FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol. 6, N° 1, 2007, pp. 89 - 95

2-π STRUCTURES ASSOCIATED TO THE LAGRANGIAN MECHANICAL SYSTEMS

UDC 531.3:532.511(045)=111

Victor Blãnuţã, Manuela Gîrţu

University of Bacău, Faculty of Sciences
email: vblanuta@ub.ro, manuelag@ub.ro

Abstract. *One defines the notion of 2-π structure on the phases space of a mechanical system and investigate its integrability.*

Key words: *2-π structures, 2-π structures associated to the Lagrangian mechanical systems.*

(*2005*) *Mathematics Subject Classification: 53C60*

INTRODUCTION

The theory of Finslerian mechanical systems has been realized by R.Miron and C. Frigioiu [7]. But, the general theory of Lagrangian mechanical systems was realized by R. Miron [4] and published also in the recent book [6] written by R. Miron and M. Bucãtaru.

In the present note we study the Lagrangian case associated to the phases space of a 2-π structure, using our ideas which we applied in the Finslerian and Lagrangian cases [1], [2], [9].

1. LAGRANGIAN MECHANICAL SYSTEMS

Consider a Lagrange space $L^n = (M, L(x, y))$ and a Lagrangian mechanical system $\Sigma = (M, L(x, y), F_i(x, y))$, where $F_i(x, y)$ are the external forces.

Following the Miron's theory we take the evolution equations of Σ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x^i}\right) - \frac{\partial L}{\partial y^i} = F_i, y^i = \dot{x}^i \; . \qquad (1.1)$$

These equations are equivalent with the system of differential equations of the second order:

Received February 26, 2005

$$\frac{d^2x^i}{dt^2}+2G^i(x,\dot{x})=\frac{1}{2}F^i \tag{1.2}$$

where

$$F^i = g^{ij}F_j \tag{1.3}$$

and

$$G^i=\frac{g^{is}}{2}\left(\frac{\partial^2 L}{\partial y^s \partial x^j}y^j-\frac{\partial L}{\partial x^s}\right). \tag{1.3'}$$

The system of differential equations (1.2) defines a dynamical system of the second order.

R.Miron characterises this system by means of a vector field on the phases space *TM*.

So, he proves the following theorem:

Theorem 1.1. (Miron [4])

1^0 The following operator

$S=y^i\frac{\partial}{\partial x^i}-2(G^i-\frac{1}{4}F^i)\frac{\partial}{\partial y^i}$ is a vector field on the manifold $\widetilde{TM}=TM\setminus\{0\}$ which depend only of the Lagrangian mechanical system Σ.

2^0 S is a semispray on the phases space $\widetilde{TM}$.

3^0 The integral curves of S are the solution curves of evolution equations (1.2).

The proof of this theorem can be found in the papers [4], [8].

2. CANONICAL NONLINEAR CONNECTIONS OF Σ

The geometry of Lagrangian mechanical systems is determinated by the geometry of the pair $(\widetilde{TM},S)$.

So, the nonlinear connection N of the mechanical system Σ is given by the coefficients

$$N^i_j=\frac{\partial}{\partial y^j}\{G^i-\frac{1}{4}F^i\}=\overset{\circ}{N}{}^i_j-\frac{1}{4}\frac{\partial F^i}{\partial y^j} \tag{2.1}$$

where $\overset{\circ}{N}{}^i_j=\frac{\partial G^i}{\partial y^j}$ are the coefficients of the canonical nonlinear connection $\overset{\circ}{N}$ of the associated Lagrange space $L^n=(M,L(x,y))$ of the mechanical system Σ.

Now, we remark that the distribution N of the canonical nonlinear connection N give rise to the direct decomposition:

$$T_u(\widetilde{TM})=N_u\oplus V_u \tag{2.2}$$

Let $\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\right)$ the local adapted basis to the distributions N and V:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} = \frac{\overset{\circ}{\delta}}{\delta x^i} + \frac{1}{4}\frac{\partial F^j}{\partial y^i}\frac{\partial}{\partial y^j}, \tag{2.3}$$

where

$$\frac{\overset{\circ}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \overset{\circ}{N}{}_i^j \frac{\partial}{\partial y^j}. \qquad (2.3)'$$

The dual adapted basis (dx^i, δy^i) has the 1-forms δy^i given by

$$\delta y^i = \overset{\circ}{\delta} y^i - \frac{1}{4}\frac{\partial F^i}{\partial y^j} dx^j. \tag{2.4}$$

The tensor of weak torsion of the nonlinear connection N is $t_{jk}^i = \frac{\partial N^i}{\partial y^k} - \frac{\partial N_k^i}{\partial y^j}$.

We have, by means at (2.1)

$$t_{jk}^i = -\frac{1}{4}\left(\frac{\partial^2 F^i}{\partial y^k \partial y^j} - \frac{\partial^2 F^i}{\partial y^j \partial y^k}\right) = 0. \tag{2.5}$$

Consequently, the nonlinear connection N of Σ is symmetric (because $t_{jk}^i = 0$).

The curvature tensor R_{jk}^i of system Σ is as follows

$$R_{jk}^i(x,y) = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}.$$

Thus, the following formula holds:

$$R_{jk}^i(x,y) = \overset{\circ}{R}{}_{jk}^i(x,y) + \frac{1}{4}\left(\frac{\partial F^m}{\partial y^k} B_{jm}^i - \frac{\partial F^m}{\partial y^j} B_{km}^i\right) - \frac{1}{4}\frac{\overset{\circ}{\delta}}{\delta x^k}\left(\frac{\partial F^i}{\partial y^j}\right) + \frac{1}{4}\frac{\overset{\circ}{\delta}}{\delta x^j}\left(\frac{\partial F^i}{\partial y^k}\right), \tag{2.6}$$

where

$$B_{jk}^i = \frac{\partial N_j^i}{\partial y^k} \tag{2.7}$$

are the coefficients of the Berwald connections of mechanical system Σ and $\overset{\circ}{R}{}_{jk}^i$ is the curvature tensor of the Cartan nonlinear connection of Finsler space F^n.

Therefore, we have,

Proposition 2.1.

The nonlinear connection of the mechanical system is integrable, if and only if the d-tensor field $R_{jk}^i(x,y)$ defined by (2.6) vanishes.

3. 2-π STRUCTURES ON THE PHASES SPACE OF Σ

Following our methods from the papers [1],[2], we define the 2-π structure on the manifold $\widetilde{TM}$, for the case of the Lagrangian mechanical systems Σ .

Definition 3.1.

An almost 2-π structure **F** of the mechanical system Σ is a tensor field **F** of type (1.1) which has the following property:

$$\mathbf{F}^2(X) = -\lambda^2 X, \forall X \in \chi.\,(TM) \tag{3.1}$$

where λ is one of the numbers {1,-1,*i*,-*i*}.

But, the canonical nonlinear connection N of the mechanical system Σ determines by the natural way such a 2-π structure.

Indeed, we define **F** on the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ by

$$\mathbf{F}\left(\frac{\delta}{\delta x^i}\right) = -\lambda\frac{\partial}{\partial y^i}, \mathbf{F}\left(\frac{\partial}{\partial y^i}\right) = \lambda\frac{\delta}{\delta x^i}. \tag{3.2}$$

We remark that the definition has the geometrical meaning because this respect to change of local coordinates of the manifold $\widetilde{TM}$ the equations (2) are invariants.

F is a tensor of type (1.1) given by

$$\mathbf{F} = -\lambda\frac{\partial}{\partial y^i}\otimes dx^i + \lambda\frac{\delta}{\delta x^i}\otimes \delta y^i. \tag{3.3}$$

Using (3.2) we can prove without difficultes (3.1) and (3.2).

Let us consider also 2-π structures $\overset{\circ}{\mathbf{F}}$ defined by the canonical nonlinear connection $\overset{\circ}{N}$ of the Lagrange space L^n.

By means of (2.2) and (2.3) we have

$$\mathbf{F} = \overset{\circ}{\mathbf{F}} + \frac{\lambda}{4}\left[\frac{\partial F^i}{\partial y^j}\frac{\partial}{\partial y^j}\otimes\overset{\circ}{\delta} y^i - \frac{\overset{\circ}{\delta}}{\delta x^i}\otimes\frac{\partial F^i}{\partial y^k}dx^k - \frac{1}{4}\frac{\partial F^i}{\partial y^j}\frac{\partial}{\partial y^j}\otimes\frac{\partial F^i}{\partial y^k}dx^k\right]. \tag{3.4}$$

Also, we have

$$\begin{cases} \mathbf{F}\left(\frac{\overset{\circ}{\delta}}{\delta x^i}\right) = -\lambda\left[\frac{\partial}{\partial y^i} + \frac{1}{4}\frac{\partial F^j}{\partial y^i}\mathbf{F}\left(\frac{\partial}{\partial y^j}\right)\right] \\ \mathbf{F}\left(\frac{\partial}{\partial y^i}\right) = \lambda\left[\frac{\overset{\circ}{\delta}}{\delta x^i} + \frac{1}{4}\frac{\partial F^j}{\partial y^i}\frac{\partial}{\partial y^j}\right] \end{cases} \tag{3.5}$$

The condition of integrability for the 2-π structure **F** is given by

$$N_{\mathbf{F}}(X,Y)=\mathbf{F}^2[X,Y]+[\mathbf{F}X,\mathbf{F}Y]-\mathbf{F}[\mathbf{F}X,Y]-\mathbf{F}[X,\mathbf{F}Y]=0 \quad (3.6)$$

where $N_{\mathbf{F}}$ is the Nijenhuis tensor. Let us calculate the integrability conditions $N_{\mathbf{F}}(X,Y) = 0$, by considering the following values for the pair (X,Y):

$$\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^h}\right),\left(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^h}\right),\left(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^h}\right).$$

At first time, we have

$$\mathbf{F}^2\left(\frac{\delta}{\delta x^i}\right)=-\lambda^2\frac{\delta}{\delta x^i},\mathbf{F}^2\left(\frac{\partial}{\partial y^i}\right)=-\lambda^2\frac{\partial}{\partial y^i} \quad (3.7)$$

and for $\left[\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^h}\right],\left[\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^h}\right]$ and $\left[\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^h}\right]$ one obtains

$$\begin{cases}\left[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^h}\right]=R^i_{jh}\frac{\partial}{\partial y^i}\\ \left[\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^h}\right]=\frac{\partial N^i_j}{\partial y^h}\frac{\partial}{\partial y^i}=B^i_{jh}\frac{\partial}{\partial y^i}\\ \left[\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^h}\right]=0\end{cases} \quad (3.8)$$

where N^i_j, R^i_{jh} and B^i_{jh} have the expression (2.1),(2.6) and (2.7).

Consequently, the following partial results are valid:

$$\mathbf{F}^2\left[\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^h}\right]=-\lambda^2R^i_{jh}\frac{\partial}{\partial y^i},\quad \left[\mathbf{F}\frac{\delta}{\delta x^j},\mathbf{F}\frac{\delta}{\delta x^h}\right]=0$$

$$\mathbf{F}\left[\mathbf{F}\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^h}\right]=\lambda^2\frac{\partial N^i_j}{\partial y^h}\frac{\delta}{\delta x^i},\quad \mathbf{F}\left[\frac{\delta}{\delta x^j},\mathbf{F}\frac{\delta}{\delta x^h}\right]=-\lambda^2\frac{\partial N^i_h}{\partial y^j}\frac{\delta}{\delta x^i}$$

$$\mathbf{F}^2\left[\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^h}\right]=-\lambda^2\frac{\partial N^i_j}{\partial y^h}\frac{\partial}{\partial y^i},\quad \left[\mathbf{F}\frac{\delta}{\delta x^j},\mathbf{F}\frac{\partial}{\partial y^h}\right]=\lambda^2\frac{\partial N^i_h}{\partial y^j}\frac{\partial}{\partial y^i}$$

$$\mathbf{F}\left[\mathbf{F}\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^h}\right]=0,\quad \mathbf{F}\left[\frac{\delta}{\delta x^j},\mathbf{F}\frac{\partial}{\partial y^h}\right]=-\lambda^2R^i_{jh}\frac{\delta}{\delta x^i}$$

$$\mathbf{F}^2\left[\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^h}\right]=0,\quad \left[\mathbf{F}\frac{\partial}{\partial y^j},\mathbf{F}\frac{\partial}{\partial y^h}\right]=\lambda^2R^i_{jh}\frac{\partial}{\partial y^i}$$

$$\mathbf{F}\left[\mathbf{F}\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^h}\right]=-\lambda^2\frac{\partial N_j^i}{\partial y^h}\frac{\delta}{\delta x^i}\ ,\ \mathbf{F}\left[\frac{\partial}{\partial y^j},\mathbf{F}\frac{\partial}{\partial y^h}\right]=\lambda^2\frac{\partial N_h^i}{\partial y^j}\frac{\delta}{\delta x^i}$$

Applying them, we conclude that

$$\begin{cases} N_{\mathbf{F}}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^h}\right)=-\lambda^2\left[R_{jh}^i\frac{\partial}{\partial y^i}+\left(\frac{\partial N_j^i}{\partial y^h}-\frac{\partial N_h^i}{\partial y^i}\right)\frac{\delta}{\delta x^i}\right] \\ N_{\mathbf{F}}\left(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^h}\right)=\lambda^2\left[R_{jh}^i\frac{\delta}{\delta x^i}-\left(\frac{\partial N_j^i}{\partial y^h}-\frac{\partial N_h^i}{\partial y^j}\right)\frac{\partial}{\partial y^i}\right] \\ N_{\mathbf{F}}\left(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^h}\right)=\lambda^2\left[R_{jh}^i\frac{\partial}{\partial y^i}+\left(\frac{\partial N_j^i}{\partial y^h}-\frac{\partial N_h^i}{\partial y^i}\right)\frac{\delta}{\delta x^i}\right] \end{cases} \tag{3.9}$$

Taking into account the expression of the torsion tensor $t_{jh}^i=\frac{\partial N_j^i}{\partial y^h}-\frac{\partial N_h^i}{\partial y^j}$ the following result follows:

Theorem 3.1.

An almost 2-π structure **F** of the mechanical system Σ is integrable if and only if the curvature tensor R_{jk}^i is given by:

$$R_{jk}^i(x,y)=\overset{\circ}{R}{}^i_{jk}(x,y)+\frac{1}{4}\left(\frac{\partial F^m}{\partial y^k}B_{jm}^i-\frac{\partial F^m}{\partial y^j}B_{km}^i\right)-\frac{1}{4}\frac{\overset{\circ}{\delta}}{\delta x^k}\left(\frac{\partial F^i}{\partial y^j}\right)+\frac{1}{4}\frac{\overset{\circ}{\delta}}{\delta x^k}\left(\frac{\partial F^i}{\partial y^k}\right) \tag{3.10}$$

is equal to zero.

§4. THE METRICAL 2-π STRUCTURE ASSOCIATED TO MECHANICAL SYSTEM Σ

Let us consider the N-lift G of the fundamental tensor g_{ij} of L^m

$$G=g_{ij}dx^i\otimes dx^j+g_{ij}\delta y^i\otimes\delta y^j \tag{4.1}$$

We remark that G is a Riemannian structure on $\widetilde{TM}$ which depend only of the mechanical system Σ.

By means of (2.4) G can be written in the form:

$$G=g_{ij}dx^i\otimes dx^j+g_{ij}\left(\overset{\circ}{\delta}y^i-\frac{1}{4}\frac{\partial F^i}{\partial y^k}dx^k\right)\otimes\left(\overset{\circ}{\delta}y^j-\frac{1}{4}\frac{\partial F^j}{\partial y^h}dx^h\right). \tag{4.2}$$

We have

Theorem 4.2. The pair (**F**, G) is an almost metric 2- π structure of the mechanical system Σ.

Proof. It is sufficient to prove the formula $G(\mathbf{F}X, \mathbf{F}Y) = \lambda^2 G(X, Y)$ using the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$.

We get

$$G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}\,,\; G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = 0\,,\; G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}\,,$$

$$G\left(\mathbf{F}\left(\frac{\delta}{\delta x^i}\right), \mathbf{F}\left(\frac{\delta}{\delta x^j}\right)\right) = \lambda^2 G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \lambda^2 g_{ij} = \lambda^2 G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)$$

$$G\left(\mathbf{F}\left(\frac{\delta}{\delta x^i}\right), \mathbf{F}\left(\frac{\partial}{\partial y^j}\right)\right) = 0$$

$$G\left(\mathbf{F}\left(\frac{\partial}{\partial y^i}\right), \mathbf{F}\left(\frac{\partial}{\partial y^j}\right)\right) = \lambda^2 G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \lambda^2 g_{ij} = \lambda^2 G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) \quad \text{q.e.d}$$

REFERENCES

1. Blănuţă, V., Hassan B.T. *Metrical homogeneous 2-π structure determined by a Finsler metric in tangent bundle.* Kluwer Academic Publishers FTPH (2003), 63-68.
2. Blănuţă, V., Gîrţu ,M., Yawata, M. *Natural n-almost 2-π structures on Cartan spaces of order 2* (to appear).
3. Miron, R. *The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics.* Kluwer Academic Publishers FTPH no. 82 (1997).
4. Miron, R., Anastasiei, M. *The Geometry of Lagrange Spaces. Theory and Applications.* Kluwer Academic Publishers FTPH no 59,(1994).
5. Miron, R., *A Lagrangian theory of relativity* , (I,II). Anal.St.Univ "Al.I.Cuza" Iaşi XXXII S.1 Math. f2,f3, (1986), 7-16, 37-62.
6. Miron, R., Bucătaru M. *The Finsler-Lagrange Geometry. Applications to Dynamical Systems* (to appear)
7. Miron, R., Frigioiu C. *Finslerian Mechanical Systems*. Algebras, Groups and Geometries, Nr.4, 2005
8. Miron, R, Hrimiuc,D., Shimada, H., Sabău V. *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publishers, FTPH nr. 118 (2000).
9. Sandovici, A., Blănuţă, V. *A class of metrical almost 2-π structures on tangent bundle*. Hadronic Press.Ims.Palm.Harbor F.C. 34682 (U.S.A.) (2002).

2-π STRUKTURA PRIDRUŽENA LAGRANE-OVOM MEHANIČKOM SISTEMU

Victor Blãnuțã, Manuela Gîrțu

Definisano je kretanje dinamičkog sistema 2-π structure u faznom prostoru mehaničkog sistema i izučavana je njegova integrabilnost.

Ključne reči: *2-π structure, 2-π structure pridružene Lagrange- ovom mehaničkom sistemu.*