**FACTA UNIVERSITATIS**
Series: **Mechanics, Automatic Control and Robotics** Vol. 6, N° 1, 2007, pp. 65 - 73

# TWO REAL BODIES PROBLEM: COMPLEX HARMONY OF MOTIONS

*UDC 531.5+534.24(045)=111*

**Milutin Marjanov**

Faculty of Forestry, University of Belgrade

**Abstract**. *The gravitational interaction of two arbitrary shaped bodies, moving on the closed, periodic orbits is considered in this work. The stability of motions requires that every variable defining an aspect of the state has to be periodic and that any ratio between two arbitrary chosen periods of these variables has to be rational. It was shown in this work that such dynamical system implicates the perfect harmony of motions, with more than two hundred resonances.*

**Key words**: *periodicity, resonances*

## 1. Introduction

The necessary mathematical model for the classical two body problem in which only orbital motions are to be determined is a homogeneous gravitational field. In such a model intensities and the directions of the elementary forces acting on the body particles depend on the position of the mass center in the gravitational field only. All the elementary forces acting on the body particles are parallel to the direction relating the center of mass of the body and the center of gravitational attraction. Their sum, the "weight" of the body coincides with that line, passing trough the center of mass. Consequentially, these bodies may be replaced by the point mass particles situated in their centers of mass.

However, the study of rotational, together with orbital motions of the bodies under mutual interaction requires that an inhomogeneous gravitational field was adopted and that the real bodies, instead of the point mass particles were included into the corresponding model. In such a model intensities and the directions of the elementary forces depend on the position of the mass center in the gravitational field, on the position of the body in one relative reference frame and on position of the corresponding particle within the body. All these forces converge toward the center of attraction and so does their resultant.

The definition and classification of the inhomogeneous gravitational fields was made in the works [10,11]. The planar motion (rotational – orbital resonance) of the body

---

Received February 21, 2005

around the dominant center of gravitation (the center of attraction) in such a field was investigated in [12], and nonplanar motion in [13].

One contribution of this work is the formulation of the gravitational interaction, and its potential for an isolated system of two arbitrary shaped bodies of comparable masses, moving on the closed stable orbits around their common centre of mass**.** The model of the planar orbital and rotational motions of the bodies is assumed again.

The correspondent differential equations of motions represent the formulation of the two - real bodies problem and it was pointed out that the inherent symmetry of these expressions indicates that periodicity demands the existence of a complex harmony in these motions.

## 2. The Inertial Frame of Reference and Orbital Plane

Consider two, arbitrary shaped bodies of masses *m* and *m′* moving through the space, with velocities of their centers of mass *C* and *C'* $\vec{v}\circ$ and $\vec{v}'\circ$ respectively**.** These bodies represent one dynamic system with the mass center *C****.** When they arrive at the distance of the noticeable gravitational interaction, under the influence of the gravitational force, the motions of *C* and *C′* cease to be uniform, their paths start to bend and their velocities to change: bodies begin to orbit around the centre *C** as the common centre of gravitation.

The paths of two centers depend on the $\min\overline{CC'}$ and on the mechanical energy *E* (the sum of the energy of motion and of the gravitational potential energy) of the system at that point.

If $E(\min\overline{CC'})\geq 0$ the trajectories are unbound (hyperbolic and parabolic) and the gravitational interaction eventually fades away. On the other hand, negative *E* at that point produces closed orbits and continual interaction.

We restrict our considerations to the closed orbits.

Since the gravitational force represents an internal force of the system, in accordance with the linear momentum conservation law, the total momentum vector of the system remains constant: $\vec{K}^* = m\vec{v}\circ + m'\vec{v}'\circ = \overrightarrow{const}$. Therefore, the mass center *C** moves uniformly through the space with velocity $\vec{v}^* = \vec{K}^*/\ (m+m')$. Besides, the gravitational force is a central one, therefore the angular momenta of the bodies and consequently, the total angular momentum vector remains constant: $\vec{L}^* = \overrightarrow{CC^*}\, x\ m\vec{v}\circ + \vec{L}^C + \overrightarrow{C'C^*}\, x\ m'\vec{v}'\circ + \vec{L}^{C'} = \overrightarrow{const}$. Needless to say that the planar model requires that orbital and spin parts of the angular momenta have to be parallel.

It is convenient to display the position of the system in one inertial reference frame ρ, ψ, *z*, to direct 0z axis along the vector $\vec{L}^*$ and to take *C** for its origin. Corresponding unit vectors are $\vec{e}_1, \vec{e}_2$ and $\vec{e}_3$. This Galilean frame *moves translatory* with constant transport velocity $\vec{v}^*$ through the space. It is easy to show that $\vec{L}^*$ can remain constant only if the orbital plane stays at the right angle to this vector. This is, the so cold, *invariable or Laplace's plane* (Figs. 1a and 1b).

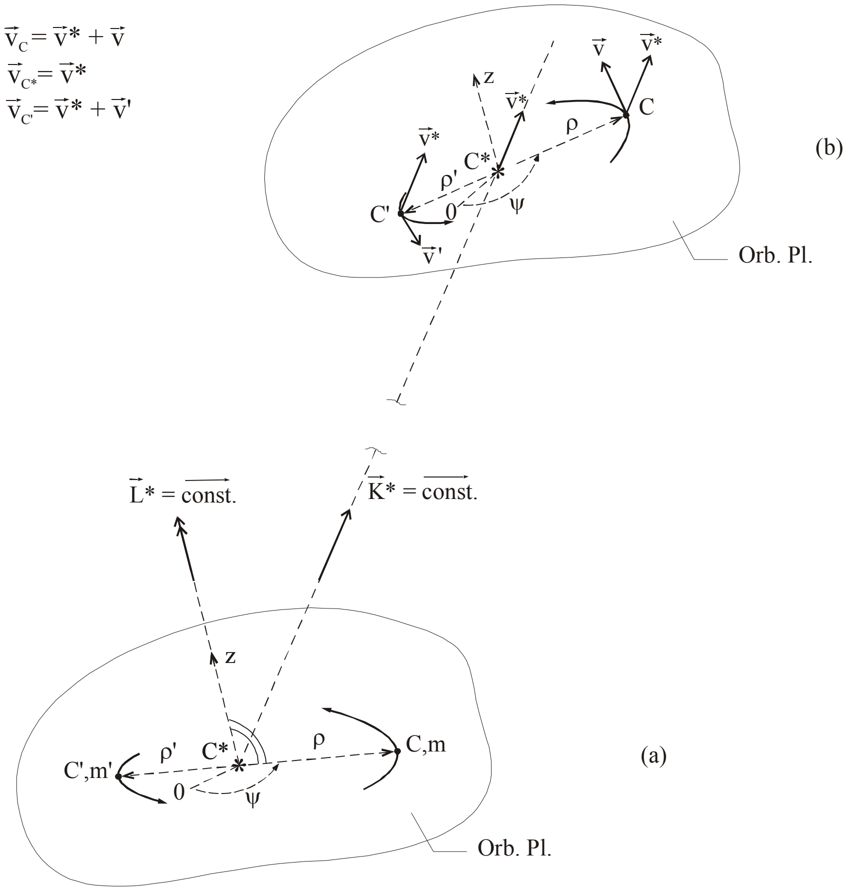


Fig. 1a $\vec{L}^*, \vec{K}^*$, Inertial Frame of Reference and Orbital Plane.
1b Transport and Relative Velocities of $C$, $C'$ and $C^*$.

## 3. THE MOTION ON STABLE ORBITS AND SIX FRAMES OF REFERENCE

The relative motions of the mass centers $C$ and $C'$ in the adopted frame are defined by the vectors

$$\vec{r} = \rho\vec{e}_1 \text{ and } \vec{r}' = \rho'\vec{e}_1,$$

where the total distance between two centers is $\overline{CC'} = \rho + \rho' = R$ . Since $C^*$ is the mass center of the system, these vectors and their derivatives have to be related in the same way

$$m\vec{r} = -m'\vec{r}', \tag{1^a}$$

$$m\vec{v} = -m'\vec{v}', \tag{1^b}$$

$$m\vec{a} = -m'\vec{a}'. \tag{1^c}$$

Dissipative forces (friction with the cosmic dust, tidal effects...) acting on two (haevenly) bodies are neglected in this model because of their smallness. But the work of these forces gradually slows down the orbital and rotational motions, so that eventually, the closed orbits become stable and the motions of the bodies along them periodic. The stable orbits of two bodies are represented in the Figure 2. Radii ρ and ρ′ of two similar ellipses are related in the proportion ($1^a$). The major axes of both orbits (the lines of apsides) coincide and the common centre of gravitation *C** is situated at their inversely disposed foci. If the orbits are circular, *C** lies in their common centre**.** The relation ($1^b$) requires the same directions of the orbital motions and the opposite directions of the velocities of *C* and *C*′. Since the orbital velocities are related in the same proportion as the radii of the orbits are, it is evident that the periods of revolution of both bodies have to be equal.

First of all, instead of one cylindrical, we shall use two polar inertial frames for each body separately, $C^*\rho\psi$ and $C^*\rho'\psi$ as it is shown in the Fig. 2.

Regarding the rotational motions, we assume that the bodies revolve around principal axes **(1)** and **(1')** of their ellipsoids of inertia and that those axes stay at the right angles with the orbital plane. Thus, we adopt planar motion for our model, as it was done in the work [12]. The mass centers of the bodies *C* and *C*′ are chosen to be the origins of two pairs of moving frames of reference *Cxy*, *C′x′y′* and *C*ξη, *C*′ξ′η′. The first two translational (along radial directions) displaced polar frames are related to the geometries of the corresponding orbits and the second two are related to the geometries of the corresponding masses: *C*ξ and *C*′ξ′ are directed along the principal axes **(3)** and **(3'),** while *C*η and *C*′η′ are directed along the principal axes **(2)** and **(2')** of the ellipsoids of inertia.

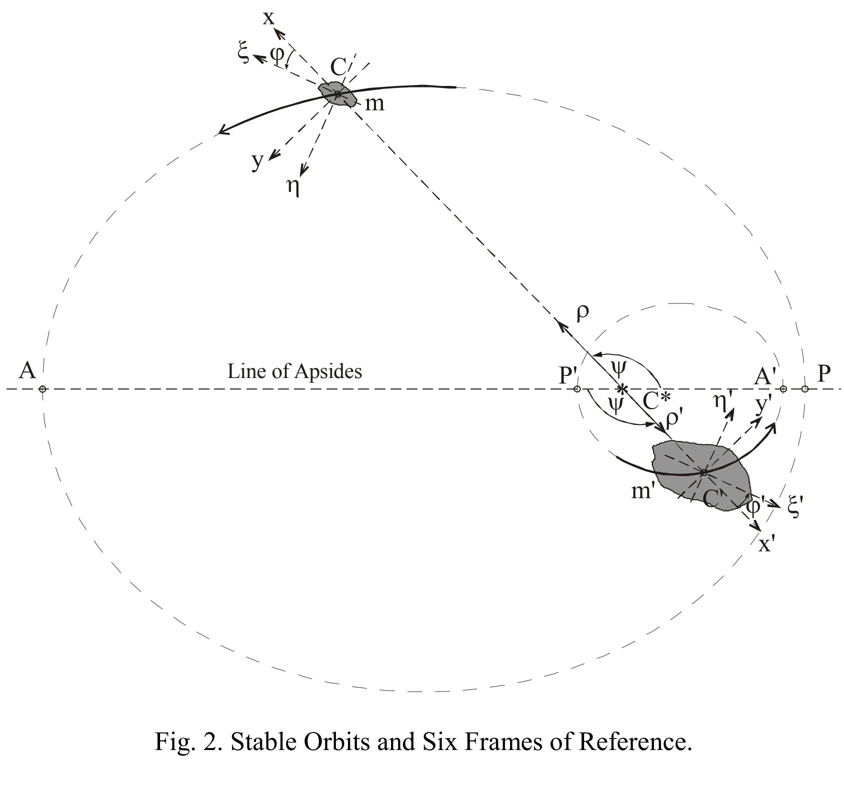


Fig. 2. Stable Orbits and Six Frames of Reference.

The positions of the second pair of frames with respect to the first one are defined by the angles of relative rotations $\varphi = \angle xC\xi$ and $\varphi' = \angle x'C'\xi'$ (Fig. 2).

## 4. Newton's Gravitational Force and its Potential

The gravitational interaction between two elementary masses *dm* and *dm'* (Fig. 3) is given by the following expression

$$d\vec{F} = \frac{Gdm\,dm'}{r_m^2}\frac{\vec{r}_m}{|\vec{r}_m|}, \quad (2)$$

Where $G$ represents the gravitational constant and $\vec{r}_m(-R-x-x',-y-y')$ is the vector relating the elementary masses of two bodies (Fig. 3). Note that in two, oppositely oriented frames of reference *Cxy* and *C'x'y'*, $\vec{r}_m$ (and, consequentially, $\vec{F}(X,Y)$) have the same form.

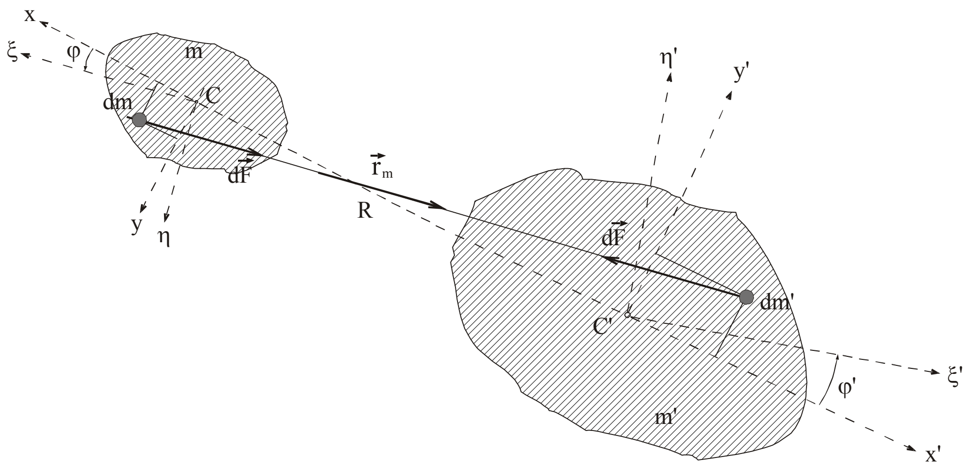


Fig. 3. Elementary Gravitational Force.

If dimensions of the bodies are small compared to their distance $R$, the expansion of the components of this elementary force into Newton's binomial series, in which terms up to the exponent two were retained (inhomogeneous gravitational field GF(2), see [12] the integration over the masses of both bodies, and introduction of the transformations of **x, y** and **x', y'** axes into ξ, ηand ξ′, η′, respectively, leads to the following components of the gravitational load:

$$X = -\frac{Gmm'}{R^2} - \frac{3G}{4R^4}(m'I_1 + mI_1') - \frac{9G}{4R^4}[m'(I_2 - I_3)\cos 2\varphi + m(I_2' - I_3')\cos 2\varphi'] \quad (3)$$

$$Y = \frac{3G}{2R^4}[m'(I_2 - I_3)\sin 2\varphi + m(I_2' - I_3')\sin 2\varphi'] \quad (4)$$

$$M^C = -\rho Y, \qquad M^{C'} = -\rho' Y. \tag{5}$$

In these expressions $I_1$, $I_2$, $I_3$ and $I'_1, I'_2, I'_3$ are the central principal moments of inertia of the respective bodies.

The gravitational interactions of two real bodies are represented in the Fig. 4.

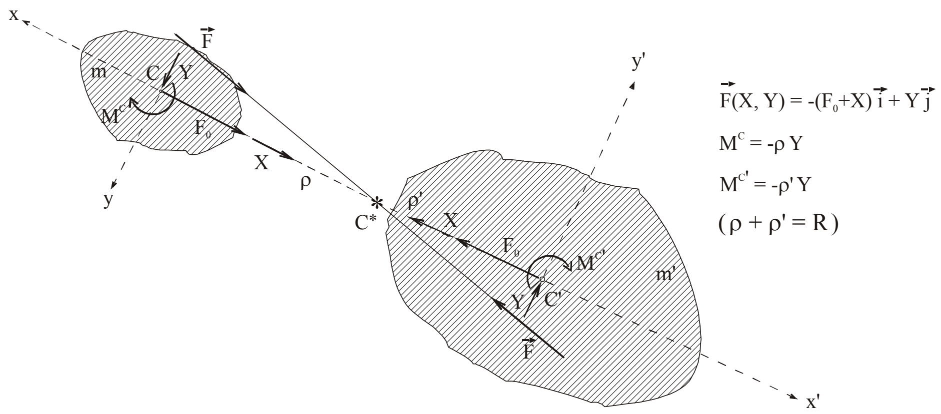


Fig. 4. Gravitational Interactions of Two Bodies.

There is one fact we want to emphasize here: in an inhomogeneous gravitational field, gravitational interactions are not symmetric, as a whole. Whereas the gravitational forces acting upon two bodies are opposite vectors, the gravitational moments $M^C$ and $M^{C'}$ have the same direction, while their magnitudes are inversely proportionate to the mass ratio of the bodies $|M^C| \,/\, |M^{C'}| = m' / m$. Thus, the sun, attracts Jupiter with the same force as Jupiter attracts the sun of course, but being about thousand times more massive than Jupiter, it produces a thousand times greater gravitational moment upon this planet, then vice versa.

The work of the gravitational force on motions of the bodies consists of the work of the component $X$ on the displacements along $R$ and of the work of the total moment $-RY$ on the relative rotations φ and φ′ of the bodies. It's evident that the potential energy existence condition

$$\frac{\partial(RY)}{\partial R} = \frac{\partial X}{\partial \varphi} + \frac{\partial X}{\partial \varphi'} \tag{6}$$

is fulfilled. The potential energy of the gravitational load acting on the two body system is given by the formula

$$U = -\frac{Gmm'}{R} - \frac{G}{4R^3}(m'I_1 + mI'_1) - \frac{3G}{4R^3}[m'(I_2 - I_3)\cos 2\varphi + m(I'_2 - I'_3)\cos 2\varphi'] \,. \tag{7}$$

Now, one can obtain the components of the gravitational force (3), (4) simply, finding respective negative partial derivatives of this expression

$$X = -\frac{\partial U}{\partial R} \text{ and} \tag{8}$$

$$Y = -\frac{1}{R}\left(\frac{\partial U}{\partial \varphi} + \frac{\partial U}{\partial \varphi'}\right). \tag{9}$$

As far as we know, (3), (4), (5) and (7) represent the first formulation of the Newton's gravitational interactions and their potential for two arbitrary shaped bodies

## 5. Differential Equations of Motion

By the use of the equations (3), (4) and (5) it is possible to write down the differential equations of motions for two real bodies. The equations for each body are written separately and in that way the inherent symmetry of these expressions is emphasized:

$$m(\ddot{\rho} - \rho\dot{\psi}^2) = X, \tag{10}$$

$$m'(\ddot{\rho}' - \rho'\dot{\psi}^2) = X, \tag{10'}$$

$$m(\rho\ddot{\psi} + 2\dot{\rho}\dot{\psi}) = Y, \tag{11}$$

$$m'(\rho'\ddot{\psi} + 2\dot{\rho}'\dot{\psi}) = Y, \tag{11'}$$

$$I_1(\ddot{\varphi} + \ddot{\psi}) = -\rho Y, \tag{12}$$

$$I_1'(\ddot{\varphi}' + \ddot{\psi}) = -\rho' Y. \tag{12'}$$

The classical two – body problem contains only equations describing orbital motions. In that case, on the right hand sides of the first pair (10) and (10') is Newton's gravitational force, the second pair (11) and (11') is homogeneous, and the third pair (12) and (12') does not exist.

From the expressions (1) it follows out that the equations (10) and (10'), as well as (11) and (11') are identities, so that only four equations, say (10), (11), (12) and (12'), with four variable functions of time, $\rho$, $\psi$, $\varphi$ and $\varphi'$ are sufficient to describe the motions of the system on the stable orbits.

It makes no sense, of course, to speak of the 'initial' conditions, because the entrance into stable orbit takes a certain period of time. Instead, observable data $\rho_0, \dot{\rho}_0, \psi_0, \dot{\psi}_0, \varphi_0, \dot{\varphi}_0, \varphi'_0$ and $\dot{\varphi}'_0$ at the moment $t_0$ have to be used to solve these equations.

## 6. Resonances

The periods of two functions characterizing the periodic motions of certain heavenly bodies are related as rational numbers. This phenomenon is denominated resonance in celestial mechanics [1, 2, 3, 4, 5…].

The results obtained in this work expose the fact that the stability of motions of two gravitationally interacting bodies requires the existence of resonance between any pair of constituents of the state of the system. Let us enumerate six principal resonances.

First of all, it was already mentioned, proportions ($1^a$) and ($1^b$) imply that orbital motions of two bodies are unison. So, the resonance in orbital motions is 1/1. The revolutions are in, the so-called, "ideal resonance".

It was shown in the works [9, 12] that the periodic motion of the arbitrary shaped body around the center of gravitation is possible only if the kinematical and dynamical extreme conditions were fulfilled at perigee and apogee of the orbit. These conditions require that the principal axes of inertia $\xi$ and $\xi'$ or $\eta$ and $\eta'$ (see Fig. 2) become coincident with the line of apsides every time when the bodies cross that line. In that case the relation between rotational and orbital periods has to be a rational number. For two bodies, it makes two more resonances.

Since the relation between two rational numbers has to be rational number again, the rotational periods of both bodies have to be in resonance, as well.

At last, at least formally, the rotational period of the first body has to be in resonance with the orbital period of the second one and *vice versa.*

On the whole, it makes six resonances between periods of variables describing the motions of this dynamic system.

But the problem of the number of resonances may be considered from the other viewpoint, too.

If we take all variable constituents of the differential equations of two bodies and their first and second derivatives, we shall have 2x3x3 = 18 characteristics of the state of this dynamic system. We can join to this number the other variables, for instance, sectorial velocities and accelerations for both bodies as well. Namely, if the ellipsoid of inertia is not rotational symmetric with respect to the axis of rotation, these quantities are variable, because of the existence of the right hand sides in the equations (11) and (11'), as it was shown in the work [12]. All in all, it makes 22 functions defining some aspect of state of this system. It is obvious that all these functions have to be periodic. Since all the variables are coupled, the stability of the orbits, that is, the periodicity of the motions requires $\binom{22}{2} = 231$ resonances.

In conclusion, it is possible to make a general statement that periodicity of the dynamic system requires that

- *all inherent attributes of motions are periodic and that*
- *any pair of periods is related as a rational fraction***.**

In other words, periodicity requires the perfect harmony between all variable elements of the system.

## 8. Conclusion

The gravitational interaction of two arbitrary shaped bodies moving on the stable orbits was studied in this work. Since the rotational motions were included into consideration it was necessary to adopt an inhomogeneous gravitational field as a mathematical model. In such a way, the gravitational load and its potential were derived.

After that, the differential equations of the orbital and rotational motions were established. The symmetry of the obtained expressions suggests that the stable motions of the bodies implies that all variables and their derivatives defining movements of this dynamic system have to be periodic and that every pair of these variables has to be in resonance.

## REFERENCES


1. Garfinkel B., (1982), *Celes. Mech.,* 28, pp. 275 – 290.
2. Sergysels R., (1989), *Celes. Mech.,* 44, 155.
3. Howland R. A., (1989) *Celes. Mech*., 44, 209.
4. De Moraes R. V. Da Silva & P.A.F., (1990) *Celes. Mech*., 47, 225.
5. Beletskii V. V., (1974), Resonance Phenomena at Rotations of Artificial and Natural Celestial Bodies, *G. E.O. Giacaglia, Springer Verlag*, Berlin, pp. 192 – 232.
6. Hawking S., Israel W., (1989), *300 Years of Gravitation*, Cambridge University Press.
7. Pars L. A., (1956), A Treatise of Analytical Dynamics, *Heinemann*, London.
8. Strom R. G., (1987), Mercury, *Smithsonian Institution Press*, London
9. Marjanov M., (1997), Gravitational Resonances, *Bull. Astron*., Belgrade, 156, 9 – 19.
10. Marjanov M., (2001), Inhomogeneous Gravitational Field and the Body without Centre of Gravity, *XXIII JUMEH Congress*, Belgrade.
11. Marjanov M., (2004), Homogeneous and Inhomogeneous Gravitational Fields, *Zbornik radova Gradjevin-skog fakulteta u Subotici.*
12. Marjanov M., (2005), On the Cause of Resonant Motions of Celestial Bodies, *Facta Universitatis, Series: Mechanics, Automatic Control and Robotics*, vol. 4, N° 17, 2005, Niš, 2005 (http//www.ni.ac.yu/Facta/index.htm).
13. Marjanov M., (2005), Motion of the Body in an Inhomogeneous Gravitational Field, *Facta Universitatis*, *MACAR,* vol.5, No1, Nis, 2006 pp. 79-89 .


# PROBLEM DVA REALNA TELA: SLOŽENA HARMONIJA KRETANJA

**Milutin Marjanov**

*U radu se razmatra gravitaciona interakcija dva tela proizvoljnih oblika koja se kreću u zatvorenim, periodičnim orbitama. S obzirom da stabilna orbitalna i rotaciona kretanja zahtevaju da sve promenljive koje definišu stanje sistema moraju biti periodične i da se svaki par perioda mora stajati u odnosu celih brojeva, pokazano je da u posmatranom dinamičkom sistemu mora postojati preko dve stotine rezonanci.*

Ključne reči: *periodičnost, rezonance*