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THE ACTION OF FORCE AND COUNTERACTION PRINCIPLE

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Abstract. *In this article, the standpoint is the terms action and principle in classical mechanics. The ontological clarification of the term "action" is defined through expressions (1), (2) and (3). The example of the linear body motion is showing uselessness of the standard action integral. Then, using the integral (4), the term "Action of force", is defined. The action of the inertia force will be named "counteraction". Similarly, the term the "principle of mechanics" is analyzed and defined. The principle of action and counteraction is formulated. The independence of this principle is proved, with the comparison of generalized variational principles of mechanics. We give the very simple, but essentially important examples.*

Key words: *Analytic mechanics, principle, action, counteraction*

1. INTRODUCTION

The terms indicated in the title, *action* and principle, are not uniformly and unambiguously defined in classical mechanics. Therefore, for the sake of a clear understanding of this paper, it is necessary to clarify this statement and then to proceed to a generalization and definition of the concepts of *action* and *principle*.

The first concept of *action* can be found in the work of Leibnitz (1669) (Ref. [1], p.782) as the *actio formalis*, whose dimension is a product of mass, velocity and path. According to Newton's definition, [2] it is written that: "Vis impressa est *actio* in corpus exercita, ad mutandum eius statum vel movendi *actio*", (An impressed force is exerted upon a body, in order to change its state of uniform motion, in a right line). ([2a], 4).

Accordingly, the force is defined by means of the concept of *action* that has not been previously defined and thus it is assumed to be clear and known. The concept of *actio* was later used by Christian Wolff (1726): " *actio consists of mass, velocity and space*."([1], pp. 111, 750, 787). Maupertuis P. L. wrote (1746): "The quantity of action is a product of the body masses, their velocities and the distances they are traversing" ([1], pp. 53, 881). Euler L. found (1748) that the sum of all momentary actions has the form

$$\int dt\ (\int Vdv + V'dv' + V''dv'' + ...)\ , \tag{1}$$

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where V, V', V'' are forces expressed as functions of distances v, v', v'', ([1], pp. 76, 791, 882). Let's also quote Hamilton (Ref. 1, p.179) who wrote (1834) that the integral

$$A_1 = \int \sum (x'dx + y'dy + z'dz) = \int 2Tdt$$

namely accumulate living force (vis viva) often termed as the *action of system* from its initial to its final position. In the Van Nostrand's Scientific Encyclopedia, Second Edition (New York, 1947) it is formulated in the following way: "ACTION: In certain discussions of dynamics there is a need of an expression for the product of twice the mean total kinetic energy of a system of particles, during a specified interval of time, by the duration of the interval. This product is called *action.* Mathematically, it is expressed by

$$A_1 = 2\int_{t_0}^{t_1} E_k dt \tag{2}$$

in which E_k is the kinetic energy and t_0, t_1 are the times of beginning and ending of the interval" The concept of action has been much more or quite sufficiently discussed in the above-mentioned anthology [1]. Still, for our purpose here let's also quote a few sentences from A. Sommerfeld's book [3], (1944):" . . . just as power is defined as energy magnitude time, so *action* is defined as energy magnitude x time". "When we speak about the cause and action, we imply that the action is a consequence or result" . . . "However, afterwards, if the term action is sanctioned by Hemholtz and Planck, each attempt to substitute it by some other term will be without perspective" . . . "As an example, the elementary quantum of *Planck's action* can be used". In further development of analytical mechanics the concept of *action* was accepted in the form Pof the functional

$$A_2 = \int_{t_0}^{t_1} Ldt \tag{3}$$

where $L(q, \dot{q}, t)$ is the kinetic potential, often called Lagrange's function. The physical dimension of *action* are ML^2T^{-1}; dim mass $m = M$, dim lengt $l = L$, dim time $t = T$.

2. The Definition of the Term Action of Force and Counteraction

Here we will retain the accepted dimensions of action, but we will uniformly define the term *action of force* and the term *counteraction*.

Definition 1. *The action $A(\boldsymbol{F})$ of a force $\boldsymbol{F}$ during an interval of time $[t_1 - t_0]$ is determined by the integral*

$$\mathrm{A}(\boldsymbol{F}) = \int_{t_0}^{t} W(\boldsymbol{F})dt = \int_{t_0}^{t_1} (\int_S \boldsymbol{F}d\boldsymbol{r})dt \tag{4}$$

where $W(\boldsymbol{F})$ is the work of all resulting forces $\boldsymbol{F}_\nu$ on the ν-th dynamical point M_ν with arc S_ν of the path $S_\nu \in S$. As it is well known the work of the forces, in general case, is a curvilinear integral

$$W(\boldsymbol{F}) = \int_S \boldsymbol{F}d\boldsymbol{s} = \sum_{\nu=1}^{N} \boldsymbol{F}_\nu d\boldsymbol{r}_\nu . \tag{4a}$$

If the force $\boldsymbol{F}$ is potential then $W(\boldsymbol{F})$ is just a negative potential energy in the current point of the path S.

At first sight it may appear that it is just a different formal way of writing. However, this definition - just like Euler's action (1) has more important both mathematical and physical differences with respect to action (3) or (2). In paper [4], it is shown that functional (3) is not invariant with aspect to Lagrange's generalized coordinates $q = (q^1,...,q^n)^T \in M^n$ and Hamilton's $(p,q) \in T^*M^n$ canonical variables for rheonomic systems. Namely, for mechanical systems with time-independent constraints, the functional (2) and (3) can be written with respect to Descartes' coordinates y, curvilinear coordinates x, generalized independent coordinates q as well as with respect to Hamilton's coordinates p, q in invariant form

$$A_1 = \int_{t_0}^{t_1} 2E_k dt = \int_{t_0}^{t_1} a_{ij}\dot{q}^i\dot{q}^j dt = \int_{t_0}^{t_1} a^{ij} p_i p_j dt = \int_{q(t_0)}^{q(t_1)} p_i dq^i; \tag{5} \text{ and}$$

$$A_2 = \int_{t_0}^{t} Ldt = \rightarrow \int_{t_0}^{t_1} (E_k - E_p)dt = \int_{t_0}^{t_1} (p_i\dot{q}^i - H)dt; H := E_k + E_p. \tag{6}$$

However, if the constraints of the system depend on time, [4], relations (5) and (6) loose both mathematical and physical sense, since it is

$$A_1 = \int_{t_0}^{t_1} 2E_k dt \neq \int_{t_0}^{t_1} a_{ij}\dot{q}^i\dot{q}^j dt \neq \int_{t_0}^{t_1} a^{ij} p_i p_j dt \neq \int_{q(t_0)}^{q(t_1)} p_i dq^i;$$

and

$$A_2 = \int_{t_0}^{t} Ldt \neq \int_{t_0}^{t_1} (E_k - E_p)dt \neq \int_{t_0}^{t_1} (p_i\dot{q}^i - H)dt;$$

because in the classical interpretations

$$2E_K = a_{ij}\dot{q}^i\dot{q}^j + 2a_i\dot{q}^i + a_0,..., p_i = a_{ij}\dot{q}^j + a_i,...,\dot{q}^i = a^{ij}(p_j - a_j)..$$

The action (4) is invariant with respect to all the above-mentioned transformations, that is

$$A = \int_{t_0}^{t} W(\boldsymbol{F})dt = \int_{t_0}^{1} (\int_S \boldsymbol{F}d\boldsymbol{r})dt = \int_{t_0}^{t} (\int_S Ydy)dt = ... = \int_{t_0}^{t} (\int_S Qdq)dt$$

both for scleronomic and rheonomic mechanical systems.

Definition 2. *The action of the inertia force* ***I****, determined by integral*

$$A(\boldsymbol{I}) = \int_{t_0}^{t} W(\boldsymbol{I})dt = \int_{t_0}^{1} (\int_S \boldsymbol{I}d\boldsymbol{r})dt \tag{7}$$

it will be named counteraction.

For a material point of constant mass m, moving at the velocity $\boldsymbol{v}$, it will be

$$A(\boldsymbol{I}) = \int_{t_0}^{t_1} (\int_S -m\frac{d\boldsymbol{v}}{dt}d\boldsymbol{r})dt = \int_{t_0}^{t_1} (\int_0^v -m\boldsymbol{v}d\boldsymbol{v})dt = -\int_{t_0}^{t_1} E_k dt. \qquad (8\,)$$

Although integrals (2) and (3) seem to be similar in form to the integral (8), there is a basic difference between them. Namely, for a constant velocity motion, integral (8) is always zero,

$$A(\boldsymbol{I}) = \int_{t_0}^{t_1} (\int_{r_0} -m\frac{d\boldsymbol{v}}{dt}d\boldsymbol{r})dt = \int_{t_0}^{t_1} (\int_0^v -m\boldsymbol{v}d\boldsymbol{v})dt = 0.$$

while integrals (2) and (3) may attain arbitrarily large values. In additional, integral (2) is the action by definition, and in relation (8) it appears as the consequence of action of inertia force (7)

A very **simple example** of a mass point moving at constant velocity $v = c$ along an ideally smooth horizontal straight line very clearly points out a quantitative difference between action (2) and action of forces (4). Actions (2) and (3), in this case are

$$A_1 = \int_{t_0}^{t_1} 2E_k dt = \mathrm{mc}^2(t_1 - t_0),$$

$$A_2 = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} E_k dt = \frac{mc^2}{2}(t_1 - t_0).$$

However, action (7) is

$$A(\boldsymbol{I}) = \int_{t_0}^{t} (\int_S -m\frac{d\boldsymbol{v}}{dt})dt = 0 \qquad (9)$$

since in this example $d\boldsymbol{v}/dt = 0$. It is enough to give some thought to result (9) in order to reach the conclusion that such an action does not exist in nature while it is senseless in theory. The classical terms of action (2) and (3) differ from the new introduced ones, (4) and (7), not only by quantity, but also logically, which is seen from the following text.

A. If the trajectory is given by parameterized finite equations of the form $\boldsymbol{r} = \boldsymbol{r}(t)$ by differential equations of motion $m\dot{\boldsymbol{v}} = \boldsymbol{F}$ then the curvilinear integral of work can be reduced to definite integrals

$$W(\boldsymbol{F}) = \int_S \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^t \boldsymbol{F}(\boldsymbol{r},\boldsymbol{v},t) \cdot \boldsymbol{v}dt.$$

Example 2. A material point of mass m moves along S of the spiral:

$$x = a\cos\omega t, y = b\sin\omega t, z = ct.$$

Let us find the action of the force $\boldsymbol{F}$ in the interval $[\, t_0 = 0, t_1 = \frac{2\pi}{\omega}]$. Since $\ddot{x} = -\omega^2 a\cos\omega t = -\omega x; \ddot{y} = -\omega^2 b\sin\omega t = -\omega^2 y; \ddot{z} = 0,$

we have $F_x = -m\omega^2 x, F_y = -m\omega 2y, F_z = 0.$ Thus

$$W = \int_S F dr = -m\omega^2 (a^2 \int_{\cos 0}^{\cos 2\pi} \cos\omega t d(\cos\omega t) + b\int_0^0 \sin\omega t d(\sin\omega t)) + \int_0^h F_z c dt = 0;$$

the work on the nonclosed path of spiral is zero, and then the action in this case is

$$A(\boldsymbol{F}) = 0\,.$$

B. When one considers the general case of arbitrary forces, and the corresponding trajectories, then the problem can be treated as it is shown below.

3. On the Principle of Action of Force and Counteraction

The word *principle* (lat. principium) denotes, among other things, the following: the beginning, the basic rule, the basic teaching, the foundation of knowledge; ``Principle is only a noun, usually designating a law or rule"; (Webster's Dictionary of English Usage, 1989, p. 771). In the literature on mechanics, there is no previous determination of what it is meant by the principle; however, there are widely spread titles and subtitles such as *Principia Mathematica, Galilei's Principle of Relativity, Newton's Principle of Determinacy, Variational Principles of Mechanics, Hamilton's Principle*, *Hertz' principle of least curvature* ([6], pp.228-234) etc. In numerous books on mechanics one and the same assertion is referred to as a *principle* or *law* as well as a theorem. In order to avoid different understandings of the principles of mechanics, the concept of the principle of mechanics here implies the following: *an expression significance based on the introduced concepts and definitions of mechanics whose truthfulness is not liable to verification.*

In the book (Ref. 5, p. 91) Paul Appell writes: " **Principe de Légalité de l'action et de la réaction**. - Newton a la énonc é, sous le nom de *principe* de légatité de l'action et de la réaction}, la loi suivasnte: Si un point M est sollicité par une force **F** due à la presence d'un autre point M', sette force est dirigé suivant MM' et le second point M' é prouve de lapart de M une force égale et directement opposé à} **F**."

Likewise, the much respected scientist who explored the celestial mechanics Milankovich M. ([7], pp. 44, 435) has used the phrase "the third Newton's law of action and reaction" as well as "the principle of action and reaction". In the sense of principles of mechanics and the definition (4) of action as they are understood here the term "*the principle of action and counteraction"* can achieve the meaning of the principle if the *counteraction* is taken to be (7). Now *the sentence of the principle: to action there is always opposed an equal counteraction* could be written as:

$$A(\boldsymbol{F}) = -A(\boldsymbol{I})\,, \tag{10}$$

where " – " means "opposed". Regarding the fact that the inertia force is by definition

$\boldsymbol{I} = -\frac{d\boldsymbol{v}}{dt}$ relation (10) can be written in the form

$$\int_{t_0}^{t_1} \left(\int_S \left(\boldsymbol{F} - m\frac{d\boldsymbol{v}}{dt}\right) d\boldsymbol{r}\right) dt = 0 \tag{11}$$

As it can be seen, this is followed by important well-known assertions of theoretical mechanics. For the sake of a much clearer understanding of assertion (11) and its acceptance, let's first discuss the state of motion of a material point for which inertia force **I** is equal to zero. Accordingly, relation (11) is degenerated into equation

$$\int_{t_0}^{t_1} W(\boldsymbol{F}) dt = 0\,. \tag{11a}$$

For the case that one point is acted upon only by two forces it will follow that

$$\int_{t_0}^{t_1} \left(\int_S (\boldsymbol{F}_1 + \boldsymbol{F}_2) d\boldsymbol{r}\right) dt = 0$$

As it can be seen, for this relation it is sufficient that $\boldsymbol{F}_1 = -\boldsymbol{F}_2$ or $\boldsymbol{F}_1 + \boldsymbol{F}_2 = 0$ and this, in essence, is the assertion of the third Newton's law if the *force* and the *action of force* are equal. It is agreed that in the classical statement of the Newton's third law one has an isolated system consisting of two points, that act on each other with forces F_1 and F_2 . The law says that in this case these forces are directed along the line connecting the points and satisfy $F_1 = -F_2$. Note that if the vectors of forces, originating from the two points M_1 and M_2 have equal size and direction, and have opposite orientation, then the third point exists, where the sum of forces is zero; that point C is not moving , or it has uniform and rectilinear motion.

In order to understand better our statement, let's quote the Newton's

LEX.III: *Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in sae mutuo semper asse aequales et in partes contrarias.*

Translated in English, [2a]: That reaction is always contrary and equal to action: or, that the mutual actions of two bodies upon each other are always equal and directed to contrary parts.

Thus, in this *axiom* or *law*, the used terms are *action* and *reaction*. The words *force* or *vis*, are not mentioned here, as they are mentioned in Lex I and Lex II. We are trying to make a clear difference between the terms *force* and *action of force*, as well as between the terms *principle* and *Law*. However, for the pretension to claim that relation (11) represents the principle of mechanics it should also be related to arbitrary systems of N dynamic points. Relation (11) comprises it; thus, it can be written in a more recognizable form

$$\int_{t_0}^{t} \sum_i \left[\int_S (\boldsymbol{F}_i + \boldsymbol{I}_i) \Delta \boldsymbol{r}_i\right] dt = 0 \tag{12}$$

where $F_i = \sum_k \boldsymbol{F}_{ik}$ is the resultant force of all applied forces acting upon point i-th, and Δr is a possible displacement of point i-th, [8] N conditions would follow from here in the form of equations

$$\boldsymbol{F_i} + \boldsymbol{I}_i = 0 \ \ (i = 1, ..., N) \tag{13}$$

while this system of equations in mechanics represents D'Alembert's principle. Therefore, relation (11) or equivalent relation (12) is more general than this well-known general principle of dynamic equilibrium (13) leading to the conclusion that the relation (12) is also a principle. Yet, for relation (11) to become a principle of mechanics in our previously defined sense, it is necessary to generalize upon real displacement by any other arbitrary possible displacement, for example Δr this requires the introduction of the concept of work by any possible hypothetical trajectory S*, that is

$$W^* = \int_{S^*} \boldsymbol{F} \cdot \boldsymbol{\Delta r}$$

and adequate possible action of forces upon this trajectory

$$A^* = \int_{t_0}^{t_1} \Big(\int_{S^*} \boldsymbol{F} \cdot \boldsymbol{\Delta r} \Big) dt = \int_{t_0}^{t_1} \Big(\int_{S^*} \boldsymbol{F} \cdot \boldsymbol{\Delta s} \Big) dt = \int_{t}^{t_1} \Big(\lim \sum_{k=0}^{n-1} \boldsymbol{F}(N_k) \cdot \boldsymbol{\Delta s}_k \Big) dt, \tag{14}$$

Where is $\Delta s^* = N_k N^*_{k+1}$ N and the point N $_k \in N_k N_{k+1}$; $k = 0,1,...,n-1$.

Since this way of writing the integral is not traditional, let's clarify our understanding of the term ``possible hypothetical actual trajectory'', as well as $\Delta s^* \approx \Delta r$. The functions of trajectory consist of independent variables, and also of some geometrical and kinematic constants. For **example**, the simplest function of the straight line, which is passing through one point of the plane xy is y = $\alpha x + \beta$, ($\alpha, \beta = const.$). For $\alpha = 1$ $\beta = 1$ is the observed continuous line (Fig. 1).

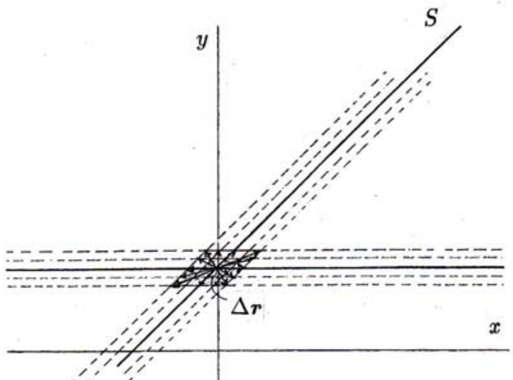

Fig. 1.

For the case $\alpha \approx 1$ and $y_0 = 1$, this S* is a set of punctual line. If $\alpha = 1$ and $y_0 \approx 1$ we obtain the bundle (sheaf) of parallel line (Fig. 1). For the more probable possibilities, that α and β are approximately close to each other in the neighborhood each point $N \in S$ there are many possible displacements Δs in the neighborhood of the observed line $y = \alpha x+y_0$. The line S symbolizes all possible lines S*. Thus the possible work W* (F) of force F on the path Δs^* differs from the standard integral

$$\int_S \boldsymbol{F} \cdot \boldsymbol{ds} = \int_S F_x dx + F_y dy + F_z dz) \, .$$

As $\boldsymbol{\Delta s}$ is a possible size, and $\boldsymbol{ds}$ is a defined differential of path in the clearly defined point of trajectory, it is possible to calculate it using the standard integral calculated. We can see that from formula (14), the work cannot be calculated, as δs is a hypothetic line which is not precisely determined. Instead, the integral form is analyzed in the same way as for the differential principle of possible displacement. Within the context of what has been said here, relation (10) would have the generalized form

$$A^*(\boldsymbol{F}) = -A^*(\boldsymbol{I}) \tag{15}$$

and relation (11) would have the form

$$\int_{t_0}^{t_1} [\int_S (\boldsymbol{F} - m\frac{d\boldsymbol{v}}{dt})\boldsymbol{\Delta r}]dt = 0 \, . \tag{15a}$$

The truthfulness and existence of this relation are confirmed by following

$$(\boldsymbol{F} - m\frac{d\boldsymbol{v}}{dt})\Delta \boldsymbol{r} = 0 \tag{15b}$$

from relations (13) as a necessary and sufficient condition. Accordingly, relation (15a) represents an operational form of the verbal expression of the principle of mechanics.

The principle of action and counteraction. *Total or possible action $A(Q)$ of generalized forces Q and generalized forces of inertia I upon a possible displacement is equal to zero as it is determined by the equation*

$$\int_{t_0}^{t_1} [W^*(Q) + W^*(I)]dt = 0 \, . \tag{16}$$

The assertion that on the basis of this principle the whole theory of dynamics can be developed is justified by the statement that it is from this principle that, as a consequence, three Newton's axioms or laws follow, just like the general differential D'Alembert-Lagrange's principle comprising all the mechanical systems. The opposite proof is also simple. If equations (13) are multiplied by scale by respective vector of possible displacement

$$\Delta \boldsymbol{r} = (\partial \boldsymbol{r} / \partial q)\Delta q$$

and if thus-obtained equations are added and the whole sum is integrated in time interval $t_0 - t_1$, relation (15) is obtained. This principle is invariant with respect to all common and usable transformations of the coordinates. With respect to the previously introduced coordinates, relation (16) can be written in the following forms: $\int_{t_0}^{t_1}[\int_{S^*}(Y+I)\Delta y]dt=\int_{t_0}^{t_1}[\int_{S^*}(X+I_x)\Delta x]dt=\int_{t_0}^{t_1}[\int_{S^*}(Q+I_q)\Delta q]dt=0,$ where I_g are generalized inertia forces.

4. THE COMPARATION WITH INTEGRAL VARIATIONAL PRINCIPLE

First of all, let's set the facts that the concepts of real displacement, possible displacement and variation differ from one another here. Real displacement is displacement of a material point in time dt along the real trajectory; possible displacement Δr is any small distance in the vicinity of the observed point that is allowed by constraints $f(r,t)=0$, ([9], p. 84) in non-singular area; the variation here assumes deviation of the observed point from the real trajectory due to a possible change of some or more parameters that final equation of motion depend on, that is,

$$\delta \boldsymbol{r} := \frac{\partial \boldsymbol{r}}{\partial \alpha}\delta\alpha = \lim_{\Delta\alpha\to 0}\frac{\boldsymbol{r}(\alpha+\Delta\alpha,t)-\boldsymbol{r}(\alpha,t)}{\Delta\alpha}\delta\alpha$$

For the sake of a more general and shorter presentation further on, let's observe the mechanical system upon configuration manifolds M^{n+1}. The elementary work $\delta W(\boldsymbol{F})$ of the force $\boldsymbol{F}$ upon variation $\delta\boldsymbol{r}$ delta is by definition

$$\delta W=\sum_{\nu=1}^{N}\boldsymbol{F}_\nu\delta\boldsymbol{r}_\nu\ .$$

Generalized variational action principle. Starting from definitions (4) and (7) and the understanding of the principle of mechanics given here, as it has previously been stated, the general variational integral principle can be formulated ([9], p.104) as: *Variations of all actions $A(F)$ of applied forces F during time $[t_0,t_1]$ is equal to variation of action $A(I)$ of inertia force I for the same amount of time, that is*

$$\delta\int_{t_0}^{t_1}W(\boldsymbol{F})dt=\delta\int_{t_0}^{t_1}W(\boldsymbol{I})dt,$$

or

$$\delta\int_{t}^{t}[W(Q)-W(I)]dt=0\ ; \tag{17}$$

and

$$\delta\int_{t_0}^{t}W(\boldsymbol{F})dt=0 \tag{18}$$

for $I=0$. Relations (17) and (18) are not obtained by formal varying of relations (4) or (11a) as can be seen by sign "-"; relation (11) represents a sum of all the actions, while relation (17) represents a difference of variational of action (4) of the imposed forces and variational of action of the inertia forces. The formula (4) represents the action of all imposed forces $\boldsymbol{F}$, while $\delta\int_{t_0}^{t_1} W(\boldsymbol{F})dt$ or $\delta\int_{t}^{t} W(\boldsymbol{F})dt$ represent a variation

$$\delta A = \int_{t_0}^{t} \delta W(\boldsymbol{F})dt = \delta\int_{t_0}^{t} \boldsymbol{F}\cdot\delta\boldsymbol{r}dt,$$

Because variation δt of independent variable t is zero. If a force $\boldsymbol{F}$ has a function of force $U(\boldsymbol{r})$, e.g. if $\boldsymbol{F} = gradU$, the following is

$$\delta\int_{t_0}^{t}\int_{r_0}^{r} gradUd\boldsymbol{r}dt = \delta\int_{t_0}^{t}\int_{U_0}^{U} dUdt = \delta\int_{t}^{t}(U-U_0)dt = \int_{t_0}\delta Udt.$$

In case of our Example 2. for motion a material point of mass m along the spiral:

$$x^2 + y^2 = R^2,\ldots, z = ct,$$

let us find the variation

$$\delta\int_{t_0} W(F)dt.$$

First approach: Since $\delta t = 0$ and $\delta W(x,y,z) = F_x\delta x + F_y\delta y + F_z\delta z$ we have

$$\delta\int_{t_0}^{t_1} W(F)dt = \int_{t_0}^{t_1}\delta W dt = \int_{t_0}^{t_1}(F_x\delta x + F_y\delta y + F_z\delta z)dt.$$

Second approach: In any point $M(x,y,z)$ of spiral, the forces are $F_x = cx, F_y = cy, F_z = 0,$ and

$$W(x,y,z) = \int_S(-cxdx - cydy) = -c\int_0^x xdx - c\int_0^y ydy = -\frac{c}{2}(x^2+y^2),$$

and then it follows

$$\delta\int_{t_0}^{t_1} W(F)dt = -\delta\int_{t_0}^{t_1}\frac{c}{2}(x^2+y^2)dt = -c\int_{t_0}^{t_1}(x\delta x + y\delta y)dt.$$

This makes the principles mutually independent. However, as it is necessary, the principles are mutually equivalent.

Hamilton--Ostrogradsky's principle. From relation (17), for I$\neq$ 0 it follows Hamilton's-Ostrogradsky principle, Ref. [2],

$$\int_{t_0}^{t_1}(\delta E_k + Q_i\delta q^i)dt = 0 \tag{19}$$

where $Q = (Q_1,\ldots, Q_n)$, $\delta\, q$:=$(\delta q^1,\ldots,q^n)^T$ and $E_k = 1/2a_{ij}\dot{q}^i\dot{q}^j$ kinetic energy of systems.

Proof: Work $W(I)$ of generalized inertia forces

$$I_i = -a_{ij}(q)\frac{D\dot{q}^j}{dt} = -a_{ij}(\frac{d\dot{q}^j}{dt} + G^j{}_{kl}\dot{q}^k\dot{q}^l)$$

is equal to negative kinetic energy E_k. Really,

$$W(I) = \int I_i dq^i = -\int a_{ij}\frac{D\dot{q}^j}{dt}dq^i = -\int a_{ij}\frac{D\dot{q}^j}{dt}\dot{q}^i dt = -\int a_{ij}\dot{q}^i D\dot{q}^j = -\frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j = -E_{k,}$$

where $\int$ is tensorial integral (See Ref. [11]) and $D\dot{q}^j = d\dot{q}^j + G^j{}_{kl}\dot{q}^k\dot{q}^l$ is natural differential of $\dot{q}^j$. Since $\delta t = 0$ it follows that $\delta\int_{t_0}^{t_1} E_k dt = \int_{t_0}^{t_1}\delta E_k dt$ for $I \neq 0$.

Hamilton's principle. In the case of the potential forces, when the work is equal to non positive potential energy E_p and $I \neq 0$, relation (17) or (19) reduces to Hamilton's principle

$$\delta\int_{t_0}^{t_1}(E_k - E_p)dt = \delta\int_{t_0}^{t_1} L dt = 0\,. \tag{20}$$

For inertial motion ($I = 0$) or for a static case the principle of action reduces to (18) or to the equivalent form

$$\int_{t_0}^{t_1} Q_i \delta q^i dt = 0, (i = 0,1,\ldots,n)$$

This is equivalent to generalized conditions of equilibrium

$$Q_i = 0.$$

For a system with scleronomic constraints all relations above have the same form except that the indexes i,j go over $1,2,\ldots,n$ (instead $0,1,\ldots,n$) when n is the number of degrees of freedom of the scleronomic system.

CONCLUSION

In this paper we introduce into Analytical Mechanics the term Counteraction as integral $\int_{t_0}^{t_1} W(I)dt$, where $W(I)$ is work of inertia force I. The new Principle of action of force and counteraction is formulated then by the relation

$$\int_{t_0}^{t}[W^*(Q) + W^*(I)]dt = 0\,,$$

where $W^*(Q)$ is the work of generalized force Q on possible trajectory S^*. In order to prove the independence of this principle from other integral principles, we compare it with the realized Variational principle of action, Ref. [12, 13, 14],

$$\delta\int_{t_0}^{t}[W(Q) - W(I)]dt = 0\,.$$

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PRINCIP DEJSTVA SILA I PROTIVDEJSTVA

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Više znamenitih autora dela iz klasične analitičke mehanike nazivaju treću Njutnovu aksiomu Princip dejstva i protivdejstva. Ovde se pokazuje da se ta aksioma ne može smatrati principom mehanike. Reafirmiše se Ojlerov pojam dejstvo sile i uvodi pojam Protivdejstvo, kao dejstvo sile inercije u toku vremena kretanja. Fizička dimenzija dejstva sile jednaka je, kao i u klasičnom shvatanju, proizvodu dimenzija rada i vremena. Medjutim, pri ravnomernom kretanju protivdejsvo jednako je nuli, što se znatno razlikuje od odgovarajućeg standardnog dejstva, koje je proporcionalno intervalu vremena kretanja. Princip dejstva sile i protivdejstva definisan je matematickom relacijom kao iskazom: Dejstvo sila je jednako i suprotno usmereno protivdejstvu. Ovaj princip ne može se identifikovati sa trećom Njutnovom aksiomom. Na osnovu ovog iskaza može se razviti skladno cela teorija o kretanju tela, što ga čini principom mehanike. U cilju boljeg poimanja stvari ovaj princip je uporedjen ovde sa integralnim varijacionim principima mehanike.

Ključne reči: *Analitička mehanika, principi, dejstvo, protivdejstvo.*