FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol. 4, N° 17, 2005, pp. 287 - 291

PERIODICAL SOLUTION OF SOME DIFFERENTIAL EQUATIONS *

UDC 517.9(045)=20

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Abstract. *In this paper we investigate the existence of periodical solutions of the first order differential equations*

$$\frac{dy}{dt} = A(t)y + f(t,y)$$

The obtained results are applied to a differential equation, where the right hand side contains a polynomial of the fourth degree with respect to y.

Key words: *Periodical solution, first-order nonlinear ordinary differential equations.*

We will investigate the differential equation of the first order

$$\frac{dy}{dt} = A(t)y + f(t,y)\,. \tag{1}$$

Definition: The right hand side of the differential equation (1) satisfies (A), if the following conditions are fulfilled:

1) Function $A(t)$ is defined in the interval $(-\infty,\infty)$, and $f(t,y)$ in $\overline{D}$ where $\overline{D}$ is a closed area of the plane (t,y);

2) $A(t)$ and $f(t,y)$ are ω - periodical with respect to t and $e^{\int_0^\omega A(t)dt} \neq 1$;

3) $|A(t)| \le B(t)$, for $t \in [0,\omega]$, where $B(t)$ is a nonnegative summarable function on $[0,\omega]$;

4) $f(t,y)$ is a measurable on $[0,\omega]$;

5) $|f(t,y)| \le \Phi(t)$ for $y(t) \in \overline{D}$, $t \in [0,\omega]$, where $\Phi(t)$ is a nonegative summarble function on $[0,\omega]$;

Received March 02, 2004
* Supported by Ministery of science, tehnology and development of Republic of Serbia, project "Structures of functional analysis and differential equations", No 1856
AMS subject classification: Primary 34A34. Secondary 34C25.

6) For each series $\{x_n(t)\}$, for every n and a function $x_0(t)$, such that if $\|x_n(t) - x_0(t)\| \to 0$, $n \to \infty$, than $\int_0^\omega |f(t, x_n(t)) - f(t, x_0(t))| dt \to 0,\ n \to \infty$, where $\|x(t)\| = \sup_{t \in [0,\omega]} |x(t)|$.

The solution of the equation (1) is absolute-continuous function $y = y(t)$ on a finite interval, which satisfies the equation (1). If $y(t)$ is a solution of the equation (1), then it is easy to show that

$$y(t) = e^{\int_0^t A(s)ds} \left(\int_0^t f(s, y(s)) e^{-\int_0^s A(t)dt} ds + C \right). \tag{2}$$

If $C = y(\omega)$, then $y(0) = y(\omega)$ and $y(t)$ is an ω – periodical solution. We will investigate the operator F, so that

$$Fy(t) = e^{\int_0^t A(s)ds} \left(\int_0^t f(s, y(s)) e^{-\int_0^s A(t)dt} ds + y(\omega) \right) \tag{3}$$

The following theorem is easy to show to be true (see [2]).

Theorem 1. A function $y = y(t)$ is an ω – periodical solution of the equation (1) if and only if it is a fixed point of the operator F.

Assume that $y = y(t)$ is a fixed point of the operator F. Then if $t = \omega$ we have

$$y(\omega) = e^{\int_0^\omega A(t)dt} \left(\int_0^\omega f(t, y(t)) e^{-\int_0^t A(s)ds} dt + y(\omega) \right),$$

or
$$y(\omega)(1 - e^{\int_0^\omega A(t)dt}) = e^{\int_0^\omega A(t)dt} \int_0^\omega f(t, y(t)) e^{-\int_0^t A(s)ds} dt, \quad 1 - e^{\int_0^\omega A(t)dt} \neq 0.$$

Then

$$y(\omega) = \frac{e^{\int_0^\omega A(t)dt}}{1 - e^{\int_0^\omega A(t)dt}} \int_0^\omega f(t, y(t)) e^{-\int_0^t A(s)ds} dt,$$

and the operator F is

$$Fy(t) = e^{\int_0^t A(s)ds} \left[\int_0^t f(s, y(s)) e^{-\int_0^s A(t)dt} ds + \frac{e^{\int_0^\omega A(t)dt}}{1 - e^{\int_0^\omega A(t)dt}} \int_0^\omega f(s, y(s)) e^{-\int_0^s A(t)dt} ds \right]. \tag{4}$$

Theorem 2. Let the conditions (A) exist. If

$$\frac{(qe^{\int_0^\omega B(t)dt}-1)e^{\int_0^\omega B(t)dt}}{e^{\int_0^\omega B(t)dt}-1}\int_0^\omega \Phi(s)e^{\int_0^s B(t)dt}ds < R, \qquad \|y(t)\| \le R$$

then there exists at least one ω – periodical solution of the equation (1).

Proof. It is possible to prove that the operator F is continuous at $\|y(t)\| \le R$. Operator transforms the sphere $\|y(t)\| \le R$ to itself. For every continuous function $y(t)$, $\|y(t)\| \le R$, we have

$$|Fy(t)| \le e^{\int_0^t |A(s)|ds}\Big[\int_0^t | f(s,y(s)) \| e^{-\int_0^s A(t)dt}|ds + |\frac{e^{\int_0^\omega A(t)dt}}{1-e^{\int_0^\omega A(t)dt}}||\int_0^\omega | f(s,y(s)) |e^{-\int_0^s A(t)dt}ds\Big] \le$$

$$\le e^{\int_0^\omega B(t)dt}\Big[\int_0^\omega \Phi(s)e^{\int_0^s B(t)dt}ds + \frac{e^{\int_0^\omega B(t)dt}}{e^{\int_0^\omega B(t)dt}-1}\int_0^\omega \Phi(s)e^{\int_0^s B(t)dt}ds\Big] =$$

$$= \frac{\Big[qe^{\int_0^\omega B(t)dt}-1\Big]e^{\int_0^\omega B(t)dt}}{e^{\int_0^\omega B(t)dt}-1}\int_0^\omega \Phi(s)e^{\int_0^s B(t)dt}ds < R,$$

According to the conditions of the theorem, where q is a corresponding non-negative value, exists at least one periodical solution, so that the theorem is proved.

Theorem 3. Let the conditions (A) be satisfied and $|f(t,y)| \le M(t)|y|$, where $M(t)$ is a nonegative summable function on $[0,\omega]$. If

$$\frac{\Big[qe^{\int_0^\omega B(t)dt}-1\Big]e^{q\int_0^\omega B(t)dt}}{e^{\int_0^\omega B(t)dt}-1}\int_0^\omega M(t)dt < R, \tag{5}$$

then, in $\|y(t)\| \le R$, there exists at least one ω–periodical solution of the equation (1).

Proof. The operator (4) is continuous in $\|y(t)\| \le R$ for $t \in [0,\omega]$, and we will prove that it transforms the sphere $\|y(t)\| \le R$ in itself. We have

$$|Fy(t)| \le e^{\int_0^t B(s)ds}\Big[\int_0^t |f(s,y(s)|e^{\int_0^s B(t)dt}ds + \frac{e^{\int_0^\omega B(t)dt}}{e^{\int_0^\omega B(t)dt}-1}\int_0^\omega |f(s,y(s)|e^{\int_0^s B(t)dt}ds\Big] \le$$

$$\le e^{\int_0^\omega B(t)dt} R\int_0^\omega M(s)e^{\int_0^s B(t)dt} ds\left[1+\frac{e^{\int_0^\omega B(t)dt}}{e^{\int_0^\omega B(t)dt}-1}\right]\le \frac{\left[qe^{\int_0^\omega B(t)dt}-1\right]e^{q\int_0^\omega B(t)dt}}{e^{\int_0^\omega B(t)dt}-1}R\int_0^\omega M(t)dt<R,$$

because of the condition (5). Therefore, in the sphere $\|y(t)\|\le R$ exists at least one ω-periodical solution, by using the results of the Theorem 1 and the Theorem 2.

Now, we consider the equation

$$\frac{dy}{dt}=p_1(t)y+p_2(t)y^2+p_3(t)y^3+p_4(t)y^4 \tag{6}$$

which is also considered in [3],[4]. Let the function $p_i(t)$ ($i=1,2,3,4,$) be ω-periodical with respect to t, summable for $t\in[0,\omega]$ and $|p_i(t)|\le p(t)$, for a nonegative summable function p(t) on [0,ω]. Define the operator

$$Ay(t)=e^{\int_0^t p_1(s)ds}\left(\int_0^t f(s,y(s))e^{-\int_0^s p_1(t)dt}ds+y(\omega)\right), \tag{7}$$

where

$$f(t,y)=p_2(t)y^2+p_3(t)y^3+p_4(t)y^4\,. \tag{7'}$$

Let $\|y(t)\|\le 1$. Than

$$y^2(t)\le|y(t)|,\quad \left|y^3(t)\right|\le|y(t)| \text{ and } |f(t,y)|\le\left[\,|p_2(t)|+|p_3(t)|+|p_4(t)|\,\right]|y(t)|.$$

For $\|y(t)\|\le 1$ ω-periodical solution satisfies the condition

$$\frac{\left[qe^{\int_0^\omega p(t)dt}-1\right]e^{q\int_0^\omega p(t)dt}}{e^{\int_0^\omega p(t)dt}-1}\int_0^\omega(|p_2(t)|+|p_3(t)|+|p_4(t)|dt<1$$

Let also $p_1(t)$ satisfy

$$1\le e^{\int_0^\omega p_1(t)dt}\left(1-\int_0^\omega p_1(s)e^{-\int_0^s p_1(t)dt}ds\right) \tag{8}$$

Then we can express the following assertion:

Theorem 4. Let the function $f(t,y)$ be defined by (7'), and let it satisfy the condition $|f(t,y)|\le p_1(t)$ for $\|y(t)\|\le R$, and $p_1(t)\ge 0$ for $t\in[0,\omega]$ satisfies the conditions (8). If

$$qe^{\int_0^\omega p_1(t)dt}-1\le R\,,$$

where $R>1$, then in $\|y(t)\|\le R$ exists at least one ω-periodical solution of the equation (6).

Proof. Under these conditions the continuous operator (7) transforms sphere $\|y(t)\| \le R$ into itself. We have,

$$|Ay(t)| = |e^{\int_0^t p_1(s)ds}\left[\int_0^t f(s,y(s))e^{-\int_0^s p_1(t)dt}ds + \frac{e^{\int_0^\omega p_1(t)dt}}{1-e^{\int_0^\omega p_1(t)dt}}\int_0^\omega f(s,y(s))e^{-\int_0^s p_1(t)dt}ds\right]| \le$$

$$\le e^{\int_0^\omega p_1(t)dt}\left[\int_0^t p_1(s)e^{-\int_0^s p_1(t)dt}ds + \frac{e^{\int_0^\omega p_1(t)dt}}{e^{\int_0^\omega p_1(t)dt}-1}\int_0^\omega p_1(s)e^{-\int_0^s p_1(t)dt}ds\right] \le$$

$$\le e^{\int_0^\omega p_1(t)dt}\int_0^\omega p_1(s)e^{-\int_0^s p_1(t)dt}ds\left[1+\frac{e^{\int_0^\omega p_1(t)dt}}{e^{\int_0^\omega p_1(t)dt}-1}\right]$$

$$\le \frac{e^{\int_0^\omega p_1(t)dt}\left[qe^{\int_0^\omega p_1(t)dt}-1\right]\left[e^{\int_0^\omega p_1(t)dt}-1\right]}{\left[e^{\int_0^\omega p_1(t)dt}-1\right]e^{\int_0^\omega p_1(t)dt}} = qe^{\int_0^\omega p_1(t)dt}-1 < R$$

and the theorem is proved.

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PERIODIČNA REŠENJA NEKIH DIFERENCIJALNIH JEDNAČINA

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U radu se razmatraju periodična rešenja diferencijalne jednačine prvog reda $\frac{dy}{dt} = A(t)y + f(t,y)$.

Dobijeni rezultat se primenjuje na diferencijalnu jednačinu čija je desna strana polinom četvrtog stepena u odnosu na y.

Ključne reči: *periodična rešenja, nelinearna diferencijalna jednačina*