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OPTIMAL DESIGN OF RING-STIFFENED SHELLS

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Abstract. *In this paper an influence of ring stiffness distribution along the shell defined through on the shell stiffness on the case on axially symmetric problem for cylindrical shell is investigated. Optimality conditions are formulated.*

Key words: *shell, homogenization, asymptotic method, optimization*

1. INTRODUCTION

In order to compute stiffened shells two approaches are usually applied. The first one is based on discretization of a studied construction using either FEM or FDM. An associated inversed problem is reduced to mathematical programming. Difficulties in getting a reliable solution increase with an increase of rings number N. Note that for non-uniformly stiffened shell N is equal to the number of design parameters.

The second approach is based on homogenization of the differential equations and it attracts a recent attention of both mathematicians and mechanical engineers (see, for instance, [1-4]). For the inversed problems this approach is reduced to an optimal design of a construction with distributed parameters [5, 6].

2. THE MAIN RESULTS

The differential equation governing deflection between rings of a considered shell has the following form

$$w^{IV} + bw = q\,, \tag{1}$$

where: $b = 12(1 - \nu^2) / R^2h^2$; $q = P(x) / D$; $D = Eh^3 / 12(1 - \nu^2)$; R is shell radius; h is shell thickness; E, ν are Young modulus and Poisson's coefficient of the shell and ring materials, respectively.

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The coupling condition of the i-th ring can be formulated in the following manner

$$w^- = w^+ ; (w')^- = (w')^+ ; (w'')^- = (w'')^+ ; (w''')^+ - (w''')^- = k(x) w_{x=is} , \tag{2}$$

where $(\ )^+$, $(\ )^-$ are intervals located to the right and to the left of the point $x = is$, where s is the distance between rings; $k(x) = EF(x) / (R^2 D)$, $F(x)$ is the area of transversal rings cross section.

The boundary conditions on edges $x = 0$, L, for the sake of simplicity, are taken in the form

$$w = w'' = 0 . \tag{3}$$

If the rings number is large ($s / L = \varepsilon << 1$), then in order to solve the problem (1)-(3), one can apply the asymptotic method of homogenization [1, 2].

Let us introduce the variable

$$\xi = x / \varepsilon \tag{4}$$

which is independent on x, and therefore, the associated differential operator reads

$$w' = w'_x + \varepsilon^{-1} w'_\xi . \tag{5}$$

The deflection w is sought in the form

$$w = w_0(x) + \varepsilon^4 w_1(x, \xi) + \varepsilon^5 w\ (x, \xi) + ..., \tag{6}$$

where w_i ($i = 1,2...$) are periodic functions with the period L and with respect to ξ.

Substituting (5), (6) into (1)-(4), and carrying out the asymptotic splitting with respect to ε powers, the following relations are obtained (periodicity conditions for w_i with respect to ξ are also applied):

$$w^{IV}_{1,\xi} + w^{IV}_{0,x} + \beta w_0 = q , \tag{7}$$

$$(w; w'_{1,\xi} ; w''_{1,\xi})_{\xi=0} = (w_1 ; w'_{1,\xi} ; w''_{1,\xi})_{\xi=L} , \tag{8}$$

$$w'''_{1,\xi/\xi=L} - w'''_{1,\xi/\xi=0} = K^*(x) w_0 , \tag{9}$$

$$w_{0/x=0,L} = w''_{0,L/x=0,L} = 0 . \tag{10}$$

Note that during derivation of relation (9), it has been assumed that $K^*(x) = LK / S \sim 1$.

Integrating (7) with respect to ξ, one gets

$$w_1 = (q - w^{IV}_{0,x} - \beta w_0)\xi^4 / 24 + C_1(x)\xi^3 + C_2(x)\xi^2 + C_3(x)\xi + C_4(x).$$

Determining $C_1 - C_4$ from conditions (8), one gets

$$w_1 = K^*(x)(q - w^{IV}_{0,x} - \beta w_0)\xi^2 (\xi - L)^2 / 24 . \tag{11}$$

Substituting (11) into (9), the following homogenized equation for w_0 is obtained

$$w^{IV}_{0,x} + (K^*(x) + \beta) w_0 = q . \tag{12}$$

Equation (12) governs the axially symmetric deformation of a structurally orthotropic shell with continuously distributed rings stiffness along the whole shell length. The corrector (11) accounts discreteness of rings distribution.

Consider first the case when variation of the rings stiffness is small, i.e.

$$K^{*}(x)+\beta = a+\varepsilon_1\varphi(x)\,, \tag{13}$$

where: $a = const$, $\varepsilon_1 << 1$.

The following series is assumed

$$w_0 = w_{00}+\varepsilon_1 w_{01}+\varepsilon_1^2 w_{02}+\ldots\ . \tag{14}$$

Substituting relations (13), (14) into equation (12) and comparing the coefficients standing by the same power of ε_1 to zero, one gets

$$w_{00,x}^{IV}+aw_{00}=q\ , \tag{15}$$

$$w_{0i,x}^{IV}+aw_{0i}=-\varphi(x)w_{0i-1},\qquad i=1,2\ldots\ . \tag{16}$$

Developing the functions $q(x)$, $\varphi(x)$, $w_0(x)$, $w_{0i}(x)$ into the Fourier series in the interval $[0, L]$ one obtains

$$q=\sum_{n-1}^{\infty}q_n\sin\alpha nx\quad \varphi=\sum_{n-1}^{\infty}\varphi_n\cos\alpha nx\quad W_{oi}=\sum_{n-1}^{\infty}A_{in}\sin\alpha nx\,, \tag{17}$$

where: q_n, φ_n, A_{in} – $const$, $\alpha = 2\pi / L$.

Substituting (17) into (15), (16) one obtains

$$A_{0n}=q_n/(\alpha^4 n^4+a)\ ,\quad A_{in}=B_{i-1n}/(\alpha^4 n^4+a)\,, \tag{18}$$

$$B_{in}=0.5\varphi_k(A_{ik+1}-A_{ik-n})\,,$$

and in result the following approximation holds

$$w_0=\sum_{n=1}^{\infty}\ \sum_{i=1}^{\infty}\varepsilon_1^i A_{in}\sin\alpha nx\,. \tag{19}$$

The corrector w_i is found from relation (11).

The solution (19) can be also extended into the case of non-small rings stiffness variations ($\varepsilon_1 \sim 1$) after an application of the Padé approximations [7].

In what follows the Padé approximation [1/1] for the series coefficients (19) gives

$$w_0=[[A_{0n}A_{1n}+\varepsilon_1(A_{1n}^2-A_{0n}A_{2n})]\ /(A_{1n}-\varepsilon_1 A_{2n})]\sin\alpha nx\,. \tag{20}$$

Consider the case $\alpha + \varepsilon_1\varphi(x) = c(1 - \varepsilon_1\cos 2\alpha x)$, c, q – $const$. The integral rings stiffness is constant for any ε_1 in this case. The coefficients of the series (18) take the form (for $\alpha = 1$):

$$A_{\ 02n-1}=4q/[(2n-1)\pi[(2n-1)^2+c]]\,,$$

$$A_{i,2n-1} = 0.5c(A_{i-1,2n-3} + A_{i-1,2n+1})/[(2n-1)^4 + c], \quad A_{i,2n} = 0 .$$

Now one can investigate how a change of ring stiffness influences a change of stiffness of the whole shell. For this purpose one must compare w_0 with w_0^* (note that the shell deflection possesses rings with the same stiffness ($\varepsilon = 0$)):

$$w_0 - w_0^* = D_{2n-1} \sin(2n-1)x ,$$

where: $D_{2n-1} = \varepsilon_1 A_{1,2n-1} / (A_{1,2n-1} - \varepsilon_1 A_{2,2n-1})$,.

The dependence D_1 / A_{01} characterizes shell stiffness variation (for $q = const$ the fundamental contribution into deflection is introduced by the first harmonic of the series (20)) reported in Figure 1.

The curves 1-5 correspond to $\varepsilon_1 = 0.1; 0.3; 0.5; 0.8; 1$, respectively.

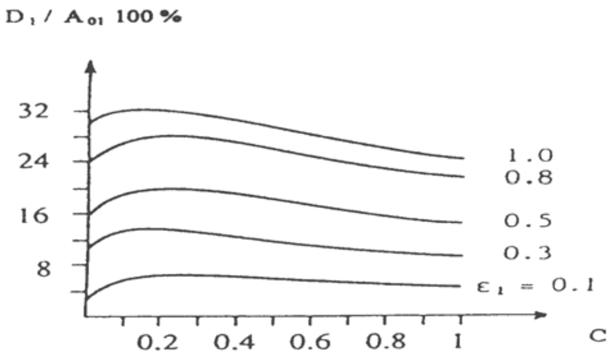

Fig. 1. Dependence D_1 / A_{01} vs c.

The analysis of the results shown in Figure 1 yields a conclusion that for given load $q = const$, the given rings stiffness distribution is particularly suitable in the interval $0.05 < c < 0.15$, and allows to decrease the largest shell deflection on amount of 30%.

Consider now the problem of optimisation, where the shell flexibility is taken as being the minimized functional of the form

$$I = \int_0^L qw dx \to \min_\kappa , \tag{21}$$

with the constraint

$$\int_0^L k \cdot \sum_{n=0}^N \delta(x - nl) dx = c . \tag{22}$$

If zero order approximation is used ($w = w_0$), then one has to add (10), (12) to the constraints. Therefore, following [5], the following new control function $\varphi(x)$ is applied

$$k = \alpha + \gamma \sin\varphi , \quad \alpha = 0.5(k_{\min} + k_{\max}) , \quad \gamma = 0.5(k_{\min} - k_{\max}) . \tag{23}$$

The inversed problem reads

$$I = \int_0^L qwdx \to \min_{\phi}\ , \quad I_1 = \int_0^L \sin\varphi dx = \frac{c-\alpha}{\gamma}\ , \tag{24}$$

$$w_{0,x}^{IV} + (\alpha + \gamma \sin\varphi) \quad w_0 = q\ , \tag{25}$$

$$w_{0/x=0,L} = w''_{0/x=0,L} = 0\ . \tag{26}$$

Following the approaches applied in the theory of optimal control with one variable, one gets the optimality condition of the problem (24)-(26). For this purpose one can write the expressions governing the first integrals (24) variations and equation in variations corresponding to (26), of the forms

$$\delta I = \int_0^L q\delta w dx\ ; \quad \delta I_1 = \int_0^L \cos\varphi\delta\varphi dx\ ; \tag{27}$$

$$\delta w_{0,x}^{IV} + (\alpha + \gamma \sin\varphi)\delta w_0 + \gamma \cos\varphi w_0 \delta\varphi = 0\ . \tag{28}$$

Notice that equation (28) is obtained first after substitution $w_0 + \delta w_0$, $\varphi + \delta\varphi$ instead of w_0 and φ_0 in (25), and after extraction of the terms linear with respect to δw_0 and $\delta\varphi$.

In what follows we are going to express the first variation of the minimized functional through the variation $\delta\varphi$.

For this purpose the conjugated variable $v(x)$ is introduced, which is defined through the condition that the expression for variation of the minimized functional does not include δw_0. Multiplying the left hand side of equation (28) by $v(x)$, and integrating it from 0 to L, one gets

$$\int_0^L v[\delta w_{0,x}^{IV} + (\alpha + \gamma \sin\varphi)\delta w_0 + \gamma \cos\varphi w_0 \delta\varphi]dx = 0\ .$$

Next carrying out the integration by parts with inclusion of (25), (26), the above integral is transformed to the following form

$$\int_0^L [(v^{IV} + (\alpha + \gamma \sin\varphi)v)\delta w_0 + \gamma\ v w_0\ \cos\varphi\ \delta\varphi]dx\ , \tag{29}$$

and the following boundary conditions are applied

$$v_{/x=0,L} = v''_{/x=0,L} = 0\ . \tag{30}$$

Following the methods of theory of optimal control with one variable, the variation δI_1 is attached to δI with a help of the Lagrange multiplier $\lambda = const$, and the relation (29) reads

$$\delta I = \int_0^L [(v^{IV} + (\alpha + \gamma\ \sin\varphi) + v + q)\delta w_0 + (\lambda + \gamma v w_0)\cos\varphi\delta\varphi]dx\ . \tag{31}$$

In order to satisfy independence of δI and δw_0, one gets

$$v^{IV} + (\alpha + \gamma \sin\varphi)v + q = 0\ . \tag{32}$$

The sought relation linking δI with $\delta\varphi$ is obtained for the first variation of the optimised functional of the form

$$\delta I = \int_0^L (\lambda + \gamma v w_0)\cos\varphi\ \delta\varphi\ dx\ .$$

Finally, it is easy to get the necessary optimization condition of the form

$$(\lambda + \gamma v w_0)\cos\varphi = 0\,. \qquad (33)$$

Observe that an inclusion of discreteness of strings distribution (11) complicates the considered problem essentially. In this case the minimized functional (21) with an account of (6), (12) has the following form

$$\mathrm{I} = \int_0^L \mathrm{q}(1 + \mathrm{k}^2 f(x)) w_0 dx\,, \qquad (34)$$

where: $f(x) = x^2(x - l) / 24$.

In this case one gets both equivalent equations to (32) and (33), respectively, of the forms

$$v^{IV} + (p + \gamma\sin\varphi)v = -q(1 + (\alpha + \gamma\sin\varphi)^2 f), \qquad (35)$$

$$\cos\varphi(\gamma v w_0 + 2(\alpha + \gamma\sin\varphi) f w_0 + \lambda) = 0 \qquad (36)$$

where: $p = \beta + \alpha$.

It is worth noticing that the considered optimization problem is reduced to that of solutions to the boundary value problems (12), (3) and (35), (30). Controlling function $\varphi(x)$ is found through optimality condition (36), whereas the constant λ is defined through izoparametric condition (25).

The obtained non-linear boundary value problem can be solved numerically either using one of the methods of successive optimisation [5] or applying a method of perturbations analogous to that used while solving the direct problem (13)-(16).

REFERENCES

1. Awrejcewicz, J., Andrianov, I.V., Manevitch, L.I. (1998) *Asymptotic Approaches in Nonlinear Dynamics: New Trends and Applications*, Springer-Verlag, Berlin.
2. Andrianov, I.V., Manevitch, L.I., Oshmyan, V.O. (2002) *Mechanics of Periodically Heterogeneous Structures,* Springer-Verlag, Berlin.
3. Bakhvalov, N., Panasenko, G. (1989) *Homogenization: Averaging Processes in Periodic Media. Mathematical Problems in Mechanics of Composite Materials*, Kluwer, Dordrecht.
4. Bensoussan, A., Lions, J.-L., Papanivolaou, G. (1978) *Asymptotic Methods for Periodic Structures*, North-Holland, New York.
5. Banichuk, N.V. (1990) *Introduction to Optimization of Structures*, Springer-Verlag, New York.
6. Lurie, K.A. (1993) *Applied Optimal Control of Distributed Systems*, Plenum Press, New York.
7. Baker, G., Graves-Morris, P. (1995) *Padé Approximants*, Addison-Wesley, London, Tokyo.

OPTIMALNO PROJEKTOVANA PRSTENASTA UKRUĆENJA LJUSKI

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U radu su predstavljeni rezultati izučavanja uticaja rasporeda prstenastih ojačanja duž ljuske definisani preko krutosti ljuske u slučaju aksijalno simetrične cilindrične ljuske. Formulisani su uslovi optimalnosti.

Ključne reči: *ljuska, homogenizacija, metoda anaizotropije, optimizacija*