphi)]\xi \tag{39}$$

which is a linear ordinary-differential equation with periodic coefficients. As $u(t+T) = u(t)$ where $T = 2\pi/\omega$, and as $u(t+T/2) = -u(t)$, $u^2(t)$ (and therefore $\cos(2\Omega t + 2\varphi)$) is periodic with the period $T/2$. Therefore, it follows from Floquet theory [2] that Eq. (39) has solutions of the form

$$\xi(t+\frac{T}{2}) = \lambda\xi(t) \tag{40}$$

where λ is an eigenvalue (also called a Floquet multiplier) of the monodromy matrix M whose elements are associated with Eq. (39) through the relations

$$\xi_1(t+\frac{T}{2}) = m_{11}\xi_1(t) + m_{12}\xi_2(t) \tag{41}$$

$$\xi_2(t+\frac{T}{2}) = m_{21}\xi_1(t) + m_{22}\xi_2(t) \tag{42}$$

where m_{ij} are constants. The functions $\xi_1(t)$ and $\xi_2(t)$ are two linearly independent solutions of Eq. (39). To generate ξ_1 and ξ_2, we use the initial conditions:

$$\xi_1(0) = 1; \dot{\xi}_1(0) = 0 \tag{43}$$

$$\xi_2(0) = 0; \dot{\xi}_2(0) = 1 \tag{44}$$

The solution $u(t)$ is a stable orbit provided that $\xi(t)$ does not grow with t. This requires that

$$|\lambda < 1| \tag{45}$$

that is, the eigenvalues of M must remain inside the unit circle in the complex plane. The monodromy matrix M can be obtained using MHPM in the Eq. (39) for the initial conditions (43) and (44). It follows from Eqs. (41)-(44) that

$$M = \begin{bmatrix} \xi_1(\frac{T}{2}) & \dot{\xi}_1(\frac{T}{2}) \\ \xi_2(\frac{T}{2}) & \dot{\xi}_2(\frac{T}{2}) \end{bmatrix} \tag{46}$$

Therefore the characteristic equation becomes:

$$\lambda^2 - 2s\lambda + \Delta = 0 \tag{47}$$

where

$$s = \frac{1}{2}[\xi_1(\frac{T}{2}) + \dot{\xi}_2(\frac{T}{2})], \Delta = \xi_1(\frac{T}{2})\dot{\xi}_2(\frac{T}{2}) - \dot{\xi}_1(\frac{T}{2})\xi_2(\frac{T}{2}) \tag{48}$$

The values of λ determine the stability of the approximate solution $u(t)$ according to equation (45).

Case 1 (Nonperiodic solutions of Eq. (39)).

We consider $\Lambda_0^{(j)} = 0, j \geq 1$. Eq. (17) becomes:

$$\ddot{\xi}_0 + \Omega^2 \xi_0 = 0 \tag{49}$$

By using the initial conditions (43), we consider $\xi_0(0) = 1, \dot{\xi}_0(0) = 0$. Thus:

$$\xi\ (t) = \cos \Omega t \tag{50}$$

Eq. (18) becomes:

$$\begin{aligned} \ddot{\xi}_0^{(1)} + \Omega^2 \xi_0^{(1)} = &- \frac{3\alpha k^2 \cos^2 \varphi}{16\mu^2\Omega^2}(2 - \cos 2\varphi)\cos \Omega t + (2\mu\Omega - \\ &- \frac{3\alpha k^2 \cos^2 \varphi \sin 2\varphi}{16\mu^2\Omega^2})\sin \Omega t + \frac{3\alpha k^2 \cos^2 \varphi \cos 2\varphi}{16\mu^2\Omega^2}\cos 3\Omega t - \\ &- \frac{3\alpha k^2 \cos^2 \varphi \sin 2\varphi}{16\mu^2\Omega^2}\sin 3\Omega t \end{aligned} \tag{51}$$

which has the solution (for $\xi_0^{(1)}(0) = \dot{\xi}_0^{(1)}(0) = 0$):

$$\begin{aligned} \xi_0^{(1)}(t) = &(\frac{3\alpha k^2 \cos^2 \varphi \sin 2\varphi}{32\mu^2\Omega^4} - \frac{\mu}{\Omega})(\Omega t \cos \Omega t - \sin \Omega t) - \\ &- \frac{3\alpha k^2 \cos^2 \varphi(2 - \cos 2\varphi)}{32\mu^2\Omega^3} t \sin \Omega t + \frac{3\alpha k^2 \cos^2 \varphi \cos 2\varphi}{128\mu^2\Omega^4}(\cos \Omega t - \cos 3\Omega t) + \\ &+ \frac{3\alpha k^2 \cos^2 \varphi(2 - \cos 2\varphi)}{32\mu^2\Omega^3} t \sin \Omega t + \frac{3\alpha k^2 \cos^2 \varphi \cos 2\varphi}{128\mu^2\Omega^4}(\cos \Omega t - \cos 3\Omega t) + \\ &+ \frac{3\alpha k^2 \cos^2 \varphi \sin 2\varphi}{128\mu^2\Omega^4}(\sin 3\Omega t - 3\sin \Omega t) \end{aligned} \tag{52}$$

The solution ξ_1 of Eq. (39) using the initial conditions (43) is given by

$$\xi_1(t) = \xi_0 + \varepsilon \xi_0^{(1)} + 0(\varepsilon^2) \tag{53}$$

By substituting Eq. (50) and (52) into Eq. (53), we obtain:

$$\xi_1(\frac{T}{2})=\xi_1(\frac{\pi}{\Omega})=-1+\varepsilon\pi(\frac{\mu}{\Omega}-\frac{3\alpha k^2\cos^2\varphi\sin 2\varphi}{32\mu^2\Omega^4})+0(\varepsilon^2) \quad (54)$$

$$\dot{\xi}_1(\frac{T}{2})=\varepsilon\pi\frac{3\alpha k^2\cos^2\varphi(2-\cos 2\varphi)}{32\mu^2\Omega^3}+0(\varepsilon^2) \quad (55)$$

The solution ξ_2 of Eq. (39) using the initial conditions (44) is given by:

$$\xi_2(t)=\frac{\sin\Omega t}{\Omega}+\varepsilon[\frac{3\alpha k^2\cos^2\varphi(2+\cos 2\varphi)}{32\mu^2\Omega^5}(\Omega t\cos\Omega t-\sin\Omega t)-$$
$$-(\frac{\mu}{\Omega}+\frac{3\alpha k^2\cos^2\varphi\sin 2\varphi}{32\mu^2\Omega^4})t\sin\Omega t+\frac{3\alpha k^2\cos^2\varphi\sin 2\varphi}{128\mu^2\Omega^5}(\cos\Omega t-\cos 3\Omega t)+ \quad (56)$$
$$+\frac{3\alpha k^2\cos^2\varphi\cos 2\varphi}{128\mu^2\Omega^5}(3\sin\Omega t-\sin 3\Omega t)]+0(\varepsilon^2)$$

Therefore:

$$\xi_2(\frac{T}{2})=-\varepsilon\frac{3\alpha k^2\cos^2\varphi(2+\cos 2\varphi)}{32\mu^2\Omega^5}+0(\varepsilon^2) \quad (57)$$

$$\dot{\xi}_2(\frac{T}{2})=-1+\varepsilon\pi(\frac{\mu}{\Omega}-\frac{3\alpha k^2\cos^2\varphi\sin 2\varphi}{32\mu^2\Omega^4})+0(\varepsilon^2) \quad (58)$$

Substituting Eqs. (54), (55), (57) and (58) into Eqs. (48), we obtain:

$$s=-1+\varepsilon\frac{\pi\mu}{\Omega}+0(\varepsilon^2) \quad (59)$$

$$\Delta=1-\varepsilon\frac{2\mu\pi}{\Omega}+\varepsilon^2\pi^2(\frac{\mu^2}{\Omega^2}+\frac{27\alpha^2k^4\cos^4\varphi}{1024\mu^4\Omega^8})+0(\varepsilon^3) \quad (60)$$

Therefore

$$s^2-\Delta=-\frac{27\alpha^2k^4\cos^4\varphi}{1024\mu^4\Omega^8}\pi^2\varepsilon^2<0 \quad (61)$$

We remark that the terms in ε^2 from Eqs. (54), (55), (57) and (58) cannot change the value of Eq. (59). The Eq. (47) has the solutions

$$\lambda_1=s+i\sqrt{\Delta-s^2};\lambda_2=s-i\sqrt{\Delta-s^2};i=\sqrt{-1} \quad (62)$$

and therefore

$$|\lambda_1|=|\lambda_2|=\sqrt{\Delta}<1 \quad (63)$$

In this case, $u(t)$ is stable.

Case 2 (Periodic solutions of Eq. (39))

For the dissipative one degree-of-freedom system described by equation (1), there are two ways in which λ can leave the unit circle, which create independent patterns of instability in a T-periodic orbit. An eigenvalue can leave the unit circle through the real axis at -1, which initiates a saddle-node (tangent) bifurcation. A second way to leave the unit circle is through the real axis +1, which starts a pitchfork bifurcation [3,5,6]. If one of the eigenvalue leaves the unit circle through -1 we have

$$\xi(t+\frac{T}{2}) = -\xi(t) \quad \text{and} \quad \xi(t+T) = \xi(t) \tag{64}$$

at the bifurcation point. Therefore, it follows from Eqs. (38) and (64) that

$$x(t+T) = x(t) \tag{65}$$

Thus, $x(t)$ is a periodic attractor with period T and the system is expected to display a saddle node instability and the solution is expected to jump either to another attractor or to an unbounded motion.

By using MHPM, we can determine T-periodic solution $\xi(t)$. Eq. (17) can be written as

$$\ddot{\xi}_0 + \Omega^2 \xi_0 = 0 \tag{66}$$

and the solution is given by

$$\xi_0(t) = C_1 \cos\Omega t + C_2 \sin\Omega t \tag{67}$$

where C_1 and C_2 are real unknown constants.

Eq. (18) becomes

$$\ddot{\xi}_0^{(1)} + \Omega^2 \xi_0^{(1)} = -2\mu\dot{\xi}_0 - \frac{3\alpha k^2 \cos^2\varphi}{8\mu^2\Omega^2}[1-\cos(2\Omega t + 2\varphi)]\xi_0 - \Lambda_0^{(1)}\xi_0 \tag{68}$$

Substituting Eq. (67) into Eq. (68) yields:

$$\begin{aligned}\ddot{\xi}_0^{(1)} + \Omega^2 \xi_0^{(1)} &= [-\frac{3\alpha k^2 \cos^2\varphi(2-\cos 2\varphi)}{16\mu^2\Omega^2}C_1 - \frac{3\alpha k^2 \cos^2\varphi \sin 2\varphi}{16\mu^2\Omega^2}C_2 - \\ &- 2\mu\Omega C_2 - \Lambda_0^{(1)}C_1]\cos\Omega t + [-\frac{3\alpha k^2 \cos^2\varphi \sin 2\varphi}{16\mu^2\Omega^2}C_1 - \\ &- \frac{3\alpha k^2 \cos^2\varphi(2+\cos 2\varphi)}{16\mu^2\Omega^2}C_2 + 2\mu\Omega C_1 - \Lambda_0^{(1)}C_2]\sin\Omega t + \\ &+ \frac{3\alpha k^2 \cos^2\varphi}{16\mu^2\Omega^2}(C_1\cos 2\varphi + C_2 \sin 2\varphi)\cos 3\Omega t - \\ &- \frac{3\alpha k^2 \cos^2\varphi}{16\mu^2\Omega^2}(C_1 \sin 2\varphi - C_2\cos 2\varphi)\sin 3\Omega t\end{aligned} \tag{69}$$

Avoiding the presence of secular terms in Eq. (69) needs:

$$[\frac{3\alpha k^2\cos^2\varphi(2-\cos 2\varphi)}{16\mu^2\Omega^2}+\Lambda_0^{(1)}]C_1+[\frac{3\alpha k^2\cos^2\varphi\sin 2\varphi}{16\mu^2\Omega^2}+2\mu\Omega]C_2=0 \tag{70}$$

$$[-\frac{3\alpha k^2\cos^2\varphi\sin 2\varphi}{16\mu^2\Omega^2}+2\mu\Omega]C_1-[\frac{3\alpha k^2\cos^2\varphi\sin 2\varphi}{16\mu^2\Omega^2}+\Lambda_0^{(1)}]C_2=0 \tag{71}$$

The non-trivial solution of Eqs. (70) and (71), there is if

$$(\Lambda_0^{(1)}+\frac{3\alpha k^2\cos^2\varphi}{8\mu^2\Omega^2})^2+4\mu^2\Omega^2-\frac{9\alpha k^4\cos^4\varphi}{256\mu^4\Omega^4}=0 \tag{72}$$

Solutions of Eq. (72) are

$$\Lambda_0^{(1)}=-\frac{3\alpha k^2\cos^2\varphi}{8\mu^2\Omega^2}\pm\sqrt{\frac{9\alpha^2k^4\cos^4\varphi}{256\mu^4\Omega^4}-4\mu^2\Omega^2} \tag{73}$$

In the case 2, the periodic solution $u(t)$ is unstable if there are conditions:

$$\Omega^2+\varepsilon(-\frac{3\alpha k^2\cos^2\varphi}{8\mu^2\Omega^2}-\sqrt{\frac{9\alpha^2k^4\cos^4\varphi}{256\mu^4\Omega^4}-4\mu^2\Omega^2})\le\omega^2\le$$
$$\le\Omega^2+\varepsilon(-\frac{3\alpha k^2\cos^2\varphi}{8\mu^2\Omega^2}+\sqrt{\frac{9\alpha^2k^4\cos^4\varphi}{256\mu^4\Omega^4}-4\mu^2\Omega^2}) \tag{74}$$

where

$$\cos^2\varphi\ge\frac{32\mu^3\Omega^3}{3\alpha k^2};32\mu^3\Omega^3\le 3\alpha k^2 \tag{75}$$

If the conditions (74) are not satisfied, the periodic solution $u(t)$ is stable. If one of the conditions (75) is not satisfied, we obtain $C_1=C_2=0$ and therefore the disturbance $\xi(t)$ does not exist.

Now, if one of the eigenvalue of M leaves the unit circle through +1, we have

$$\xi(t+\frac{T}{2})=\xi(t) \tag{76}$$

Then it follows from Eq. (38), (36) and (76) that

$$x(t+\frac{T}{2})=-u(t)+\xi(t) \tag{77}$$

and therefore a pitchfork bifurcation starts.

5. CONCLUSIONS

In this paper, we have studied analytically periodic solutions of the forced Duffing oscillator with slight viscous damping and hardening non-linearity. The modified homotopy perturbation method have been proved to be effective and have some distinct advantages over usual approximation methods (harmonic balance method, Krylov-Bogoliubov-Mitropolsky method, weighted linearization method, Lindstedt-Poincare method, Adomian decomposition method, artificial parameter method, the method of multiple scales and so on). The stability of the periodic solutions is studied using Floquet theory.

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PRINUDNI DUFING-OV OSCILATOR SA VISKOZNIM PRIGUŠIVANJEM I TVRDOM NELINEARNOŠĆU

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Izučavan je odgovor sistema sa jednim stepenom slobode i kubnom nelinearnošću u uslovima glavne rezonancije. Korišćena je modifikovana homotopska metoda poremećaja radi odredjivanja jednačine kojom se opisuju aproksimacije drugog reda periodičkog rešenja sistema. Teorija Floquel je korišćena za ispitivanje stabilnosti rešenja.

Ključne reči: *Duffing-ov oscilator, jedan stepen slobode kretanja, nelinearni, Floquet teorija, aproksimacije drugog reda.*