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ON THE CAUSE OF RESONANT MOTIONS OF CELESTIAL BODIES

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Abstract. *Resonant motions are possible only in an inhomogeneous gravitational field. This is the reason why in the beginning of the work we elucidate the principal differences between a homogeneous and an inhomogeneous field. We show that the resonant motions in closed orbits are consequences of the kinematical extrema conditions at perihelion and/or aphelion. Resonance ratios are determined and it is shown that resonance may take place after one, two and four revolutions. Finally, in the Appendix, we propose a simple criterion for ranking of the gravitational fields.*

Key words: *homogeneous and inhomogeneous gravitational fields, resonance, resonance ratio*

1. INTRODUCTION

Natural phenomenon that rotational and orbital periods of certain solid heavenly bodies moving along closed orbits are related as the rational fractions is denominated resonance in celestial mechanics. The first researchers of the resonant effects were Euler, Lagrange and Laplace, more than two centuries ago. Their work was followed by numerous authors, among whom the most significant were, at the end of the nineteenth century, Bohlin and Poincare. During the last fifty years, along with the development of the artificial satellite and orbital station techniques, interest in the resonant motions was revived (see, e.g. [1], [2], [3]). In the work [9] the second order differential equation of the relative rotation of a body in the closed orbit around the center of gravitation was derived and a hypothesis that the corresponding angular velocity must have an extremum at perihelion as the necessary resonance condition was proposed. Numerical simulations of the relative rotations (for different shapes of the body and different eccentricities of the orbit) corroborated this assumption: in the cases of the resonance, as a rule, relative angular velocities were extreme at perihelion.

Since numerical results are less convincing as a rigorous proof, here we offer an that stabilized (i.e. resonant) motion in closed orbit is possible only if one of the principal axes of inertia coincides with the direction connecting perihelion and aphelion (line of apsides), whenever body arrives at one or at both of these points. Consequences are the

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following: relative angular acceleration of the body has to be zero, so that an extremum or an inflexion must be present in the relative angular velocity function at the coincident point.

MOTION OF THE BODY AND FRAMES OF REFERENCE

Consider a body of the mass *m*, moving in the orbit around the dominant center of gravitation of the mass m^*. (Gravitational noise is excluded). Besides orbital motion, the body revolves around the principal axis (1) of its ellipsoid of inertia and this axis remains perpendicular to the orbital plane. Thus, we restrict our analysis to the planar motion of the body, only.

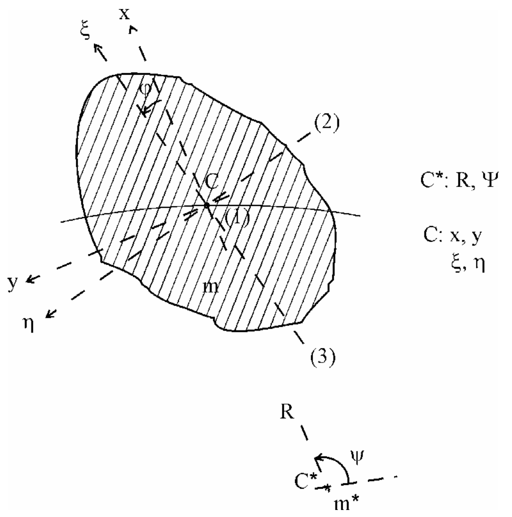

Fig. 1. Motion of the Body and Frames of Reference

As usually, the orbit is defined in the polar system of coordinates R, ψ with the center of gravitation C^* as the origin. Mass center C of the body is chosen to be the origin of two moving frames of reference Cxy and $C\xi\eta$. The first one (the translational along the radial direction displaced polar frame) is related to the geometry of the orbit. The second one is related to the geometry of the mass: $C\xi$ is directed along the principal axis (3) and $C\eta$ - along the axis (2) of the ellipsoid of inertia.

Position of the second frame with respect to the first one is defined by the angle of relative rotation $\varphi = \sphericalangle xC\xi$ (Fig. 1).

HOMOGENEOUS AND INHOMOGENEOUS GRAVITATIONAL FIELDS

Distinction between a homogeneous and an inhomogeneous gravitational field has nothing to do with its real nature, of course. The difference is only conceptual, that is, it is related to the assumed model of the field and this model has to be restricted to the active part of the field – to that part which is occupied by the body.

In a homogeneous field, intensities and directions of the elementary gravitational forces acting on the body particles depend on the position of the mass center in the gravitational field only. They are neither the functions of the position of the particles within the body, nor of the position of the body with respect to the direction C^*C (Fig. 2). All the elementary forces are parallel to this direction and their sum, the "weight" of the body, coincides with that line, regardless the relative rotation of the body in the frame of reference Cxy. The resultant of the elementary forces in a homogeneous field always passes through the mass center, so the gravitational moment with respect to that point does not exist. In fact, the mass center of a body in such a field represents the center of gravity, as conceived by Archimedes some 2.5 centuries B.C.

On the other hand, if the gravitational field is inhomogeneous one, intensities and directions of the elementary forces depend on the positions of the mass center in the gravitational field, on the positions of the corresponding particles within the body and on the relative position of the body in the "orbital" frame of reference Cxy. All these forces converge toward the gravitational center and so does their resultant. Generally, this resultant does not pass through the mass center, so it has to produce the gravitational moment at that point (Fig. 3). Hence we have a very important property of the inhomogeneous field: the existence of the gravitational moment with respect to the mass center.

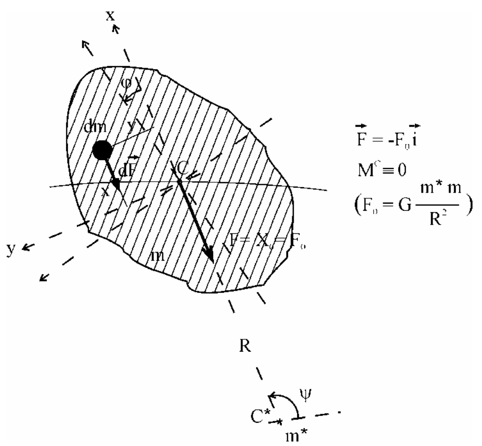

Fig. 2. Homogeneous Gravitational Field

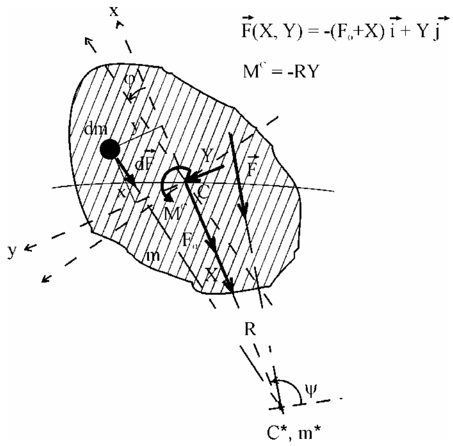

Fig. 3. Inhomogeneous Gravitational Field

Another important point is the absence of the center of gravity, as we have emphasized that before [10]. "Conversion" of a homogeneous into an inhomogeneous gravitational field implies displacement of this point from the mass center C far away from the body, into the gravitational center C^*. While the orbital motion of the body produces rotation of the radial direction C^*C around C^*, its relative rotation produces tilting of the gravitational force around this direction.

GF(2): COMPONENTS OF NEWTON'S GRAVITATIONAL LOAD AND THEIR POTENTIAL

Components of the gravitational load have the following forms in the gravitational field of the second rank (see Appendix):

$$X = -F_0\left[1 + \frac{3}{4}\frac{I_1 + 3(I_2 - I_3)\cos 2\varphi}{mR^2}\right], \tag{1}$$

$$Y = \frac{3}{2}F_0\frac{I_2 - I_3}{mR^2}\sin 2\varphi, \tag{2}$$

$$M^C = -RY = -\frac{3}{2}F_0\frac{I_2 - I_3}{mR}\sin 2\varphi. \tag{3}$$

In these expressions, F_0 is the gravitational force in a homogeneous field, G is the gravitational constant, while I_1, I_2 and I_3 are the central principal moments of inertia of the body.

It is possible now to formulate the potential energy of this load. Clearly, it has to be an approximation, valid in the *GF*(2):

$$U(R,\varphi) = -G\frac{m^* m}{R}\left[1 + \frac{I_1 + 3(I_2 - I_3)\cos 2\varphi}{4mR^2}\right]. \tag{4}$$

Obviously, negative partial derivatives with respect to the related coordinates are the correspondent components of the gravitational load

$$X = -\frac{\partial U}{\partial R},\ M^C = -\frac{\partial U}{\partial \varphi},\left(Y = -\frac{M^C}{R}\right).$$

Concerning the transversal component of the force, it may be obtained only indirectly, because it does not perform work in motion of the body along the orbit.

MOTION OF THE BODY IN AN INHOMOGENEOUS GRAVITATIONAL FIELD

Differential equations of motion of the body in GF(2) may be written as follows

$$m(\ddot{R} - R\dot{\psi}^2) = -F_0 - \frac{3}{4}F_0\frac{I_1 + 3(I_2 - I_3)\cos 2\varphi}{mR^2}, \tag{5}$$

$$m(R\ddot{\psi} + 2\dot{R}\dot{\psi}) = \frac{3}{2}F_0\frac{I_2 - I_3}{mR^2}\sin 2\varphi, \tag{6}$$

$$I_1(\ddot{\varphi} + \ddot{\psi}) = -\frac{3}{2}F_0\frac{I_2 - I_3}{mR}\sin 2\varphi. \tag{7}$$

Dots over the symbols denote the first and the second derivatives of the corresponding variable with respect to time.

It is evident that in general case, if $I_2 \neq I_1$, relative rotation of the body affects all three components of the gravitational load, causing an appearance of the periodic forcing terms on the right-hand sides of the equations (5), (6) and (7). Although these terms have very small amplitudes (of the order δ_2), they produce some immediate, but also, due to their long duration, some cumulative effects upon the motion of the body.

Let us enumerate some of them.

▪ Taking into account that the right-hand side of the equation (6) is nonzero, the sectorial velocity is not constant in an inhomogeneous field, but "almost constant" wavy line, periodic in time (Fig. 4).

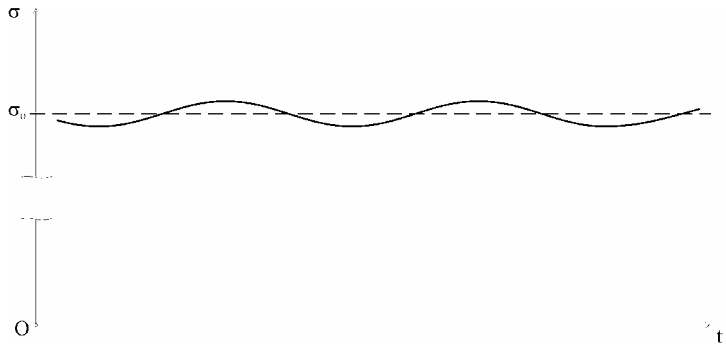

Fig. 4. Sectorial Velocity in an Inhomogeneous Gravitational Field

▪ Forcing terms, appearing in (6) and (7) produce meander course of the mass center path around the corresponding Kepler's orbit (Fig. 5).

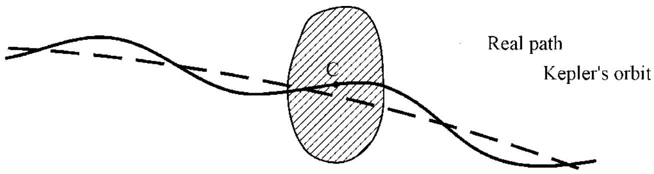

Fig. 5. Path of the Mass Center in an Inhomogeneous Field

Discrepancies are negligible if the cross section of the ellipsoid of inertia with the orbital plane is an approximate circle, but if $I_2 >> I_3$, they may be quite noticeable.

* Radial coordinate and orbital angular velocity have extrema at perihelion (and aphelion, in a closed orbit), so that right-hand sides of (6) and (7) have to be equal zero there. In such a way, equations of transversal and rotational motions become zero identities at characteristic points of the orbit:

$$R_P = \min R\,,\ \ \dot{\psi}_P = \max\dot{\psi} \ \ \rightarrow \quad \varphi_P = 0 \ \text{ or } \ \frac{\pi}{2}\,,\ \ \ddot{\varphi}_P = 0\,. \tag{8}$$

$$R_A = \max R,\ \ \dot{\psi}_A = \min\dot{\psi} \rightarrow \varphi_A = 0 \ \text{ or } \ \frac{\pi}{2},\ \ \ddot{\varphi}_A = 0\,. \tag{9}$$

Since the gravitational force is maximal at perihelion and minimal at aphelion, it is plausible to suppose that dynamic characteristics of motion, kinetic and potential energies, must also have their extrema at those points. It is obvious, from (4) that

$$U_P = \min U \text{, therefore } E_P = \max E \text{ for } \varphi_P = 0 \text{ and} \tag{10}$$

$$U_A = \max U \text{, therefore } E_P = \min E \text{ for } \varphi_P = \frac{\pi}{2}. \tag{11}$$

Such a differentiation does not follow consequentially from (8) and (9). This may be ascribed to the approximate expressions for the gravitational load and its potential, obtained in GF(2). It is more likely that only conditions (10) and (11) have to exist.

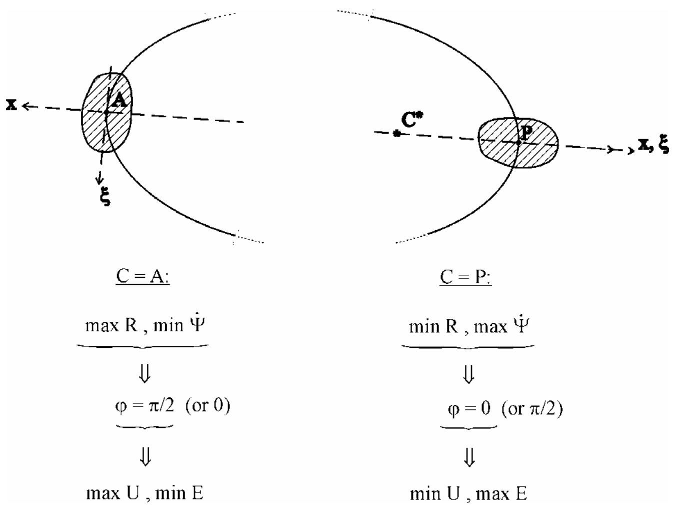

Fig. 6. Extrema Conditions and Dispositions of the Body at Perihelion

Regarding relative angular velocity, the form of this function at perihelion (and aphelion, in the case of a closed orbit) depends on several factors. By use of the equation (7) the third derivative of φ with respect to time at apsides may be written in the form

$$\dddot{\varphi}_{P/A} = -\dddot{\psi}_{P/A} - k_{P/A}\dot{\varphi}_{P/A}\cos 2\varphi_{P/A}, \tag{12}$$

where $k_{P/A} = 3\dfrac{Gm^*}{p^3}(1 \pm e)^3 \dfrac{I_2 - I_3}{I_1}$, p - is a parameter, e - eccentricity of the orbit, while P/A denotes perihelion/aphelion.

A close inspection of (12) shows that, depending on direction of the relative rotation, on the sign of $\cos 2\varphi_{P/A}$ and on the relation $\left|\dfrac{k_{P/A}\dot{\varphi}_{P/A}}{\dddot{\psi}_{P/A}}\right|$, the possibilities systematized in the Table 1 arise. Note that $\dddot{\psi}_P < 0$, while $\dddot{\psi}_A > 0$.

The possibility of the inflexion at apsides implies an asymmetrical form of the function $\dot{\varphi}(t)$ there and, since relative and orbital motions are coupled, that fact may even indicate a kind of the orbital instability.

Table 1. Relative Angular Velocity Function at Perihelion and Aphelion

<table>
<tr><th>direction of relative rotation</th><th>$\cos 2\varphi_P$</th><th>$\cos 2\varphi_A$</th><th>$\left|\dfrac{k_{P/A}\dot{\varphi}_{P/A}}{\dddot{\psi}_{P/A}}\right|$</th><th>$\dot{\varphi}_P$</th><th>$\dot{\varphi}_A$</th></tr>
<tr><td rowspan="5">$\underline{\text{progressive}}$
retrograde</td><td colspan="2">irrelevant</td><td>0 ($I_2 = I_3$)</td><td>$\min\dot{\varphi}$</td><td>$\max\dot{\varphi}$</td></tr>
<tr><td>∓ 1</td><td>± 1</td><td>irrelevant</td><td>$\min\dot{\varphi}$</td><td>$\max\dot{\varphi}$</td></tr>
<tr><td rowspan="3">± 1</td><td rowspan="3">∓ 1</td><td>> 1</td><td>$\max\dot{\varphi}$</td><td>$\min\dot{\varphi}$</td></tr>
<tr><td>$= 1$</td><td colspan="2">inflexion point</td></tr>
<tr><td>< 1</td><td>$\min\dot{\varphi}$</td><td>$\max\dot{\varphi}$</td></tr>
</table>

RESONANCE CONDITION AND RESONANCE RATIOS

As mentioned above, resonant motions are possible in the closed orbits only, and from now on we restrict our considerations to such orbits, with eccentricities $0 \leq e < 1$.

As shown before, the extrema conditions at perihelion and aphelion require that one of the principal axes ξ or η of the body becomes coincident with the line of apsides AP every time when it arrives at those points. Fulfillment of this requirement during the motion results in a resonance.

At a first impulse one would try to obtain the resonance condition by mere unification of the conditions (8) and (9), or (10) and (11). But it does not seem to be a wise approach. In order to come to the correct condition producing all the possible resonance ratios, it is necessary to take into account the very beginning of the resonant motion. Genesis of such a motion depends on the eccentricity of the orbit and on the shape of the body, mainly.

In the case of a great eccentricity and, especially if $I_2 >> I_3$, the influence of the orbit is very strong and the (maximal) gravitational force at the perihelion will, acting like a magnet, oblige the body to take the proper position and thus to produce the resonant motion. So, the capture into the resonance occurs immediately or soon after the capture into the orbit of the body. This does not mean that the corresponding extrema condition is to be immediately fulfilled at the opposite point of the orbit, where the influence of the gravitational force is minimal.

Quite a different situation arises when the eccentricity is small, or $I_2 \approx I_3$. If that is the case, once captured into the orbit, the body generally resumes chaotic rotations. During such a motion there is a chance that once, at perihelion or aphelion, one of the principal axes and direction AP become coincident. After that, the body would periodically assume the same position, having the same angular velocity and the angular acceleration zero. That event represents another scenario of the capture into the resonance. Again, it is not necessary that the extrema condition is fulfilled at the opposite point of the orbit.

Let us proceed now. We take the perihelion for the more probable capture point, with the remark that result would be the same if aphelion was the choice. If we represent the angle of relative rotation as $\varphi = \varphi(\psi)$ and assume the beginning of ψ at the line C^*P, the resonance condition may be expressed in the form of an integer argument function

$$\varphi(2n\pi) = \frac{\pi}{2}(i \pm mn)\,. \tag{13}$$

– i indicates the capture position of the body; it may take values 0 or 1;

– m represents the single revolution average angular velocity, depending on the capture angular velocity; it is equal to one of the numbers 0,1,2,3,...;

– n = 0,1,2,3,... denotes n-th passage through the capture point. Actually, it has the role of an argument in this equation.

The ± sign in (13) designates progressive/retrograde rotation.

Resonance ratio represents the relation between the angles of absolute and orbital rotation after one revolution

$$r = \frac{2\pi \pm \frac{m\pi}{2}}{2\pi} = \frac{4 \pm m}{4} \quad (m = \text{one of } 0,1,2,3,...). \tag{14}$$

Depending on the average angular velocity, resonant motions may have orbital periods of one, two and four revolutions

$$r = \cdots \underbrace{-\frac{1}{2}, -\frac{1}{4}, \frac{0}{1}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4},}_{m<0} \quad \underbrace{\frac{1}{1},}_{m=0} \quad \underbrace{\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \frac{2}{1},}_{m>0} \cdots$$

The middle term of this, in two directions infinite sequence is the so cold "ideal" resonance 1/1, related to the zero average angular velocity. It separates ratios corresponding to the retrograde and to the progressive relative rotations.

In which resonance the body is to be captured depends on many factors. The most important are

– shape of the orbit, that is, its eccentricity,
– shape of the body, characterized by its shape factor $s = (I_2 - I_3) / I_1$ and
– initial, that is, capture relative angular velocity.

There is one more point we must discuss here. Dissipative forces were neglected in the equations of motion. It was entirely justified by the fact of their smallness as well as on their insignificance regarding solution of the posed problem. However, when motions of the heavenly bodies are in question, the time measures are millennia, even eons and the work of these damping forces gradually slows down orbital and rotational motions. The effect of the friction with the cosmic dust is an increase of the radius and reduction of the eccentricity [7] of the orbit, while the tidal friction slows down relative rotation of the body. All that results in the numerical diminution of the resonance ratio and in the approach, either from the left or from the right, to the resonance situated in the middle of the sequence. As it was the case with the beginning of the resonant motion, depending on the shape of the orbit, as well as on the shape of the body, the change from the higher to the lower resonance may occur suddenly, or gradually – through chaotic motion.

After this digression, we may continue now. As seen above, resonant motions with one, two and four revolutions orbital periods were obtained from the resonance condition (13), which corresponds to the extrema condition (8). Regarding satisfaction of the remaining conditions (9), (10) and (11) we show Table 2, in which the signs + and – mean that the related condition is/is not satisfied.

Like the other motions, resonant motions may be more or less stable. Using the stability of the ball in a spherical cavity metaphor, it is possible to speak of the "shallow" and the "deep" resonance [1]. The capture into and escape from a shallow resonance should have to be relatively easy and so should have to be the capture into a deep one. On the other hand, escape from a deep resonance should have to require a great amount of the energy consumption.

If one adopts the number of the fulfilled extrema conditions as a kind of the resonant motions stability criterion, one will reach to the expected outcome that the four – cycles resonant motions with the "wrong" capture position ($\varphi_P = \pi/2$) have the least stability: excepting this capture position, none of the other conditions is to be satisfied during such motions.

And now comes a somewhat surprising result: not the single-cycle, but the two-cycles resonant motions with "correct" capture position ($\varphi_P = 0$) satisfy all four extrema conditions. On the contrary, it is impossible that during the single-cycle resonant motions the energy extrema conditions were fulfilled at both characteristic points of the orbit.

Table 2. Resonant Motions: Fulfillment of the Extrema Conditions

number of revolutions	*P*	*A*	*P*	*A*
	capture position	(9)	(10)	(11)
one	0	+ (0)	+	–
	$\frac{\pi}{2}$	$+\left(\frac{\pi}{2}\right)$	–	+
two	0	$+\left(\frac{\pi}{2}\right)$	+	+
	$\frac{\pi}{2}$	+ (0)	-	–
four	0	–	+	–
	$\frac{\pi}{2}$	–	–	–

This result is possibly an explanation of the Mercury's resonance. Rotational axis of this planet is almost perpendicular to the orbital plane and our model roughly corresponds to the Mercury motion. For almost a hundred years it was considered that Mercury moves around the Sun in the "ideal" 1/1 resonance. Subsequently, however, some forty years ago, it was established that its resonance ratio is, actually, 3/2. From the standpoint of the here proposed criterion, if the capture position of this celestial body was correct, motion in 3/2 is more stable than motion in 1/1 resonance. Taking into consideration the outstanding eccentricity of its orbit ($e = 0.206$), the chance that Mercury might escape the present and become captured into the "ideal" resonance appears insignificant. In general, 1/1 resonance is neither more nor less stable than any other single-cycle resonance.

It seems that the passage into the ideal resonance and the change in the shape of the orbit have to go together. As already mentioned, due to the continuous energy degradation, eccentricity of the orbit decreases and consequently, the orbital motion becomes uniform. Simultaneously, the relative rotation slows down and when it stops, in accordance with the minimal potential energy principle, the body assumes the stable position $\varphi = 0$, pointing its principal axis of the minimal moment of inertia toward the center of gravitation. This resonance 1/1 is the final one and, in our opinion, the term ideal should have to be reserved for such a motion only: uniform motion in a circular orbit, relative angular velocity – zero and position of the body – stable.

Earth's Moon, as well as the numerous satellites orbiting around the other solar system planets, moves, approximately, in such a resonance.

CONCLUSION

It is not possible to speak about resonant motions in a gravitational field if an inhomogeneous field as a model for this field was not adopted.

Hence, in the beginning of this work the principal features of an inhomogeneous gravitational field were emphasized and after that, the components of the gravitational load and their potential in such a field were given. Being the forcing terms in the differential equations of motions, these components produce appreciable dissimilarities between the motions in a homogeneous and in an inhomogeneous gravitational field.

One of the most important differences is that an inhomogeneous field forces the body moving in a closed and stable orbit to revolve in resonance. We have shown in this work that fulfillment of the kinematical quantities extrema condition in perihelion and/or aphelion requires that one of the body's principal axes of inertia coincides with the line of apsides whenever body arrives at one, or both these points. The result is a stable, that is, a resonant motion.

We have also determined a sequence of the resonance ratios. It was shown that the resonant motions may have orbital periods of one, two and four revolutions. An interesting conclusion is that generally, when eccentricity of the orbit differs from zero and the cross section of the ellipsoid of inertia with the orbital plane differs from the circle, the two-cycles resonance is the most stable one.

APPENDIX: RANKING OF THE GRAVITATIONAL FIELDS

According to the Figure 3, the Newton's gravitational force acting on the body in an inhomogeneous field may be represented in the form

$$\vec{F} = Gm^* \int_m \frac{\vec{r}}{r} \frac{dm}{r^2} \ ,$$

where $\vec{r}(-R-x,-y)$ denotes position of the center of gravitation with respect to the body's elementary mass.

The need for ranking of the fields follows from the fact that it is not possible to express this integral in finite form, by use of the elementary functions. Instead, it may be represented as the infinite functional series, only

$$\vec{F}(X,Y) = \sum_{k=0}^{\infty} \vec{F}_k(X_k, Y_k).$$

We propose the following form of the series representing components of this force

$$X = F_0 \sum_{k=0}^{\infty} \overline{X}_k(S_k, \varphi)\delta_k(R) , \tag{A1}$$

$$Y = F_0 \sum_{k=0}^{\infty} \overline{Y}_k(S_k, \varphi)\delta_k(R) . \tag{A2}$$

In these expressions, $F_0 = G\dfrac{m^* m}{R^2}$ and G is the gravitational constant.

The terms of the series are products of the dimensionless functions $\overline{X}_k$ and $\overline{Y}_k$ with the functional parameters δ_k . Functions $\overline{X}_k$ and $\overline{Y}_k$ depend on the geometry of mass of the body, as well as on its relative rotation. The mass distribution is characterized by the moments of mass M_{ik} (i = 1,2,3) of the order k with respect to the adopted frame of reference. It is convenient to take that the moments of the zero order determine mass of the body: $M_{i0} = m$, while the first moments define the position of the center of mass C. Since this point was chosen to be the origin of our frames of reference, it is obvious that these moments have to be zeros, identically: $M_{i1} = 0$ (i = 1,2,3)**.**

The second moments M_{i2} are the moments of inertia, of course. In the frame ξ,η,ζ those are the principal moments: $M_{i2} = I_i$ (i = 1,2,3).

Moments of mass of the higher order must have their meanings too, but their definitions have not been established, yet. Arguments S_k, denominated the shape factors in the work [9] may be defined in the following form

$$S_k = \frac{f[M_{ik}]}{\max M_{ik}} \tag{A3}$$

Regarding parameters δ_k, they may be represented as relations

$$\delta_k = \frac{\max M_{ik}}{mR^k} . \tag{A4}$$

Obviously, $\delta_0 = 1$, while $\delta_1 \equiv 0$. The other parameters are, as a rule, very small numbers in the celestial mechanics, situated in the interval

$$\frac{\max M_{ik}}{m \cdot \max R^k} \le \delta_k \le \frac{\max M_{ik}}{m \cdot \min R^k} \tag{A5}$$

Note that $\min R = C^* P$ (P- perihelion), while $\max R = \infty$, for an open orbit and $\max R = C^* A$ (A-aphelion), in the case of the closed one. As an illustration, for Earth in the gravitational field of the Sun (considering small eccentricity of its orbit) one may adopt that $O(\delta_2) = 10^{-9}$. An effect of smallness of these parameters is a fast convergence of the sums of series toward the exact solutions.

Convenience of the forms (A1) and (A2) becomes evident. Parameters δ_k in both expressions indicate the terms of the comparable order, so that we may safely truncate

series in these equations now and use the sums of the finite series as the approximate values of the force components. The term after which the series are to be truncated may be used as a criterion for ranking of the gravitational fields. The rule is quite simple:
- **rank of the field is K if the components of the gravitational force are approximated by the finite series with K terms.** Such a field may be denominated **GF(K).**

If we substitute $\overline{X}_0 = -1$ and $\overline{Y}_0 = 0$ into (A1) and (A2) we shall get

$$X = -F_0,\ Y = 0 \text{ and } M^C = -RY = 0.$$

According to the given criterion, as well as to what was said about the homogeneous and inhomogeneous fields, the $GF(0)$ is obviously a homogeneous field. Since $\delta_1 \equiv 0$, $GF(1)$ would be a homogeneous field, also. Actually, $GF(1) \equiv GF(0)$.

All the other fields are inhomogeneous. If we take into account one more term, we "enter" into the $GF(2)$ which has all the characteristics of an inhomogeneous field.

Here follow the components of the gravitational force in that field, written in the form (A1), (A2):

$$X = F_0\left[-1 - \frac{3}{4}(1 + 3S_2\cos 2\varphi)\frac{I_1}{mR^2}\right], \tag{A6}$$

$$Y = F_0\frac{3}{2}S_2\sin 2\varphi\frac{I_1}{mR^2}, \tag{A7}$$

and the gravitational moment with respect to the mass center is, of course

$$M^C = -RY\ . \tag{A8}$$

Clearly, $\overline{X}_2 = -\frac{3}{4}(1 + 3S_2\cos 2\varphi)$, $\overline{Y}_2 = \frac{3}{2}S_2\sin 2\varphi$, the shape factor corresponds to that one, obtained in the work [9] $S_2 = \frac{I_2 - I_3}{I_1}$, while the parameter $\delta_2 = \frac{I_1}{mR^2}$.

And that is all we have at this moment:

– $GF(0) = GF(1)$ – homogeneous gravitational field and

– $GF(2)$ – inhomogeneous gravitational field.

Gravitational fields of the higher rank will have to wait until definitions of the moments of mass of the higher order were established.

REFERENCES

1. B. Garfinkel: 1982, Celes. Mech., 28, pp. 275 – 290.
2. R. Sergysels: 1989, Celes. Mech., 44, 155.
3. R. A. Howland : 1989, Celes. Mech., 44, 209.
4. R. V. de Moraes & P.A.F. da Silva: 1990, Celes. Mech., 47, 225.
5. V. V. Beletskii: 1974, Resonance Phenomena at Rotations of Artificial and Natural Celestial Bodies, G. E.O. Giacaglia, Springer Verlag, Berlin, pp. 192 – 232.
6. S. Hawking, W. Israel: 1989, 300 Years of Gravitation, Cambridge University Press.
7. L. A. Pars: 1956, A Treatise of Analytical Dynamics, Heinemann, London.
8. R. G. Strom: 1987, Mercury, Smitsonian Institution Press, London.
9. M. Marjanov: 1997, Gravitational Resonances, Bull. Astron., Belgrade, 156, 9 – 19.
10. M. Marjanov: 2001, Inhomogeneous Gravitational Field and the Body without Center of Gravity, XXIII JUMEH Congress, Belgrade.

O UZROKU REZONANTNIH KRETANJA NEBESKIH TELA

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Fenomen gravitacione rezonance može se ispoljiti samo u nehomogenom gravitacionom polju. Stoga su na početku rada istaknute osnovne razlike između homogenog i nehomogenog polja. Izvedene su komponente gravitacionog opterećenja i njegov potencijal pri ravnom kretanju u nehomogenom polju. Pokazano je da su rezonantna kretanja (u zatvorenim orbitama) posledice postojanja ekstremuma kinematičkih i dinamičkih atributa kretanja u apsidama. Dat je beskrajni niz rezonantnih brojeva, to jest, razlomaka čiji su imenioci jedan, dva ili četiri. Naime, pokazalo se da su to brojevi orbitalnih revolucija u kojima se može uspostaviti rezonanca s rotacionim kretanjem. U prilogu je prikazan jednostavan kriterijum za razvrstavanje gravitacionih polja koji je korišćen u ovome radu.

Ključne reči: *homogeno i nehomogeno gravitaciono polje, rezonanca, rezonantni broj*