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AN ITERATION PROCEDURE FOR SOME NON-LINEAR OSCILLATIONS

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Vasile Marinca, Nicolae Herişanu

"Politehnica" University of Timişoara, Romania, Department of Mechanics and Vibration

Abstract. *The purpose of this paper is to give a simple but effective iteration procedure to search for the solution of a non-linear differential equation. This procedure is a powerful tool for determination of periodic solutions of an equation of motion. A correction functional is constructed by a general Lagrange multiplier which can be identified optimally via the variational theory. This analytical research is verified with numerical examples and a very good agreement is found, which shows the applicability of the method.*

Key words: *non-linear oscillation, iteration procedure*

1. INTRODUCTION

The perturbation methods are a kind of powerful tools for treating weakly nonlinear problems, but they are less effective for analysing strongly nonlinear problems [1]. They provide the most versatile tools available in nonlinear analysis of engineering problems, and they are constantly being developed and applied to even more complex problems. But, like other nonlinear asymptotic techniques, perturbation methods have their own limitations: almost all perturbation methods are based on such an assumption that a small parameter must exist in an equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques, as it is well known, an overwhelming majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all.

There exist some alternative analytical asymptotic approaches, such as the nonperturbation method, weighted linearization method, homotopy analysis method, Adomian decomposition method, modified Lindsted-Poincare method, interpolation perturbation methods, and parameterized perturbation method [5]. There also exists a wide body of literature dealing with the problem of approximate solutions to nonlinear equations with various different methodologies. It is very difficult to solve non-linear problems either numerically or theoretically. This is possible due to the fact that various discredited meth-

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ods or numerical simulations apply iteration techniques to find their numerical solution of non-linear problems and nearly all iterative methods are sensitive to initial solutions.

In this paper, a new kind of analytical technique for a general non-linear problem is presented. The problem is initially approximated with unknown constants, which can be further determined. The iterative process is constructed by a general Lagrange multiplier, which can be identified optimally via variational theory. This method is effective and accurate for non-linear problems with approximations converging rapidly to accurate solution by imposing conditions.

2. THE ITERATION PROCEDURE [10]

We will consider the following nonlinear equation

$$\ddot{x} + \omega^2 x = \varepsilon f(\Omega t, x, \dot{x}, \ddot{x}) \tag{1}$$

where ω and Ω are positive constants; in general f is assumed to be non-linear function of both x and $\dot{x}$, periodic of $\Omega\, t$ which may be expanded in a Fourier series; the parameter ε does not need to be small and $\dot{x} = \dfrac{dx}{dt}$.

We construct the following iteration formula:

$$x_n(t) = x_{n-1}(t) + \int_0^t \lambda(\tau,t)[x''_{n-1}(\tau) + \omega^2 x_{n-1}(\tau) - \varepsilon f(\Omega\tau, \tilde{x}_{n-1}(\tau), \tilde{x}'_{n-1}(\tau), \tilde{x}''_{n-1}(\tau))]\, d\tau \tag{2}$$

where $\lambda(\tau,t)$ is called a general Lagrange multiplier, which can be identified optimally via variational theory; $\tilde{x}_{n-1}$ is considered as a restricted variation, i.e. $\delta\tilde{x}_{n-1} = \delta\tilde{x}'_{n-1} = 0$ and $' = \dfrac{d}{d\tau}$.

The Lagrange multipliers can be readily identified:

$$\lambda(\tau,t) = \frac{1}{\omega}\sin\omega(\tau - t) \tag{3}$$

and as a result, we obtain the following iteration formula (named correction functional):

$$\begin{aligned} x_n(t) &= x_{n-1}(0)\cos\omega t + \frac{1}{\omega}\dot{x}_{n-1}(0)\sin\omega t + \\ &+ \frac{\varepsilon}{\omega}\int_0^t \sin\omega(t-\tau) f(\Omega\tau, x_{n-1}(\tau), x'_{n-1}(\tau), x''_{n-1}(\tau)) d\tau \end{aligned} \tag{4}$$

Assume that $\omega \approx \Omega$ and let us denote (σ is detuning parameter):

$$\Omega^2 - \omega^2 = \varepsilon\sigma \ , \ (0 < \varepsilon << 1) \tag{5}$$

The equation (1) may then be written as:

$$\ddot{x} + \Omega^2 x = \varepsilon[\sigma x + f(\Omega t, x, \dot{x}, \ddot{x})] \tag{6}$$

For ε=0, Eq.(6) has the solution $x = A\cos(\Omega t + \varphi)$ where A and φ are constants. For $\varepsilon \neq 0$ we try the input of starting function as:

$$x_0 = A_0 \cos(\Omega t + \varphi_0) \tag{7}$$

According to Ref.[9] we propose the following iteration formula for Eq.(6):

$$\begin{aligned} x_n(t) = A_n \cos(\Omega t + \varphi_n) + \frac{\varepsilon}{\Omega}\int_0^t \sin\Omega(t-\tau)[\sigma x_{n-1}(\tau) + \\ + f(\Omega\tau, x_{n-1}(\tau), x'_{n-1}(\tau), x''_{n-1}(\tau))]d\tau, n = 1,2,... \end{aligned} \tag{8}$$

where A_n and φ_n are constants and x_0 is given by Eq.(7).

Expanding $\sigma x_{n-1} + f$ in a Fourier series, we have:

$$\begin{aligned} \sigma x_{n-1}(\tau) + f(\Omega t, x_{n-1}(\tau), x'_{n-1}(\tau), x''_{n-1}(\tau)) = \\ = \sum_{p=0}^{P} a_p^{n-1}(A_{n-1}, \varphi_{n-1}, \Omega, \sigma)\cos p\Omega\tau + \sum_{r=0}^{R} b_r^{n-1}(A_{n-1}, \varphi_{n-1}, \Omega, \sigma)\sin r\Omega\tau \end{aligned} \tag{9}$$

and therefore, the approximation of n-th order (8) becomes:

$$\begin{aligned} x_n(t) = A_n \cos(\Omega t + \varphi_n) + \frac{\varepsilon}{\Omega}\Bigg[a_0^{n-1} + \frac{1}{2}a_1^{n-1}(t\sin\Omega t) + \frac{1}{2}b_1^{n-1}\left(t\cos\Omega t + \frac{1}{2\Omega}\sin\Omega t\right) + \\ + \sum_{p=2}^{P}\frac{a_p^{n-1}(\cos\Omega t - \cos p\Omega t)}{(p^2-1)\Omega} + \sum_{r=2}^{R}\frac{b_r^{n-1}(r\sin\Omega t - \sin r\Omega t)}{(r^2-1)\Omega}\Bigg] \end{aligned} \tag{10}$$

The solution (10) is chosen so that it contains no secular terms, which require that coefficients a_1^{n-1} and b_1^{n-1} into (10) disappear, i.e.:

$$a_1^{n-1}(A_{n-1}, \varphi_{n-1}, \Omega, \sigma) = 0\,,\; b_1^{n-1}(A_{n-1}, \varphi_{n-1}, \Omega, \sigma) = 0 \tag{11}$$

For real systems, the expansion of the function $\sigma x_{n-1} + f$ usually contains only a small number of harmonics.

3. THE EXAMPLE

We consider the equation [2]

$$\ddot{x} + \frac{x}{1+\varepsilon x^2} = 0 \tag{12}$$

with the initial conditions:

$$x(0) = A, \dot{x}(0) = 0 \tag{13}$$

We rewrite (12) in the form:

$$\ddot{x} + x = -\varepsilon x^2 \ddot{x} \tag{14}$$

Assume that $\Omega \approx 1 (= \omega^2)$ and let us denote

$$\Omega^2 - 1 = \varepsilon\sigma \tag{15}$$

The Eq.(14) may be written as:

$$\ddot{x} + \Omega^2 x = \varepsilon F(x, \ddot{x}) \tag{16}$$

where

$$F(x, \ddot{x}) = \sigma x - x^2 \ddot{x} \tag{17}$$

For ε=0 and in the initial conditions (13), we propose the input function as

$$x_0(t) = A\cos\Omega t \tag{18}$$

Expanding $F(x_0, \ddot{x}_0)$ in a Fourier series, we have:

$$F(x_0, \ddot{x}_0) = \left(\sigma A + \frac{3}{4}\Omega^2 A^3\right)\cos\Omega t + \frac{1}{4}\Omega^2 A^3 \cos 3\Omega t \tag{19}$$

In order to ensure that no secular terms appear in the next iteration, resonance must be avoided. To do so, the coefficient a_0^1 of the $\cos\Omega t$ in Eq.(19) requires to be zero i.e.:

$$\sigma = -\frac{3}{4}\Omega^2 A^2 \tag{20}$$

From Eq.(15), we obtain

$$\Omega^2 = \frac{1}{1 + \frac{3}{4}\varepsilon A^2} \tag{21}$$

and from Eq.(10) we have the following first-order approximate solution:

$$x_1(t) = \left(A + \frac{\varepsilon A^3}{32}\right)\cos\Omega t - \frac{\varepsilon A^3}{32}\cos 3\Omega t \tag{22}$$

The formula (21) works well for small ε but breaks down quickly when ε becomes large. With the aid of Eq.(21) we define a new expansion parameter $\eta = \eta(\varepsilon, A)$ as follows:

$$\eta = \frac{\frac{3}{4}\varepsilon A^2}{1 + \frac{3}{4}\varepsilon A^2} \tag{23}$$

The relation (23) is quickly convergent regardless of the magnitude of εA^2, since $\eta<1$ for all εA^2. In terms of η the original parameter ε is given by

$$\varepsilon = \frac{\eta}{\frac{3}{4}A^2(1-\eta)} \tag{24}$$

and Eq.(14) can be rewritten as:

$$\ddot{x} + x = \eta\left(\ddot{x} + x - \frac{4x^2\ddot{x}}{3A^2}\right) \tag{25}$$

Assume that $\Omega \approx 1$ and denoting

$$\Omega^2 - 1 = \eta\sigma \tag{26}$$

therefore Eq.(25) becomes:

$$\ddot{x} + \Omega^2 x = \eta G(x, \ddot{x}) \tag{27}$$

where

$$G(x, \ddot{x}) = \sigma x + \ddot{x} + x - \frac{4x^2\ddot{x}}{3A^2} \tag{28}$$

For Eq.(27) and η=0, we propose the input function as

$$x_0(t) = A\cos\Omega t \tag{29}$$

such as Eq.(28) becomes:

$$G(x_0, \ddot{x}_0) = (\sigma+1)A\cos\Omega t + \frac{1}{3}A\Omega^2\cos 3\Omega t \tag{30}$$

Avoiding the secular term needs that

$$\sigma_1 = -1 \tag{31}$$

For n=1 into Eq.(10) we obtain the first-order approximate solution:

$$x_1(t) = A\left(1+\frac{\eta}{24}\right)\cos\Omega t - \frac{1}{24}\eta A\cos 3\Omega t \tag{32}$$

and therefore

$$G(x_1, \ddot{x}_1) = \left(1+\frac{\eta}{24}\right)\left[\sigma A - A\Omega^2 + A + \frac{4}{3}\Omega^2 A\left(\frac{1}{2}+\frac{1}{24}\eta+\frac{1}{576}\eta^2\right) + \right.$$
$$\left.\frac{1}{3}\Omega^2 A\left(1-\frac{1}{576}\eta^2\right) - \frac{1}{8}\Omega^2 A\left(1-\frac{1}{24}\eta\right) + \frac{1}{96}\eta^2\Omega^2 A\right]\cos\Omega t +$$

$$
\begin{aligned}
&+\left[-\frac{1}{24}\sigma\eta A+\frac{3}{8}\eta\Omega^2 A-\frac{1}{24}\eta A-\frac{1}{2}\eta\Omega^2 A\left(\frac{1}{2}+\frac{\eta}{24}+\frac{\eta^2}{576}\right)+\right.\\
&\left.\frac{1}{3}\Omega^2 A\left(1+\frac{\eta}{24}\right)\left(1-\frac{\eta^2}{576}\right)-\frac{1}{36}\eta\Omega^2 A\left(1+\frac{\eta}{24}\right)^2-\frac{\eta^3\Omega^2 A}{4608}\right]\cos 3\Omega t+\\
&+\left[\frac{\eta^2\Omega^2 A}{1728}\left(1+\frac{1}{24}\eta\right)-\frac{1}{8}\eta\Omega^2 A-\frac{1}{36}\eta\Omega^2 A\left(1+\frac{1}{24}\eta\right)^2\right]\cos 5\Omega t+\\
&+\left(1+\frac{\eta}{24}\right)\frac{19}{1728}\eta^2\Omega^2 A\cos 7\Omega t-\frac{1}{4608}\eta^3\Omega^2 A\cos 9\Omega t
\end{aligned}
\tag{33}
$$

Avoiding the presence of a secular term needs

$$
\sigma_2=\Omega^2\left(\frac{5}{72}\eta-\frac{5}{288}\eta^2\right)-1 \tag{34}
$$

Substituting Eq.(34) into Eq.(33), we have:

$$
\begin{aligned}
G(x_1,\ddot{x}_1)&=\Omega^2 A\left(\frac{1}{3}+\frac{1}{9}\eta-\frac{23}{864}\eta^2-\frac{1}{2304}\eta^3\right)\cos 3\Omega t+\\
&+\Omega^2 A\left(-\frac{11}{72}\eta-\frac{1}{576}\eta^2-\frac{1}{41472}\eta^3\right)\cos 5\Omega t+\\
&+\Omega^2 A\left(\frac{19}{1728}\eta^2+\frac{19}{41472}\eta^3\right)\cos 7\Omega t-\frac{1}{4608}\eta^3\Omega^2 A\cos 9\Omega t
\end{aligned}
\tag{35}
$$

where Ω^2 is obtained from Eqs.(34) and (26):

$$
\Omega^2=\frac{1-\eta}{1-\frac{5}{72}\eta^2+\frac{5}{288}\eta^3} \tag{36}
$$

Substituting Eq.(23) into (30) we have:

$$
\Omega^2=\frac{1152\varepsilon^2 A^4+3072\varepsilon A^2+2048}{819\varepsilon^3 A^6+3376\varepsilon^2 A^4+4608\varepsilon A^2+2048} \tag{37}
$$

For n=2 into (10) we obtain the second-order approximate solution:

$$
\begin{aligned}
x_2(t)&=A\left(1+\frac{1}{24}\eta+\frac{13}{1728}\eta^2-\frac{263}{82944}\eta^3+\frac{17}{1990656}\eta^4\right)\cos\Omega t+\\
&+A\left(-\frac{1}{24}\eta-\frac{1}{72}\eta^2+\frac{23}{6912}\eta^3+\frac{1}{18432}\eta^4\right)\cos 3\Omega t+\\
&+A\left(\frac{11}{1728}\eta^2+\frac{1}{13824}\eta^3+\frac{1}{995328}\eta^4\right)\cos 5\Omega t+
\end{aligned}
\tag{38}
$$

$$+A\left(-\frac{19}{82944}\eta^3-\frac{19}{1990656}\eta^4\right)\cos 7\Omega t+\frac{1}{368640}\eta^4 A\cos 9\Omega t$$

To illustrate the remarkable accuracy of this method, we compare the approximate period given by means of Eq.(37):

$$T_{approx}=2\pi\sqrt{\frac{1152\varepsilon^2A^4+3072\varepsilon A^2+2048}{819\varepsilon^3A^6+3376\varepsilon^2A^4+4608\varepsilon A^2+2048}} \quad (39)$$

with the exact one [2]:

$$T=4\sqrt{\varepsilon}\int_0^A\frac{du}{\sqrt{\ln(1+\varepsilon A^2)-\ln(1+\varepsilon u^2)}}=4\sqrt{\varepsilon}\int_0^A\frac{du}{\sqrt{\ln[(1+\varepsilon A^2)/(1+\varepsilon u^2)]}} \quad (40)$$

In case $\varepsilon A^2\to\infty$, Eq.(40) reduces to

$$\lim_{\varepsilon A^2\to\infty}T=4\sqrt{\varepsilon}\int_0^A\frac{du}{\sqrt{2(\ln A-\ln u)}}=$$
$$=2\sqrt{2\varepsilon}A\int_0^1\frac{du}{\sqrt{\ln\frac{1}{u}}}=4\sqrt{2\varepsilon}A\int_0^\infty\exp(-s^2)ds=2\sqrt{2\pi\varepsilon}A \quad (41)$$

It is obvious that the approximate period (39) has the same feature as the exact one for ε>>1. In case of $\varepsilon\to\infty$, we have

$$\lim_{\varepsilon A^2\to\infty}\frac{T}{T_{aprox}}=\frac{2\sqrt{2\pi\varepsilon}A}{2\pi\sqrt{\frac{91}{128}}\varepsilon A}=0{,}94629 \quad (42)$$

Therefore, for any values of ε, it can be easily proved that the maximal relative error is less than 5,4% on the whole solution domain ($0<\varepsilon<\infty$).

This rather extraordinary virtue of the technique has first been exploited in this paper.

4. Conclusions

The proposed method is effective and has some distinct advantages over usual approximation methods in that the approximate solutions obtained in the present paper are valid not only for weakly nonlinear oscillations, but also for strongly nonlinear ones.

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PROCEDURA ITERACIJE ZA NEKE NELINEARNE OSCILACIJE

Vasile Marinca, Nicolae Herişanu

Cilj ovog rada je da da jednostavnu, ali efektivnu proceduru iteracije da bi se tražilo rešenje nelinearnih diferencijalnih jednačina. Ovaj postupak je moćno sredstvo za odredjivanje periodičnih rešenja jednačina kretanja. Funkcija korekcije je konstruisana opštom Lagranžovom metodom množilaca, kojom se može identifikovati kao optimalna preko teorije varijacija. Ovo analitičko istraživanje je dokazano mnogim primerima i postignuta je veoma dobra saglasnost što pokazuje primenljivost ove metode.

Ključne reči: *nelinearne oscilacije, procedura iteracije*