THE CONSTRUCTION OF THE LAGRANGE MECHANICS OF THE DISCRETE HEREDITARY SYSTEMS

UDC 531.3:531.3.011:532.511(045)=111

Oleg Aleksandrovich Gorosko¹, Katica (Stevanović) Hedrih²

 ¹Ukrainian Higher Education Academy of Sciences
 Kiev National University named Taras Shevchenko, KIEV (Kiïv) 2166
 Vul. Lísoviy prospekt No 22. kv. 32. (Lesnoy prospekt No 22. kv. 32.) Tel: (044) 5181508 e-mail: p_natalia@univ.kiev.ua
 ²Faculty of Mechanical Engineering University of Niš Mathematical Institute SANU Belgrade
 Yu-18 000- Niš, ul. Vojvode Tankosica 3/22, Telefax: 381 18 241 663, Mobile 063 8 75 75 99
 e-mail: katica@masfak.ni.ac.yu * khedrih@eunet.yu

Abstract. The research results in the area of mechanics of hereditary discrete systems, obtained by the authors of this paper, are generalized and presented in the monograph [4] which contains the first completed presentation of the analytical dynamics of hereditary discrete systems. Two classes of dynamically defined and undefined hereditary systems are defined and considered by introducing corresponding restrictions. The main results of mechanics of hereditary discrete systems are presented with new applications important to engineering.

The approximation of expressions for the coefficients of damping and corresponding decrements as well as for the circular frequency of oscillations of hereditary oscillatory systems are obtained with high accuracy in the first and second approximations.

The analogy between hereditary interactions and reactive forces in the systems of automatic control is identified and a possibility to extend the theory of analytical dynamics of hereditary systems to mechanical systems with automatic control is pointed out.

The Lagrange's mechanics of hereditary systems is extended and generalized to thermo-rheological and piezo-rheological discrete mechanical systems as well as to discrete mechanical systems with standard light creep elements.

Key words: hereditary system, rheological element, rheological and relaxational kernels, standard hereditary element, integro-differential equation, fractional order derivative, material particles, rheonomic coordinate, rheological pendulum, rheological coordinate, covariant coordinate, thermo-rheological and piezo-rheological hereditary elements

Received Ocober 20, 2006

1. INTRODUCTOIN

Analytical dynamics as general science of mechanical system motions was founded by Lagrange (*Joseph Luis Lagrange* (1735-1813)) in the period of his work at the Berlin Academy. The Lagrange's book "*Mècanique Analytique*" [14] contains basic analytical methods of mechanics and was published in France in 1788. The introduced analytical methods in Mechanics by Lagrange are main and the first base of analytical mechanics in general. Lagrange's equations of the second kind and Lagrange's equations of the first kind with unwoven Lagrange's multipliplicators of constraints are the main fundament of Analytical Dynamics.

After first period of Analytical Mechanics foundation, canonical equations obtained by Hamilton (Wiliam Rowan Hamilton (1805 - 1865)) were the main advanced results of fundamental progress of analytical methods in mechanics.

In Lagrange's opinion his equations of the first and second kind are universal and applicable to all mechanical systems. In 1985 С.А.Чаплыгин's (S.A. Chapligin's) analysis of a paper written by Э. Линделефа (E. Lindelef) follows to the conclusions that Lagrange's equations are not applicable to nonholonomic mechanical systems [15]. Then, С.А.Чаплыгин (S.A. Chapligin) proposed the beginning of research in a new area of Analytical mechanics under the name Nonholonomic Mechanics – Mechanics of nonholonomic mechanical systems.

Lagrange's equations of first kind with unwoven Lahrange's multiplicators of constraints are really general and universal. At the beginning of the XX century nonholonomic mechanics was founded as a separate science discipline. Equations of C.A. Чаплыгин (S.A. Chapligin), B. Вольтерра (V. Voltera), П.B. Воронца (P.V. Voronc), Г. Маджи (G.Madzi), П. Аппеля (P.Apela) and others are considered to be great contributions in the area of Analytical dynamics of nonholonomic systems. Separate parts of mechanics of nonholonomic systems are applicable to the control of systems and to systems containing deformable bodies.

Analytical dynamics is largely applied and used in engineering system dynamics and in natural sciences as well as for investigation of mechanical system dynamics and in the physics of the microworld.

The appearance and distribution of new materials for construction on the basis of synthetics with clear rheological properties are source and inspiration for development of a new area of mechanics – *Mechanics of hereditary systems*. The mechanics of hereditary continuum is presented by the series of fundamental publications and monographs [1] and is applied for estimations of the construction built by new material.

The new material with high rigidity parallel possesses series of unequal properties like dielectric and radio properties as well as the properties and possibilities of high deformability and "*lightibility*". These properties of new materials give them advantages for application in engineering systems over classical metals and materials. Thanks to advanced knowledge in the area of chemistry and technology of materials, new materials with new properties are produced. In accordance with previous Mechanics of deformable rheological (hereditary materials), the intensive development continues.

In current literature term "*hereditary*" and "*rheological*" systems are equivalent. In opinion of Работнов Ю.Н.(Rabotnov Yu..H.) [1], the name "*hereditary*" system or continuum proposed by В.Вольтерра (V. Voltera), is more precise as well as suitable. By using this name, the property of rheological systems "*to remember*" the history of loading

is fully described. By series of the fundamental papers and monographs, Mechanics of hereditary continuum is presented. Also, in numerous references, many examples with applications in engineering, biological and other areas [2] are published. The pioneer research results in the area of mechanics of discrete hereditary systems are presented in a publication written by a talented scientist A.P.Pжаницин (A.R.Rzanicin) [2]. Also, numerous applications in this area grow.

The mechanics of discrete hereditary systems up to a few years before was presented only by separate single papers [3] containing only solutions of partial problems.

The research results in the area of mechanics of hereditary discrete systems, obtained by authors of this paper, are generalized and presented in the monograph [4], published in 2001, which contains the first presentation of analytical dynamics of hereditary discrete systems. We can conclude that this monograph contains complete foundation of analytical dynamics theory of discrete hereditary systems and by using these results, numerous examples are obtained and solved (see Refs. [5-12]). In this analytical mechanics of hereditary discrete systems, modified Lagrange's differential equations of the second kind in the form differential and integro-differential forms with the kernels of relaxation or rhelogogy are derived.

This paper based on research results from monograph [4] and new authors' advanced research is published [5-13 and 17] with unpublished results.

As we say, in the name "*hereditary*" properties of rheological body "*to remember*" history of loading is fully described for both cases: for the first case of the short time loading period when rheological body possesses property to obtain quickly the previous unloaded body form after unloading and the second case of the long time loading when rheological body possesses property to obtain the previous unloaded body form after unloading for long period when the material property "*to remember*" is present as history of loading, and name "hereditary elasticuisty" of the corresponding name.

The hereditary properties are present as property in every solid body. For example, the stainless steel spring excited by forces during numerous years obtained a deformed state, and after the break of the excitation needs some time to obtain previous undeformed state. In this example it is visible that a long time period is necessary for identification of hereditary properties of rheological material.

For numerous visco-plastic synthetic materials the necessary time for identification of hereditary properties is expressed by minutes or by seconds. Then hereditary theory is suitable to describe internal tribological in material for small deformations.

2.1. TERMINOLOGY AND DEFINITIONS OF DYNAMICS OF HEREDITARY SYSTEMS

The hereditary discrete mechanical system can be a mechanical system containing one or more hereditary interconnections or inter-influence between bodies and material particles. In a similar way, the hereditary connection is possible to be realized by rheological element like a synthetic string or by rubber band.

In this paper term "*hereditary*", "*rheological*" and "*viscoelastic*" will be considered synonimous.

For the presentation of the hereditary body we used symbols and graphics usually used in the world literature in the area of Rheology. In this paper we used two kinds (groups) of rheological bodies – viskoelastic with properties to obtain previous

undeformed forms after the ceasing of external excitations and elastoviscose bodies with property for non bounded deformation under external excitation during the time. The accepted names and terms of rheological bodies in the world literature are different and numerous.and not uniform. In our paper we use terminology accepted in publications written by Γ .H. Савин. and Я.Я. Рущицкий [2] (G.N. Savin and Ya..Ya. Ruschickij). For the dynamical equivalent bodies of the types Kelvin and Thompson-Poyting, a therm "standard .visco-elastic body" is used.

For the schematic presentation of hereditary multibody discrete systems the general presentations of rheological elements are used. It is necessary to point out that rheological elements in analytical dynamics, like a pure elastic element introduced in discrete mechanical system, do not appear as constraints, and the numbers of degrees of the system freedom are not smaller.

2.2. THE MODELS OF HEREDITARY ELEMENTS IN ANALYTICAL DYNAMICS OF HEREDITARY DISCRETE DYSTEMS

The hereditary system is every system which contains mutual hereditary interaction between material particles in the form of one or more constraints with hereditary properties.

The simple viscoelastic element is Foight's type element. In the state of extension the resultant force appears by two components, one by visco and one by elastic properties in the deformation of viscoeelastic element and constitutive stress-strain relation given as relation between force and the extension of element in the following form:

$$P(t) = cy(t) + \mu \dot{y}(t) \tag{1}$$

In Mechanics of hereditary continuum in the case of axial (in one direction) stressed and deformed Foight's type body stress strain constitutive relation is expressed by following relation:

$$\sigma(t) = E\varepsilon(t) + \mu\dot{\varepsilon}(t) \tag{a}$$

The model of a Foight's type body approximately describes viscoelastic properties of real bodies. And due to its simple form and description large applications to the engineering systems are obtained. Basic disadvantage of this Foight's type body model is strong dependence of damping coefficients with the change of frequency and the dependence of frequency on the energy degradation that aren't compatible with experimental data.

More acceptable and precise and better compatible with experimental data with real hereditary body properties is the model of the standard viscoelastic body (Kelvin and Poyting-Thompson's body). The constitutive stress strain relation given as the relation between the force and extension of element in the following form:

$$n\dot{P}(t) + P(t) = nc\dot{y}(t) + \tilde{c}y(t)$$
(2)

In the mechanics of hereditary system constants n, c and \tilde{c} obtain special names: the time of relaxation, rigidity coefficients, one momenteneous and the prolonged one.

For generalized hereditary element model relation between force and deformation it is possible to describe by differential equation high order derivative in the following form:

The Construction of the Lagrange Mechanics of the Discrete Hereditary Systems

$$\sum_{r=1}^{m} a_k \frac{d^k P}{dt_k} + P(t) = b_0 x + \sum_{k=1}^{m} b_k \frac{d^k x}{dt^k}$$
(3)

For example of generalized hereditary body we can analyze a typical industrial rubber amortizer which is applied for vibroisolation of heavy machines. The relation between the deformation and external excitation for this amortizer obtained by Goroshko's experiment has the following form:

$$n_{1}n_{2}n_{3}\ddot{P}(t) + (n_{1}n_{2} + n_{2}n_{3} + n_{1}n_{3})\ddot{P}(t) + (n_{1} + n_{2} + n_{3})\dot{P}(t) + P(t) =$$

$$= n_{1}n_{2}n_{3}c\ddot{y}(t) + (n_{1}n_{2} + n_{2}n_{3} + n_{1}n_{3})c_{2}\ddot{y}(t) + (n_{1} + n_{2} + n_{3})c_{1}\dot{y}(t) + \tilde{c}y(t)$$
(4)

where times of relaxations are $n_1 = 36[\text{sec}]$, $n_2 = 220[\text{sec}]$, $n_3 = 2218[\text{sec}]$ and the rigidity coefficients of momentary and the prolonged one are: $c = 2,57 \cdot 10^5 [N/m]$, $\tilde{c} = 1,91 \cdot 10^5 [N/m]$ and coefficients of the rigidity of particular components are: $c_1 = 0,865c$, $c_2 = 0,926c$.

The stress-strain state of viscous element (Maxwell's body) is described by the following constitutive relation:

$$n\dot{P}(t) + P(t) = nc\dot{y}(t) \tag{5}$$

where *c* is the coefficient of momentary rigidity. For the case of long time loading of the element by constant intensity load $P(t) = P_0$ the deformation of the Maxwell's element is:

$$P_0(t) = nc\dot{y}(t)$$
 or $y(t) = \frac{P_0}{nc}t$.

It is visible that deformation increases unbounded.

For more complex viscose elements (represented by the Jeffreys' bidy (J-body) and Lethersich's body) stress-strain state is described by differential equation in the form:

$$nP(t) + P(t) = b_1 \dot{y}(t) + nb_2 \ddot{y}(t)$$
(6)

For the generalized viscous element the constitutive relation of the stress-strain state is described by the following differential equation:

$$a_{k}\frac{d^{k}P}{dt^{k}} + a_{k-1}\frac{d^{k-1}P}{dt^{k-1}} + \dots + a_{l}\dot{P} + P = b_{l}\dot{y} + \dots + b_{m-1}\frac{d^{m-1}y}{dt^{m-1}} + b_{m}\frac{d^{m}y}{dt^{m}}$$
(7)

For all viscous bodies as well as viscous elements the term $b_0 y(t)$ is not included in the equation because material does not have elastic properties, only viscous properties.

It is necessary to point out that the properties of Maxwell's elements in discrete hereditary systems are the source of cyclic coordinates appearing.

The equivalency and analogy of hereditary interactions and reactive forces in the systems of automatic control give the possibility to extend the theory of analytical dynamics of hereditary systems to mechanical systems with automatic control. For example, automaton with transfer function presented in the following form

$$W(p) = \frac{b_0 + b_1 p + \dots + b_n p^n}{1 + a_1 p + \dots + a_n p^n}$$
(8)

presents a hereditary interaction (3) between material particles of the discrete mechanical system with one degree of freedom.

The parameters of the automaton of arbitrary structures are defined in an experimental way and it is possible to obtain amplitude-phase characteristic. In our opinion there are real possibilities and the perspective to use method of amplitude-phase characteristic for the experimental obtaining of mechanical characteristic of the hereditary discrete mechanical systems. It is possible to solve some difficulties with identification coefficient of the momenteneous rigidity which appears in the mechanical investigation of the hereditary forms and shortened longtime experiments.

2.3. THE INTEGRAL MODELS OF THE STRESS-STRAIN STATE OF THE HEREDITARY ELEMENTS

There are three mathematical forms for the description of the constitutive relations of the hereditary properties of hereditary interaction [2], in the building of hereditary system's mechanics. These forms are:

1* Differential equation, expressed in the form of dependence reaction force P of the rheological coordinate x, usually presented as deformation or relative displacement of the hereditary constraint in the form (3).

 2^* Integral equation, expressed in the form of dependence reaction force *P* of the rheological coordinate *y*, usually presented as deformation or relative displacement of the hereditary constraint:

$$P(t) = c \left[y(t) - \int_{0}^{t} \Re(t - \tau) y(\tau) d\tau \right]$$
⁽⁹⁾

where $\Re(t-\tau) = \frac{c-\tilde{c}}{nc}e^{-\frac{1}{n}(t-\tau)}$ is relaxation kernel,

and $\beta = \frac{1}{n}$ is coefficient of the element relaxation.

This integral relation (9) can be obtained by solving equation (2) with respect to the force P. By this integral equation, the relaxation of the reaction force P depending on the rheological coordinate y, is presented and expressed.

For the case of the generalized standard hereditary element (3) integral equation is possible to obtain in the form (9) in which relaxation kernel $\Re(t - \tau)$ presents sum by sum of exponents.

 3^* Integral equation, expressed in the form of dependence rheological coordinate *y*, usually presented deformation or relative displacement of the hereditary constraint and reaction force *P*:

$$y(t) = \frac{1}{c} \left[P(t) + \int_{0}^{t} \aleph(t - \tau) P(\tau) d\tau \right]$$
(9a)

where $\aleph(t-\tau) = \frac{c-\tilde{c}}{nc}e^{-\frac{\tilde{c}}{nc}(t-\tau)}$ is kernel of rheology and

 $\beta_1 = \frac{\tilde{c}}{nc}$ is the coefficient of the creep or retardation or rheology.

6

This integral relation (9a) can be obtained by solving equation (2) with respect to the rheological coordinate y. By this integral equation, the relaxation of the reaction force P depending of the rheological coordinate y, is presented and expressed.

By this integral equation, the retardation of the rheological coordinate y of the reaction force P is presented and expressed.

In the previous integral equations $\Re(t-\tau)$ and $\aleph(t-\tau)$ - are relaxational and rheological kernel. The history of stress strain state up to beginning of the motion of the system t = 0 is described by the following integrals:

$$\int_{-\infty}^{t} \Re(t-\tau) y(\tau) d\tau \text{ and } \int_{-\infty}^{t} \aleph(t-\tau) P(\tau) d\tau$$

and previous integral equations (9) and (9a) take the following forms: $\begin{bmatrix} t \\ t \end{bmatrix}$

$$P(t) = c \left[y(t) - \int_{-\infty}^{t} \Re(t - \tau) y(\tau) d\tau \right]$$
$$y(t) = \frac{1}{c} \left[P(t) + \int_{-\infty}^{t} \aleph(t - \tau) P(\tau) d\tau \right]$$
(10)

and

where by integral operators, the histories of the previous interactions of the hereditary constraints are expressed. The model of the standard hereditary element is one of the used models, but it is possible to use the model of a weak singular element in which the models of the kernel of relaxation as well as of the rheology are in the form:

$$\Re(t-\tau) = \frac{ae^{-\beta(t-\tau)}}{(t-\tau)^{\alpha}}, \text{ and } \Re(t-\tau) = \frac{a_1 e^{-\beta_1(t-\tau)}}{(t-\tau)^{\alpha}}$$
(11)

where $0 < \alpha < 1$. (and usually $0 < \alpha << 1$) and the integral members obtained from the final bounded values in the order compared with $\frac{t^{1-\alpha}}{1-\alpha}$. By introducing that $t - \tau = s$, the constitutive equations of hereditary elements take the following forms:

$$P(t) = c \left[y(t) - \int_{0}^{\infty} \Re(s)y(t-s)ds \right] \text{ and } y(t) = \frac{1}{c} \left[P(t) + \int_{0}^{\infty} \Re(s)P(t-s)ds \right]$$
(12)

It was accepted [1] that in the initial motion moment of time, constitutive stress strain relation with weak singular kernel is more precise description of the stress-strain state of a hereditary element. Ю.Н. Работнов (Yu.N. Rabotnov) proposed a special function known under the name of Rabotnov's function in the following form:

$$\mathfrak{s}_{\alpha} (-\beta, t) = t^{-\alpha} \sum_{k=0}^{\infty} \frac{(-\beta)^{k} t^{k(1-\alpha)}}{\Gamma[(k+s)(1-\alpha)]}$$
(13)

with properties for generating a class of fractional-rational functions for weak singular kernels (resolvents).

During the research of particular problems with hereditary elements in analytical dynamics of hereditary discrete systems in the forms with weak singular integral

equations (12) is useful concerning reason that is possible to use special Euler's Gama functions. Then methods for solving problems of dynamics of hereditary systems are considered with special Euler's Gama functions.

At the end of this part it is necessary to make one comparison between equations of stress-strain state for the cases of the standard hereditary element and for the case of the weak singular element. In the first case equations of the stress-strain state of standard hereditary element in the integro-differential forms (8) and (9) are equivalent to the equation (1) in the differential form. For the case of weak singular hereditary element integro-differential forms of stress-strain state equations (11) are unique and there aren't corresponding differential forms.

3. THREE FORMS OF EQUATIONS OF MOTIONS OA A HEREDITARY OSCILATOR

THE Simple model of a hereditary discrete system is hereditary oscillator with one degree of freedom which contains one material particle with mass m and one standard hereditary element P with material viscoelastic properties defined by the following coefficients: n, c and \tilde{c} constitutive stress-strain relation expressed by relation (2) between force P(t) and generalized and rheological coordinate y(t). Then by using principle of dynamical equilibrium of the oscillator it is possible to obtain the equation of the oscillator motion in the following form:

$$m\ddot{y}(t) + P(t) = F(t) \tag{14}$$

where P(t) is resistive reaction of the rheological element, F(t) external forced excitation. Using constitutive relation (2) or (10) for stressed and deformed standard hereditary (rheological) element for eliminating resistive reaction of the rheological element P(t)from the last equation (14) we obtain three corresponding forms of the equation of motion of the rheological – hereditary oscillator with one degree of freedom listed as follow: one in differential form:

$$m\ddot{y}(t) + m\ddot{y}(t) + nc\dot{y}(t) + \tilde{c}y(t) = n\dot{F}(t) + F(t)$$
(15)

and two in integrodifferential forms

$$m\ddot{y}(t) + c \left[y(t) - \int_{-\infty}^{t} \Re(t-\tau) y(\tau) d\tau \right] = F(t)$$
(16)

$$m\ddot{y}(t) + c\left[y(t) + \int_{-\infty}^{t} \aleph(t-\tau)m\ddot{y}(\tau)d\tau\right] = F(t) + c\int_{-\infty}^{t} \aleph(t-\tau)F(\tau)d\tau$$
(17)

For the case of the weak singular hereditary oscillator equation of the dynamic equilibrium (oscillator motion) in the differential form is not possible to obtain, but in the integro-differential forms it is possible.

3.1. THE FORMS OF INITIAL VONDITIONS FOR SOLVING EQUATIONS OF THE MOTIONS OF THE HEREDITARY OSCILLATOR

The initial condition for solving integro-differential equations (16) or (17) are in the classical form

$$v(0) = y_0 \quad \dot{y}(0) = \dot{y}_0 \tag{18}$$

In these cases the initial conditions are defined in classical way by initial position y(0) and initial velocity $\dot{y}(0)$ of the material particle.

The history of the rheological standard element loading in these integro-differential equations is taken into account by integral members in the period of integration $(-\infty,0)$. For solving differential equation (15) in every case, initial conditions are defined by three initial conditions y(0), $\dot{y}(0)$ and $\ddot{y}(0)$. In these cases initial conditions are defined by initial position y(0), initial velocity $\dot{y}(0)$ and initial acceleration $\hat{y}(0)$ of the material particle. The last initial condition, initial acceleration $\hat{y}(0)$ of the material particle is directly defined from stress-strain state of the standard hereditary (rheological) element on the basis of element loading history. Particular examples to obtain or to define the third initial condition in accordance with the different loaded element history are presented in the Refs. [3,4].

In Ref. [3] a detailed schema to obtain initial conditions of the hereditary oscillator in the case of the impulse external excitations is presented.

3.2. THE ESTIMATIONS OF THE FREQUENCY, DECREMENT AND COEFFICIENT OF THE RHEOLOGY OF THE HEREDITARY OSCILLATOR

The characteristic equations for equation of oscillation of the hereditary oscillator have the following form:

$$nm\lambda^3 + m\lambda^2 + nc\lambda + \tilde{c} = 0 \tag{19}$$

Let's present the roots of the previous equation in the complex form

$$\lambda_0 = -\delta_0$$
, $\lambda_{1,2} = -\delta \pm i\omega$

and after their introduction in the characteristic equation (19) we obtain:

$$(\lambda + \delta_0)(\lambda + \delta + i\omega)(\lambda + \delta - i\omega) = 0$$

or in the form

$$\lambda^3 + (\delta_0 + 2\delta)\lambda^2 + (\omega^2 + \delta^2 + 2\delta\delta_0)\lambda + \delta_0(\omega^2 + \delta^2) = 0$$
⁽²⁰⁾

After we put that coefficients of equations (19) and (20) of the corresponding exponents we obtain relations between kinetic parameters of the hereditary oscillator in the following forms:

$$\frac{\delta_0(\omega^2 + \delta^2)}{(\omega^2 + \delta^2 + 2\delta\delta_0)} = \frac{\tilde{c}}{nc}, \quad \delta_0 + 2\delta = \frac{1}{n}, \quad (\omega^2 + \delta^2 + 2\delta\delta_0) = \frac{c}{m}$$
(21)

from which follows:

$$\delta_0 = \frac{\tilde{c}}{nc} \left(1 + \frac{2\delta\delta_0}{\omega^2 + \delta_0^2} \right)$$

$$\delta = \frac{1}{2n} - \frac{\delta_0}{2} = \frac{c - \tilde{c}}{2nc} - \frac{\tilde{c}}{nc} \frac{\delta \delta_0}{\omega^2 + \delta_0^2}$$

$$\omega^2 = \frac{c}{m} \left[1 - \frac{\delta(\delta + 2\delta_0)}{\omega^2} \right]$$
(22)

In the first approximation, taking into account that ratio $\left(\frac{\delta}{\omega}\right)^2$ is small, the kinetic

parameters δ_0 , δ , ω of the hereditary oscillator in the first approximation are obtained in the forms:

$$\delta_0 = \frac{\tilde{c}}{nc} \qquad \delta = \frac{c - \tilde{c}}{2nc} \qquad \omega^2 = \frac{c}{m} \tag{23}$$

By using expressions (23) of the first approximation and putting them in the expressions (22), the kinetic parameters δ_0 , δ , ω of the hereditary oscillator in the second approximation are obtained in the forms:

$$\delta_{0} = \frac{\tilde{c}}{nc} \left[1 + \frac{(c-\tilde{c})\tilde{c}}{c^{2}} \frac{1}{n^{2}\omega^{2}} \right]$$

$$\delta = \frac{c-\tilde{c}}{2nc} \left[1 - \left(\frac{\tilde{c}}{c}\right)^{2} \frac{1}{n^{2}\omega^{2}} \right]$$

$$\omega^{2} = \frac{c}{m} \left[1 - \frac{c-\tilde{c}}{c} \frac{c+3\tilde{c}}{4c} \frac{1}{n^{2}\omega^{2}} \right]$$
(24)

For many visco-elastic hereditary materials the time of hereditary element relaxation is $n \sim 50$ [sec]. For the frequency of the hereditary oscillator $f \sim 1$ [*hertz*] or $\omega = 2\pi f = 6,28$ [sec⁻¹] dimensionless ratio takes the following value $1 / (n^2 \omega^2) \approx 4 \cdot 10^{-5}$. By this way, the values of hereditary oscillator coefficients δ_0 , δ and circular frequency ω are defined by expressions (23) with high degree of precision.

By using previous considerations and the approximation of the standard hereditary oscillator coefficients δ_0 , δ and circular frequency ω defined by expressions (23), the solution of the equation (14) or (15), (16) and (17) for the standard hereditary oscillator, we can write in the following form:

$$y(t) = mg\left[\frac{1}{\tilde{c}} + \left(\frac{1}{c} - \frac{1}{\tilde{c}}\right)e^{-\delta_0 t} - \frac{e^{-\delta t}}{c}\cos\omega t - \frac{3}{2}\frac{c-\tilde{c}}{c^2}\frac{1}{n^2\omega^2}\sin\omega t\right]$$
(25)

for initial conditions y(0) = 0, $\dot{y}(0) = 0$, $\ddot{y}(0) + P(0) = mg$, where P(0) = cy(0), corresponding to applied heavy material particle with weight mg and with zero initial velocity of the hereditary oscillator material particle corresponding to the unstressed and undeformed natural state of the hereditary element in the hereditary oscillator.

The motion of this considered hereditary oscillator in defined initial conditions represents damped oscillations in accordance with the curve of rheology.

An estimation of precision expressions of values of the hereditary oscillator with the weak singular rheological element for the coefficients: δ_0 of rheology, δ for decrement and ω for circular frequency expressed by Γ - Euler function give us conclusion, presented in reference [4], that these expressions are with the high steep of the precision.

The kernel of the relaxation in the governing integro-differential equation of the rheological oscillator oscillations in the form

$$m\ddot{y}(t) + c\left[y(t) - \int_{0}^{\infty} \Re(\tau)y(t-\tau)d\tau\right] = mg$$

is taken in the form $\Re(\tau) = ae^{-\beta\tau}\tau^{-\alpha}$ [1]. The expressions of the weak singular hereditary oscillator coefficients: δ_0 of rheology, δ for decrement and ω for circular frequency are obtained in the form:

$$\delta_0 \approx \beta - \frac{1-\alpha}{\sqrt{a\Gamma(1-\alpha)}}, \quad \delta \approx \frac{a_1 \Gamma(1-\alpha)\omega^{\alpha}}{2} \quad \text{and} \quad \omega^2 \approx \frac{c}{m}$$
 (26)

For the weak singular hereditary oscillator with the kernel of rheology in the form $\aleph(t-\tau) = \frac{a_1 e^{-\beta_1(t-\tau)}}{(t-\tau)^{\alpha}}$ (the kernel defined by Ржаницын (Rzanicin) [1,3[) the approximations

of the oscillator coefficients: δ_0 of rheology, δ for decrement and ω for circular frequency by use of Γ - Euler function are obtained in the following forms:

$$\delta_0 \approx \beta_1$$
, $\delta \approx \frac{a_1 \Gamma (1 - \alpha) \omega^{\alpha}}{2}$ and $\omega^2 \approx \frac{c}{m}$ (27)

The precision of expressions of values of the hereditary oscillator with the weak singular rheological element for the coefficients: δ_0 of rheology, δ for decrement and ω for circular frequency expressed by Γ - Euler function give us the conclusion, presented in reference [4], that these expressions are with high steep of the precision like in the previous examples of the standard hereditary oscillators.

4. LAGRANGE'S EQUATIONS OF THE SECOND KIND FOR THE HEREDITARY DISCRETE SYSTEMS

Let us consider discrete mechanical systems composed of N material particles constrained by s ideal holonomic constraints expressed by

$$f_{v}(x_{1}, x_{2}, \dots, x_{3N}, t) = 0 \qquad (v = 1, 2, 3, \dots, s)$$
(28)

where x_i , (i=1,2,3,...3N) are Descartes' coordinates of the mechanical system material particles. The position of the material system as well as material particle configuration are defined by n = 3N - s generalized coordinates $q_1,q_2,...,q_n$. The interaction between material particles, material bodies and is realized by *K* heological light elements in the system. The reaction of the rheological light elements are noted by $P_k = P_k(y)$, in which $y = (y_1, y_2, ..., y_K)$, y_k , k = 1, 2, ..., K and $y_k = y_k(q_1, q_2, ..., q_K)$. There, y_k , k = 1, 2, ..., K are deformation of the rheological elements. Number *K* f the rheological elements is arbitrary

and it is possible that this number K is greater then number n of the degrees of the hereditary system K > n.

The generalized equation of the hereditary discrete system is in the form:

$$\sum_{i=1}^{3N} \left[m_i \ddot{x}_i - X_i(t) + \sum_{k=1}^{K} P_k(y_k) e_{ik} \right] \delta x_i = 0$$
(29)

where $e_{ik} = e_{ik}(x_1, x_2, ..., x_{3N}) P_k(y_k)$ cosine of the rheological elements and their reaction directions in the Descartes' system of the coordinates; $X_i(t)$ external forces components (projections) in the Descartes' system coordinates; $y_k(x_1, x_2, ..., x_{3N})$ rheological coordinates of the *k*-rheological element. By using the expression of the variations of the Descartes' coordinates through the generalized coordinates in the following forms $\delta x_i = \sum_{j=1}^n \frac{\partial x_i}{\partial q_j} \delta q_j$ and

taking into account that Lagrange's identities are in the forms: $\frac{\partial x_i}{\partial q_j} = \frac{\partial \dot{x}_i}{\partial \dot{q}_j}$, and

 $\frac{d}{dt} \left(\frac{\partial x_i}{\partial q_j} \right) = \frac{\partial \dot{x}_i}{\partial q_j}$ the previous equation (29) is transformed into the following form:

$$\sum_{j=1}^{n} \left[\frac{d}{dt} \frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{j}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{j}} - Q_{j} + \sum_{k=1}^{K} b_{kj} P_{k} \right] \delta q_{j} = 0$$
(30)

where coefficients b_{kj} are defined by expressions $b_{kj}(q) = \sum_{i=1}^{3N} e_{ik} \frac{\partial x_i}{\partial q_j}$, (j = 1, 2, 3, ..., n). The

kinetic energy E_k and the generalized forces Q_j correspond to the generalized coordinates q_j in the forms: $E_{\mathbf{k}} = \sum_{i=1}^{3N} \frac{m_i \dot{x}_i^2}{2}$ and $Q_j = \sum_{i=1}^{3N} X_i \frac{\partial x_i}{\partial q_j}$ (j = 1, 2, 3, ..., n).

Taking into account previous considerations and the independency of the generalized coordinates q_j , j = 1,2,...,n and their variations δq_j , (i = 1,2,3,...,n) from the basic generalized equation of system dynamics (30) we obtain the equations of the motion of the hereditary discrete system in the following form:

$$\frac{d}{dt}\frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{j}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{j}} + \sum_{k=1}^{K} b_{kj}P_{k} = Q_{j}, \quad (j = 1, 2, ..., n)$$
(31)

which contain the generalized reactions of the hereditary elements in the forms: $\sum_{k=1}^{K} b_{kj} P_k$. These reactions are basic for founding Lagrange's equations in the analytical dynamics of the hereditary discrete systems. (For the classical discrete systems, Lagrange's equations don't contain like that reactions). For the elimination of these reactions from the previous system of equations it is possible to use one of the three previously defined and presented forms of the constitutive relation for stress-strain state of the used standard rheological hereditary elements. For obtaining Lagrange's equations from initial equations (31) is connected with possibility to solve these equations with respect to the equations of the reactions P_k of the hereditary elements.

4.1. THE UNIVERSAL FORM OF LAGRANGE'S EQUATIONS FOR HEREDITARY DISCRETE SYSTEMS

Expressed in the equations (31) reactions P_k of the hereditary elements through (10) is not difficult to obtain Lagrange's equations in the integro-differential (relaxation) forms

$$\frac{d}{dt}\frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{j}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{j}} + \sum_{k=1}^{K} b_{kj} c_{k} \left[y_{k}(t) - \int_{-\infty}^{t} \Re_{k}(t-\tau) y_{k}(\tau) d\tau \right] = Q_{j}, \quad (j = 1, 2, ..., n)$$
(32)

The universality of the obtained Lagrange's equations in the form (32) it is in the possibility to construct (build) these equations in the case of an arbitrary number K of rheological elements into discrete hereditary system. The history of the hereditary element loading in the time period $t \in (-\infty, 0)$ before the start of the considered and described hereditary discrete system motion is taken into account by integral members in the equations.

The initial conditions for solving the system of equations (32) are defined in the classical form by the generalized coordinate initial values and generalized velocities initial values:

$$q_{j}(0) = q_{j}^{0} \qquad \dot{q}_{j}(0) = \dot{q}_{j}^{0} \tag{33}$$

Let's consider an example by the model of a vibroisolation system of an object with rheological foundation for the case with kinematics excitation as a suitable hereditary discrete system for application of the previous derived equations (Figure 1). For determining the position of the object two generalized coordinates are taken in the form of x(t) for object translation and $\varphi(t)$ for object rotation. The number of rheological elements is K = 4 and number of freedom degrees is n = 2 and one kinematics excitation. $\xi(t)$ by motion of the object. Lagrange's equations in the relaxation form are:



Fig. 1. Model of a vibroisolation system of the object with rheological foundation for the case with kinematics excitation



Fig. 2. a) Model presentation of the standard piezo-rheological hereditary element.;
b) Model presentation of the pizo-modified Maxwell's elasto-viskosic hereditary element;
c) Model presentation of the pizo-modified Kelvin-Foight's visko-elastic hereditary element;
d) Model presentation of the piezo-modified Burgers's hereditary element.



Fig. 3. Thermorheological pendulum

$$m\ddot{x}(t) + (c_1 + c_2 + c_3 + c_4)x(t) + (2c_1 + c_2 - c_3 - 2c_4)a\varphi(t) - \int_{-\infty}^{t} [\Re_{11}(t - \tau)x(\tau) + \Re_{12}(t - \tau)a\varphi(\tau)]d\tau = = (c_1 + c_2 + c_3 + c_4)\xi(t) - \int_{-\infty}^{t} \Re_{11}(t - \tau)\xi(\tau)d\tau$$
(34.1)

$$\mathbf{J}\ddot{\varphi}(t) + (c_1 + c_2 + c_3 + c_4)a^2\varphi(t) + (2c_1 + c_2 - c_3 - 2c_4)ax(t) - \int_{-\infty}^{t} [\Re_{21}(t - \tau)x(\tau) + \Re_{22}(t - \tau)a^2\varphi(\tau)]d\tau =$$

= $(2c_1 + c_2 - c_3 - 2c_4)a\xi(t) - \int_{-\infty}^{t} \Re_{21}(t - \tau)a\xi(\tau)d\tau$ (34.2)

where generalized kernel of the relaxation are in the following forms:

$$\begin{aligned} \Re_{11}(t-\tau) &= [c_1 \Re_1(t-\tau) + c_2 \Re_2(t-\tau) + c_3 \Re_3(t-\tau) + c_4 \Re_4(t-\tau)] \\ \\ \Re_{22}(t-\tau) &= [4c_1 \Re_1(t-\tau) + c_2 \Re_2(t-\tau) + c_3 \Re_3(t-\tau) + 4c_4 \Re_4(t-\tau)] \\ \\ \Re_{12}(t-\tau) &= \Re_{21}(t-\tau) = [2c_1 \Re_1(t-\tau) + c_2 \Re_2(t-\tau) - c_3 \Re_3(t-\tau) - 2c_4 \Re_4(t-\tau)] \end{aligned}$$

4.2. DYNAMICALLY DEFINED HEREDITARY DISCRETE SYSTEMS

From all previously considered theories of hereditary discrete systems and numerous examples as well as presented in the main key points hereditary discrete systems we must separate two groups of hereditary discrete systems on the basis of the possibilities to solve governing equations (31) with respect to rheological reactions P_k . By solving of the governing equations (31) with respect to rheological reactions P_k is possible under the following two conditions:

1* Number *K* of the rheological elements must be less or equal to the number *n* of the degrees of the hereditary discrete system freedom, $K \le n$;

2* Structure of the mechanical hereditary discrete system must be like that, that there is the possibility of choices of the generalized coordinate that is possible to obtain inequities with zero defined by

$$\left| b_{kj}(q) \right| = \left| \sum_{i=1}^{3N} e_{ik}(q) \frac{\partial x_i}{\partial q_j} \right| \neq 0 \text{ for } K \le n$$
(35)

These conditions are generalizations of the known conditions of the static defined mechanical system, applied widely for solving problems in the strength of materials.

4.2.1. LAGRANGE'S EQUATIONS FOR THE HEREDITARY DISCRETE SYSTEMS WITH STANDARD RHEOLOGICAL ELEMENTS

1* Case for K = n. Governing system equations (31) of the hereditary discrete system is possible presently in the matrix form:

$$(b_{jk})\{P_j\} = -\{L_k\}$$
(36)

where (b_{jk}) is $n \times n$ matrix composed by elements b_{jk} , $\{P_j\}$ matrix column with elements P_j and $\{L_j\}$ matrix column with elements $L_j = \frac{d}{dt} \frac{\partial E_k}{\partial \dot{q}_j} - \frac{\partial E_k}{\partial q_j} - Q_j$. In the case that conditions (36) of the dynamical defined hereditary discrete system are satisfied and possess solvability with respect of the reactions P_k it is easy to obtain the following explicate solution for P_k :

$$P_k = -\sum_{\nu=1}^{\nu=n} a_{k\nu} L_{\nu}(q, \dot{q}, t), \quad (k = 1, 2, 3, ..., K)$$
(37)

Using differential equations – constitutive relations of the stress-strain state of rheological standard light elements (6) and solutions (37), from governing equations (31) it is easy to obtain equations in the Lagrange's form:

$$\left(1+n_k\frac{d}{dt}\right)_{j=l}^{j=n}a_{kj}(q)\left[\frac{d}{dt}\frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_j}-\frac{\partial E_{\mathbf{k}}}{\partial q_j}-Q_j\right]+n_kc_k\dot{y}_k(q)+\tilde{c}y_k(q)=0,\ (k=1,2,3,...,n)$$
(38)

The previous system of equations is a system of Lagrange's equations of the second kind containing derivatives of the third order, of the generalized coordinates, with respect to time.

The initial conditions for the generalized coordinates for the case natural (nondeformed and unstressed) state of the rheological elements at initial moment of the system motion are defined by following expressions:

$$q_{k}(0) = q_{k}^{0} \quad \dot{q}_{k}(0) = \dot{q}_{k}^{0}$$

$$\left\{ P_{k} + \sum_{\nu=1}^{n} a_{k\nu}(q) \left[\frac{d}{dt} \frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{\nu}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{\nu}} - Q_{\nu} \right] \right\}_{t=0} = 0, \quad (k = 1, 2, 3, ..., n) \quad (39)$$

and contain, in addition to classical cases, a second derivative of the generalized coordinate with respect to time at initial moment of the system motion: $\ddot{q}_k(0), (k = 1, 2, 3, ..., n)$,

2* Case for K < n. Governing system of equations (31) of the hereditary discrete system is separable into two groups: first contains *K* equations for which system determinate satisfies the following condition $|b_{kj}| \neq 0$ and by analogy with (36)-(37) it is easy to obtain the following:

$$P_{k} = -\sum_{\nu=1}^{\nu=n} a_{k\nu} L_{\nu}(q, \dot{q}, t), \quad (k = 1, 2, 3, ..., K)$$
(40)

For that group of equations by using constitutive relations (6) for stress-strain rheological element state and solutions (40) by analogy with (38) it is easy to construct modified Lagrange's equations of the second kind and with members of the third order derivatives with respect to time in the following form:

$$\left(1+n_k\frac{d}{dt}\right)_{\nu=1}^{\nu=K}a_{k\nu}(q)\left[\frac{d}{dt}\frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{\nu}}-\frac{\partial E_{\mathbf{k}}}{\partial q_{\nu}}-Q_{\nu}\right]+n_kc_k\dot{y}_k(q)+\tilde{c}y_k(q)=0, \quad (k=1,2,3,\dots,K) \quad (41)$$

The remaining of the system governing equations (31) without equations transformed into system (41) with *K* equations, contain the rest of the n - K equations which are possible to transform into modified Lagrange's equations of the second kind expressed by:

$$\frac{d}{dt}\frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{j}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{j}} - \sum_{\nu=1}^{\nu=K} d_{j\nu}(q) \left[\frac{d}{dt}\frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{\nu}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{\nu}} - Q_{\nu} \right] = Q_{j}'(j = K + 1, K + 2, ..., n)$$
(42)

where $d_{j\nu} = \sum_{k=1}^{k=K} b_{kj} a_{k\nu}$.

In this way, for dynamically defined rheological discrete systems in which the number of the contained rheological elements is less the number of the degrees of the system freedom, one part of equations is with the members of the second order and the rest are equations with members of the third order derivatives.

In accordance with previous considerations and the conclusion pointed out for different hereditary system cases is right to derive conclusion that possibilities to obtain modified Lagrange's equations for the hereditary discrete system containing generalized hereditary elements with constitutive stress-strain relations in the form of the equations (3) exist.

4.2.2. THE MODIFIED LAGRANGE'S EQUATIONS IN THE INTEGRODIFFERENTIAL FORM FOR HEREDITARY DISCRETE SYSTEMS WITH STANDARD RHEOLOGICAL ELEMENTS OF HEREDITARY SYSTEMS

The construction of integro-differential equations in rheological forms is realized under the conditions of the dynamically defined hereditary discrete systems of which dynamics are described by (30). For that, we used expressions of P_k in the form (37) and rheological states of the rheological elements are described by (12) and for K = nLagrange's equations in rheological forms can be written as:

$$\sum_{\nu=1}^{N-n} a_{k\nu}(q) \left[\frac{d}{dt} \frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{\nu}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{\nu}} - Q_{\nu} \right] + \qquad (k = 1, 2, 3, ..., n)$$

$$+ \int_{-\infty}^{t} \aleph_{k}(t - \tau) \left\{ \sum_{\nu=1}^{N-n} a_{k\nu}(q) \left[\frac{d}{dt} \frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{\nu}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{\nu}} - Q_{\nu} \right] \right\} d\tau + c_{k} y_{k}(q) = 0 \qquad (43)$$

For the case K < n in the system (30), as in the case (38), (39) and (37) two groups of equations are obtained in the following forms:

$$\sum_{\nu=1}^{\nu=K} a_{k\nu}(q) \left[\frac{d}{dt} \frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{\nu}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{\nu}} - Q_{\nu} \right] + \qquad (k = 1, 2, 3, ..., K)$$
$$+ \int_{-\infty}^{t} \aleph_{k}(t-\tau) \left\{ \sum_{\nu=1}^{\nu=K} a_{k\nu}(q) \left[\frac{d}{dt} \frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{\nu}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{\nu}} - Q_{\nu} \right] \right\} d\tau + c_{k} y_{k}(q) = 0 \qquad (44)$$

$$\frac{d}{dt}\frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{j}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{j}} - \sum_{\nu=1}^{\nu=K} d_{k\nu}(q) \left[\frac{d}{dt}\frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{\nu}} - \frac{\partial E_{\mathbf{k}}}{\partial q_{\nu}} - Q_{\nu} \right] = Q_{j} \quad (j = K + 1, K + 2, ..., n)$$
(45)

16

For the systems with different forms (a number with differential and a number with integrodifferential forms) of the defined stress-strain constitutive relations of the rheological elements contained in the hereditary discrete mechanical system expressed by (6) and (10) in the case of the dynamically defined hereditary system of the generalized Lagrange's equation consists of equations in the forms (32), (38), (43) and (45).

In the case that hereditary discrete system K > n is a part of the system of constitutive equations containing K - n equations of stress-strain states of hereditary elaments are in need of being transformed into equations of relaxational forms.

4.3. LAGRANGE'S EQUATIONS FOR HEREDITARY DISCRETE SYSTEMS WITH VISCOSE (MAXWELL'S) ELEMENTS

Let's consider a hereditary discrete system with *n* degrees of freedom which contains K < n rheological elements of Maxwell's type, with stress-strain states described by differential equations in the following form:

$$n_k \dot{P}_k + P_k = n_k c_k \dot{y}_k(q) \quad (k = 1, 2, 3, ..., K_1)$$
 (46)

From the basic governing system of equations (31), for the first, a group of K equations for which the condition $|b_{kj}| \neq 0$ and by solving these equations explicit with reactions P_k of the viscose elements, and the expressions of these reactions P_k are expressed in the forms (40). By elimination of reactions P_k from the separated group of the equations for the governing system with the use of constitutive relations expressed by equations (46) it is easy to obtain the following groups of equations:

$$\left(1+n_k\frac{d}{dt}\right)\sum_{\nu=1}^{\nu=n}a_{k\nu}(q)\left\{\frac{d}{dt}\frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{\nu}}-\frac{\partial E_{\mathbf{k}}}{\partial q_{\nu}}-Q_{\nu}\right\}+n_kc_k\dot{y}_k(q)=0, (k=1,2,3,...,K_1)$$
(47)

The remaining of the system governing equations (40) without equations transformed into system (47) with *K* equations, contain the rest of the n - K equations, which can be transformed into modified Lagrange's equations of the second kind as expressed by (42).

In the special case when expression of the system kinetic energy E_k and the generalized forces Q_k explicit do not contain generalized coordinates q_k and coefficients don't depend explicitly on time, $a_{k\nu}(q)$, then equations (47) are integrable and it is easy to obtain K first integrals in the forms:

$$\left(1 + n_k \frac{d}{dt}\right)_{\nu=1}^{\nu=n} a_{k\nu}(q) \frac{\partial E_{\mathbf{k}}}{\partial \dot{q}_{\nu}} + n_k c_k y_k(q) = \int_0^t \left(1 + n_k \frac{d}{d\tau}\right)_{\nu=1}^{\nu=n} a_{k\nu}(q) Q_{\nu}(\tau) d\tau + h_k \qquad (48)$$

$$(k = K_1 + 1, K_2 + 2, \dots, n)$$

where $h_k = const$ constant of the integration.

The generalized coordinates q_k for which the conditions $\partial E_k / \partial q_k = 0$ are satisfied, and when the expression of the system kinetic energy E_k explicitly does not contain these generalized coordinates q_k , are cyclic coordinates. In this way, under the corresponding constrictions, as in the case of the classical systems with cyclic coordinates, for the hereditary systems with rheological elements maxwell's types, the first integral appears and exists.

5. THERMO-RHEOLOGICAL PENDULUM

V. 1. Light standard thermo-rheological hereditary element

When standard hereditary element is modified by two temperatures $T_K(t)$ and $T_M(t)$, which are introduced by thermo-modification of visco-elastic properties by temperature $T_K(t)$, and by thermo-modification of elasto-viscosic properties by temperature $T_M(t)$, than constitutive relation between stress and strain state of the thermo-rheological hereditary element (see Figure 2, and Refs. [4, 10, 11]) is:

$$n\dot{P}(t) + P(t) + n\dot{F}_{M}(t) + F_{K}(t) = nc\dot{\rho}(t) + \tilde{c}[\rho(t) - \rho_{0}]$$
(49)

in which (see Refs. [4] and [8-13])

$$F_M(t) = c_M \alpha_M T_M(t), \ F_K(t) = c_K \alpha_K T_K(t)$$
(50)

are thermoelastic forces, and $\rho(t)$ is rheological coordinate, c_M, c_K are the coefficients of thermo-elastic rigidity, α_M , α_K are coefficients of thermo-elastic dilatations, *n* is time of relaxation, and *c*, \tilde{c} an instantaneous rigidity and a prolonged one of an element.

Constitutive relation (49) of the thermo-rhelogical hereditary element from differential form we can rewrite in two integro-differential forms as in the previous chapters.

V. 2. Light standard piezo- and thermo-rheological hereditary element

When the standard hereditary element is modified by two polarization voltages $U_K(t)$ and $U_M(t)$, which are introduced by piezo-modification of visco-elastic properties of the subelement of piezoceramics, by $U_K(t)$ and by piezo-modification of elasto-viscosic properties by $U_M(t)$, and thermo-modified by two temperatures $T_K(t)$ and $T_M(t)$, than constitutive relation between stress and the strain state of the piezo-rheological hereditary hybrid element is in the form (49) in which (see Figure 2. and Refs. [4] and [8-13])

$$F_{M}(t) = c_{UM} \alpha_{UM} U_{M}(t) + c_{TM} \alpha_{TM} T_{M}(t) \qquad F_{K}(t) = c_{UK} \alpha_{UK} U_{K}(t) + c_{TK} \alpha_{TK} T_{K}(t)$$
(51)

are thermoelastic forces, and $\rho(t)$ is rheological coordinate, c_{TM} , c_{TK} are coefficients of thermo-elastic rigidity, α_{TM} , α_{TK} are coefficients of thermo-elastic dilatations, c_{UM} , c_{UK} are coefficients of piezo-elastic rigidity, α_{UM} , α_{UK} are coefficients of piezo-elastic dilatations, n is time of relaxation, and c, \tilde{c} an instantaneous rigidity and the prolonged one of an hybrid element.

V.3. Pendulum with standard thermorheological hereditary element

The thermo-rheological hereditary pendulum (Figure 3) has two degrees of freedom, one degree of motion freedom defined by angular coordinate ϑ and one degree of deformations freedom defined by changeable length of thread as a coordinate $\rho(t)$.

Let us compose the equations of the thermo-rheological pendulum dynamics (see Figure 3) with thread in which the standard thermo-rheological hereditary element with constitutive stress-strain relation (49) is incorporated. Now, by introducing force P(t) of

the extension of the thermo-rheological hereditary thread from constitutive relation (49) presented into integral form, the equations of the pendulum motion are in the forms:

$$\ddot{\rho} - (\rho_o + \rho(t))\dot{\theta}^2 + g\cos\theta + \frac{c}{m} \left[\rho(t) - \int_0^t \rho(\tau)\Re(t - \tau)d\tau \right] =$$

$$= \frac{1}{m} F_M(t) - \frac{c}{m(c - \tilde{c})} \int_0^t [F_M(\tau) - F_K(\tau)] \Re(t - \tau)d\tau P(t)$$

$$(\rho_0 + \rho(t))^2 \ddot{\theta} + 2(\rho_0 + \rho(t))\dot{\theta}\dot{\rho}(t) + g(\rho_0 + \rho(t))\sin\theta = \mu(t)(t)$$
(52.2)

This system is a system with one integro differential and one differential equation of the thermo-rheological hereditary pendulum with motion in vertical plane.

If the thermo-rheological pendulum is in the horizontal plane, from the second differential equation of the previous system, we can obtain the relation between the length of the pendulum thread and of the angular velocity in the following form:

$$\dot{\theta}(t) = \dot{\theta}(0) \left[\frac{\rho_0 + \rho(0)}{\rho_0 + \rho(t)} \right]^2$$
(53)

By introducing this previous expression (53) in the first equation of the system (52 (for the case - horizontal plane) the following integro-differential equation for the pendulum length thread is obtained:

$$\ddot{\rho}(t) - [\dot{\theta}(0)]^2 \frac{[\rho_0 + \rho(0)]^4}{[\rho_0 + \rho(t)]^3} + \frac{c}{m} \left[\rho(t) - \int_0^t \rho(\tau) \Re(t - \tau) d\tau \right] = = \frac{1}{m} F_M(t) - \frac{c}{m(c - \tilde{c})} \int_0^t \left[F_M(\tau) - F_K(\tau) \right] \Re(t - \tau) d\tau P(t)$$
(54)

6. COVARIOAN TENSOR INTEGRO-DIFFERENTIAL EQUATIONS OF HEREDITARY SYSTEMS

We obtain the following system of equations in the covariant coordinates:

$$a_{\alpha\beta} \frac{D\dot{q}^{\beta}}{dt} = Q_{\alpha} + Q_{\alpha}^{*} + P_{\alpha}^{H} + P_{\alpha}^{C}$$

$$\alpha = 1, 2, 3, ..., n; \quad n = 3N - S$$
(55)

where, by analyzing the members, we have the following expressions of the generalized fictive, active and reactive forces in the tensor covariant form for the corresponding curvilinear gereralized covaqriant coordinates(see Refs. [5, 7, 10 and 11]) :

$$I_{\alpha} = -\sum_{\nu=1}^{\nu=N} m_{\nu} \left(\ddot{\vec{r}}_{\nu}, \frac{\partial \vec{r}_{\nu}}{\partial q^{\alpha}} \right) = -\sum_{\nu=1}^{\nu=N} m_{\nu} \left(\frac{d}{dt} \sum_{\beta=0}^{\beta=n} \frac{\partial \vec{r}_{\nu}}{\partial q^{\beta}} \dot{q}^{\beta}, \frac{\partial \vec{r}_{\nu}}{\partial q^{\alpha}} \right) = -\left[a_{\alpha\beta} (\ddot{q}^{\beta} + \Gamma^{\alpha}_{\gamma\delta} \dot{q}^{\gamma} \dot{q}^{\delta}) \right] = -a_{\alpha\beta} \frac{D\dot{q}^{\beta}}{dt}$$
(56)

$$\alpha = 1, 2, 3, ..., n; \quad n = 3N - S$$

$$Q_{\alpha} = \sum_{\nu=1}^{\nu=N} \left(\vec{F}_{\nu}(t), \frac{\partial \vec{r}_{\nu}}{\partial q^{\alpha}} \right)$$
(57)

$$Q_{\alpha}^{f} = \sum_{\nu=1}^{\nu=N} \sum_{\mu=1}^{\mu=S} \lambda_{\mu} \left(grad_{\nu} f_{\mu}(\vec{r}_{1},...,\vec{r}_{N}), \frac{\partial \vec{r}_{\nu}}{\partial q^{\alpha}} \right) = 0$$
(57*)

$$\mathbf{P}_{\alpha}^{H} = \sum_{\nu=1}^{\nu=N} \left\{ \sum_{j=1}^{j=N} \sum_{k=1}^{k=K_{\nu}} P_{(\nu,\nu+j)k}(t) \frac{\left(\vec{\rho}_{(\nu,\nu+j)k}, \frac{\partial \vec{r}_{\nu}}{\partial q^{\alpha}}\right)}{\left|\vec{\rho}_{(\nu,\nu+j)k}\right|} \right\}$$
(58)

$$\mathbf{P}_{\alpha}^{C} = \sum_{\nu=1}^{\nu=N} \left\{ \sum_{j=1}^{j=N} \sum_{c=1}^{c=C_{\nu}} P_{(\nu,\nu+j)c}\left(t\right) \frac{\left(\vec{\rho}_{(\nu,\nu+j)k}, \frac{\partial \vec{r}_{\nu}}{\partial q^{\alpha}}\right)}{\left|\vec{\rho}_{(\nu,\nu+j)k}\right|} \right\}$$
(59)

$$Q_{\alpha}^{*} = \sum_{\nu=1}^{\nu=N} \left(\vec{R}_{\nu T}(t), \frac{\partial \vec{r}_{\nu}}{\partial q^{\alpha}} \right)$$
(60)

5. CONCLUDING REMARKS

On the basis of the construction of Lagrange's mechanics of hereditary discrete systems, the classical mechanics principles are used. These principles are: the Principle of the work of forces along corresponding possible system displacements, as well as the Principle of dynamical equilibrium.

By using the Principle of the work of system forces along the corresponding possible system displacements we obtain governing system equations of the hereditary discrete system dynamics.

In this paper, the series of schemes for construction of Lagrange's equations for hereditary discrete mechanical systems with rheological interactions between bodies and material particles in the system are given, expressed by constitutive stress-strain relations in the three different forms: by differential equations and two forms by integrodifferential equations with resolvents as kernels of the relaxation and of kernel of rheology. A class of dynamically defined hereditary systems is defined and investigated. For this class of dynamically defined hereditary systems, it is possible to eliminate reactions of rheological elements and to obtain modified Lagrange's equations in differential and integrodifferential forms. In the case that stress-strain relations of hereditary elements can be expressed in all three forms, and also only in the forms by used relaxational kernels, it is possible to construct Lagrange's equations for every arbitrary type of hereditary systems.

Also, a class of dynamically undefined hereditary systems is defined and considered basic. The initial conditions of hereditary system dynamics are very important, containing the history of rheological interactions of the system. Then, it is important to take into account the stress-strain history of viscoelastic elements – interactions between hereditary system material particles.

The analogy between hereditary interactions and reactive forces in the systems of the automatic control give possibility to extend theory of analytical dynamics of hereditary systems to mechanical systems with automatic control.

A hereditary discrete system with n degrees of freedom which contains rheological elements of Maxwell's type, with stress-strain states described by differential equations is considered.

For the description of properties of dynamics of a hereditary system by using relaxational or rheological kernel (resolvent), these kernels are expressed by exponential or fractional-exponential forms [1]. The descriptions of hereditary properties of the system by using differential forms (1) and integral form (2) and (3) with exponential kernels are equivalent. For the case of fractional-exponential forms of the kernel (2) and (3) in the integral form corresponding equivalent differential forms do not exist.

The Lagrange's mechanics of hereditary systems is extended and generalized to thermo-rheological [4, 8] and piezo-rheological [4, 9] mechanical systems.

Acknowledgment: Parts of this research were supported by Ukrainian Higher Education Academy of Sciences and Kiev National University named Taras Shevchenko and also by the Ministry of Sciences, Technologies and Development of Republic Serbia trough Mathematical Institute SANU Belgrade Grants No. ON144002 Theoretical and Applied Mechanics of the Rigid and Solid Body, Mechanics of Materials, and also through Faculty of Mechanical Engineering University of Niš. Parts of this research results are presented as Openeing Lecture of Minisymposia Theory and Aplications, measyres if Dynamical Integrity at International Summer school Advances of Theoretical and Applied Mechanics.

REFERENCES

- 1. Работнов Ю.Н. Элементы наследственной механики твердых тел.-М.:Наука, 1977.-384 с.
- 2. Савин Г.Н., Рущицкий Я.Я. Элементы механики наследственных сред.-К.:Вища школа, 1975 -252 с
- Ржаницын А.Р. Некоторые вопросы механики систем, деформирующихся во времени.– М.:ГИТГЛ, 1949.–242 с.
- Oleg Aleksandrovič Goroško (Ukrajina) i Katica (Stevanović) Hedrih (Srbija): Analitička dinamika (mehanika) diskretnih naslednih sistema, (Analytical Dynamics (Mechanics) of Discrete Hereditary Systems), University of Niš, 2001, Monograph, p. 426 (in Serbian), YU ISBN 86-7181-054-2.
- Hedrih (Stevanović) K., (1994), *The power of the rheonomic constraints change*, Zbornik radova JDM, Simpozijum iz Opšte mehanike, Novi Sad, 1994. pp. 177-185, .(in Serbian)
- Goroshko, O.A., Puchko, N.P., Lagrangian equations for the multibodies hereditary systems, Facta Universitatis, Series Mechanics, Automatic Control and Robotics, Vol. 2, No 7, 1997, pp. 209-222.
- Hedrih (Stevanović) K., Rheonomic Coordinate method Applid to Nonlinear Vibration Systems with Hereditary Elements, Facta Universitatis, Series, Mechanics, Automatic Control and Robotics, Vol. 2, No. 10, 2000, pp. 1111-1135. YU ISSSN 0354-2009.
- Hedrih (Stevanović) K., *Thermorheological Hereditary Pendulum*, (Ref. No. TVP-11) Thermal Stresses 99, Edited by J.J. Skrzypek and R. B. Hetnarski, Cracow 1999, pp.199-202.
- Hedrih (Stevanović) K., Differential Equations of Two Mass Particles, Constrained with a Piezo-thermo-rheological Hereditary Element, Dynamics, Proceedings of full papers, Edited by Dragutin Veličković, 5 th International Conference on Applied Electromagnetics, PES 2001., pp. 77-80.

O. ALEKSANDROVICH GOROSKO, K. (STEVANOVIĆ) HEDRIH

- Hedrih (Stevanović) K., Discrete Continuum's Models and Thermo-rheological Elements -Basic Idea and Tensor Equations, Proceedings of full papers, Edited by Dragutin Veličković, 6 th International Conference on Applied Electromagnetics, PES 2003, Stability Pact for South Eastern Europe, DAAD, pp. 127-130.
- Hedrih (Stevanović) K., Discrete Continuum's Models and Thermo-rheological Elements Homogeneous Linear Chain and Plane/Space Material Nets, Proceedings of full papers, Edited by Dragutin Veličković, 6 th International Conference on Applied Electromagnetics, PES 2003, Stability Pact for South Eastern Europe, DAAD, pp. 131-134.
- 12. Hedrih (Stevanović), K., (2007), Energy analysis in the nonlinear hybrid system containing linear and nonlinear subsystem coupled by hereditary element, Nonlinear Dynamics, Springer, DOI 10.1007/s11071-007-9197-2
- Hedrih (Stevanović) K., (2006), Modes of the Homogeneous Chain Dynamics, Signal Processing, Elsevier, 86(2006), 2678-2702. ISSN: 0165-1684 www.sciencedirect.com/science/journal/01651684
- 14. Лагранж. (1938) Аналитическая механика.-М.:ГИТТЛ.-348 с.(пер. с франц. Mechanique Analitique, Per M De La Grange. Paris)/
- 15. Чаплыгин С.А. (1949) Исследование по динамике неголономных систем.-М.-Л.: ГИТТЛ.-112 с.
- 16. Неймарк Ю.И., Фуфаев Н.А. (1967) Динамика неголономных систем.-М:Наука.-520 с.
- Goroshko, O.A., Hereditary strain theory of synthetic and steel ropes, in Fracture of Nano and Engineering Materials and Structures (p. 1416), Edited by E.E.Gdoutos, Proceedings of the 16th European Conference of Fracture (p. 1416), Springer 2006, pp. 803-804, plus CD Full paper. ISBN 1-4020-4971-4, BARCOD 9 781402 049712. http://ecf16.civil.duth.gr
- Goroshko O.A. and Hedrih (Stevanović), K., (2007), Construction of the Lagrange mechanics of the hereditary systems, Minisymposium Oppening Lecture, Proceedings of the International Summer School APM – Advanced Problem in Mechanics, Saint Petersburg 2007., pp. 133-156.

KONSTRUKCIJA LAGRANGE-OVE MEHANIKE DISKRETNIH NASLEDNIH SISTEMA

Oleg Aleksandrovich Gorosko i Katica (Stevanović) Hedrih

Istraživački rezultazi u oblasti naslednih diskretnih sistema, koje su dobili autori ovog rada, su uopšteni i predstavljeni u monografiji [4], koja sadrži prvu kompletnu predstavu annalitičke dinamike naslednih diskretnih sistema. Dve klase dinamički određenih i dinamički neodređenih sistema su definisane i razmotrene u svetlu određenih ograničenja. Glavni rezultati analitičke dinamike diskretnih naslednih sistema su provereni na novim primerima značajnim za inženjersku praksu.

Aproksimacije izraza za koeficijent prigušenja i odgovarajući dekrement, kao i za kružnu frekvenciju oscilovanja naslednog oscilatornog sistema su dobijene sa visokom tačnošću u prvoj i drugoj aproksimaciji.

Analogija između nasledne interakcije i reaktivnih sila u sistemu automatskog upravljanja je otkrivena kao i mogućnost proširenja teorije analitičke dinamike diskretnih naslednih sistema na mehaničke sisteme sa automatskim upravljanjem.

Lagrange-ova mehanika diskretnih naslednih sistema je proširena i uopštena na termoreološke [4, 8] i piezo-reološke [4, 9] diskretne sisteme, kao i na diskretne mehaničke sisteme sa puzećim standardnim lakim elementima.

Ključne reči: Nasledni sistem, reološki element, reološko i relaksaciono jezgro, standardni kali nasledni element, integro-diferencijalna jednačina, izvod necelog reda, materijalna tačka, reonomna koordinata, reološka koordinata, reološko klatno, kovarijantne coordinate, termo-reološki i piezo-reološki standardni nasledni laki element

22