FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol. 6, N° 1, 2007, pp. 97 - 106

THE FINITE STRIP METHOD IN THE ANALYSIS OF OPTIMAL RECTANGULAR BENDING BRIDGE PLATES

UDC 534.01(045)=111

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Abstract. *If the structure is moving then it is possible to reduce the dynamic problem to a static one by applying D'Alembert's principle of dynamic equilibrium in which an inertia force equal to the product of the mass and the acceleration is assumed to act on the structure in the direction of negative acceleration. For free vibration, the the system is vibrating in a normal mode, and it is possible to transform equilibrium equation into a standard eigenvalue problem. Various schemes have been developed for solving eigenvalue equations such as the one by Bishop et al. [2]. In this paper the finite strip method is used in the analysis of natural frequencies and the mode shapes of rectangular bending bridge plates.The point of our analysis was to calculate the lowest natural frequencies of different types of ribbed reinforced plates, so that we could compare them and determine which one of them is optimal. Optimal means that the plate has the lowest natural frequency for the given lenght.*

Key words: *finite strip method, free vibrations, bridge plates.*

1. THE FINITE STRIP DISPLACEMENT FUNCTIONS IN THE PROBLEM OF BENDING

The analysis of the transverse vibration of thin plates is usually performed using Kichhoff's presuppositions for plate strain. Let us observe the problem of bending of a finite strip presented in Fig. 1. The approximative function must satisfy the partial differential equation of the 4th order

$$\Delta\Delta w(x,y)=0. \tag{1}$$

If both ends simply supported, the function of deflection will be presented in the form

$$w(x,y)=\sum_{m=1}^{\infty} w_m(x)\sin(m\pi y/a), \tag{2}$$

Received February 21, 2007

where *m* represents series term, or number of the harmonic. For any single series the term we can anticipate the following polynome to represent the displacement amplitude $w(x)$:

$$w(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3, \tag{3}$$

where C_1–C_4 represent generalized displacements. This approximation enables the establishment of the compatibility of displacement w and first derivates dw/dx in the nodal lines of the discretizated structure presented in Fig. 1.

Using the condition: $\varphi = dw/dx$, after writing the polynome (3) for the nodal lines 1 and 2 with the coordinates x = 0 and x = b respectively, we obtain

$$\begin{bmatrix} w_0 \\ \varphi_0 \\ w_b \\ \varphi_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & b & b^2 & b^3 \\ 0 & 1 & 2b & 3b^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{4}$$

By the inversion of (4) we obtain the polynome coefficients

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{b^2} & -\frac{2}{b} & \frac{3}{b^2} & -\frac{1}{b} \\ \frac{2}{b^3} & \frac{1}{b^2} & -\frac{2}{b^3} & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} w_0 \\ \varphi_0 \\ w_b \\ \varphi_b \end{bmatrix}. \tag{5}$$

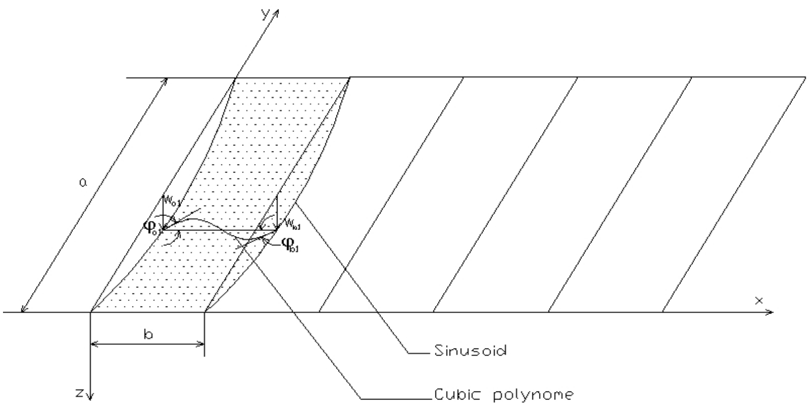


Fig. 1. Structure discretizated into a mesh of finite strips

The displacement amplitude $w(x)$ is now:

$$w(x) = \left(1 - \frac{3x^2}{b^2} + \frac{2x^3}{b^3}\right) \cdot w_0 + \left(x - \frac{2x^2}{b} + \frac{x^3}{b^2}\right) \cdot \varphi_0 + \left(\frac{3x^2}{b^2} + \frac{2x^3}{b^3}\right) \cdot w_b + \left(-\frac{x^2}{b} + \frac{x^3}{b^2}\right) \cdot \varphi_b \tag{6}$$

The final form of the part of the approximative function in the direction y for different conditions on the ends is:

a) both ends simply supported

$$Y_m(y) = \sin(\mu_m \cdot y/a), \quad \mu_m = \pi, 2\pi, ..., m\pi. \tag{7}$$

b) both ends clamped

$$\begin{aligned} &Y_m(y) = \sin(\mu_m \cdot y/a) - \sinh(\mu_m \cdot y/a) - \alpha_m \cdot \left[\cos(\mu_m \cdot y/a) - \cosh(\mu_m \cdot y/a)\right], \\ &\alpha_m = \frac{\sin(\mu_m) - \sinh(\mu_m)}{\cos(\mu_m) - \cosh(\mu_m)}, \mu_m = 4.7300, 7.8532, ..., \frac{(2m+1)\cdot\pi}{2}. \end{aligned} \tag{8}$$

c) one end clamped, the other free

$$\begin{aligned} &Y_m(y) = \sin(\mu_m \cdot y/a) - \sinh(\mu_m \cdot y/a) - \alpha_m \cdot \left[\cos(\mu_m \cdot y/a) - \cosh(\mu_m \cdot y/a)\right], \\ &\alpha_m = \frac{\sin(\mu_m) + \sinh(\mu_m)}{\cos(\mu_m) + \cosh(\mu_m)}, \mu_m = 1.8750, 4.6940, ..., \frac{(2m-1)\cdot\pi}{2}. \end{aligned} \tag{9}$$

d) one end simply supported, the other clamped

$$\begin{aligned} &Y_m(y) = \sin(\mu_m \cdot y/a) - \alpha_m \cdot \sinh(\mu_m \cdot y/a), \\ &\alpha_m = \frac{\sin(\mu_m)}{\sinh(\mu_m)}, \mu_m = 3.9266, 7.0685, ..., \frac{(4m+1)\cdot\pi}{4}. \end{aligned} \tag{10}$$

Being the mode shapes of vibration, these function satisfies the conditions of orthogonality:

$$\left.\begin{aligned} &\int_0^a Y_m \cdot Y_n \cdot dy = 0 \\ &\int_0^a Y_{,yym} \cdot Y_{,yyn} \cdot dy = 0 \end{aligned}\right\} \quad \text{for } m \neq n \tag{11}$$

2. PROBLEM OF FREE VIBRATION

The linear system of diferential equations of motion of one finite strip is:

$$M\ddot{q} + C\dot{q} + \hat{K}q = Q, \tag{12}$$

where M is the mass matrix, C is the damping matrix, $\hat{K}$ is the stiffness matrix and Q is the vector of generated forces of one finite strip, while q, $\dot{q}$ and $\ddot{q}$ represent the vectors of the generated displacements, velocities and accelerations respectively. If the external forces Q in equation (12) are equal to zero, the problem reduces to free vibration. There are two different cases: the free vibration with damping and the free vibration without damping. The second case is more simple, and it is described by the following homogeneous system of differential equations of motion

$$M\ddot{q} + \hat{K}q = 0. \tag{13}$$

In the analysis of the free vibration it is presupposed that all displacements alter in time according to the law of harmonic function, so that we shall introduce the expression:

$$q(t) = qe^{i\omega t}, \quad e^{i\omega t} = cos\,\omega t + i\,sin\,\omega t. \tag{14}$$

The matrix equation (13) is now

$$(\hat{K} - \omega^2 M)q = 0\,. \tag{15}$$

This is characteristic-value problem, so that eigenvalues and eigen vectors are obtained as results.

The various schemes have been developed for solving eigenvalue equations such as the one by Bishop et al. [2]. For more advanced techniques the mass condensation method [4], the subspace iteration method [1] and Lanczos method [3] should be used.

Note that a direct solution Eq. 15 is uneconomical because although both $[\hat{K}]^{-1}$ and $[M]$ are symmetrical, the product $[\hat{K}]^{-1}[M]$, however, is in general not symmetrical, and in practice some form of transformation similar to the process described in Chapter 6 of Reference [5] should be applied first.

For more detailed discussions on structural dynamics, readers should refer to other text such as the one by Cheung and Leung [3].

3. CONSISTENT MASS MATRIX AND STIFFNESS MATRIX OF FINITE STRIP

If we apply the same approximative function as those used for approximation of the strip displacement field to the calculation of the matrix M, we obtain the consistent mass matrix of the strip. For linear problem it is possible to study the problems of bending and plane stress separately.

Here we shall discuss the finite strip which have two degrees of freedom per nodal line in bending, and the approximative functions of displacement field:

$$A_{wm} = Y_{wm}[N_1 | N_2 | N_3 | N_4], \tag{16}$$

that is, they are defined separately,

$$A_{w1}, A_{w2}, \ldots, A_{wr}. \tag{17}$$

Now we have:

$$M_{ww} = \rho t \int_A [A_{w1} \quad A_{w2} \quad \cdots \quad A_{wr}]^T [A_{w1} \quad A_{w2} \quad \cdots \quad A_{wr}] dA =$$

$$= \rho t \int_A \begin{bmatrix} A_{w1}^T A_{w1} & A_{w1}^T A_{w2} & \cdots & A_{w1}^T A_{wr} \\ A_{w2}^T A_{w1} & A_{w2}^T A_{w2} & \cdots & A_{w2}^T A_{w1} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ A_{wr}^T A_{w1} & A_{wr}^T A_{w2} & \cdots & A_{wr}^T A_{w1} \end{bmatrix} dA. \tag{18}$$

The separate elements of the consistent mass matrix are obtained according to the expression

$$M_{wwmn} = \rho t \int_A A_{wm}^T A_{wn} dA, \quad m, n = 1, 2, ..., r. \tag{19}$$

The functions Y_{wm}, which are selected for finite strips with different boundary conditions, satisfy the conditions of orthogonality, so that the mass matrices M_{wwmn} for $m \neq n$ are equal to zero. We have only

$$M_{wwmm} = \rho t \int_A A_{wm}^T A_{wm} dA. \tag{20}$$

Series terms can be separated, and the whole procedure can be carried out for each term separately. Naturally, in this case the elements in the stiffness matrix must also be separated.

Using (16), we have

$$M_{wwmm} = \rho t \int_A N_w^T N_w dx I_{21}, \tag{21}$$

where I_{21} is integral defined with

$$I_{21} = \int_0^a Y_{wm} Y_{wm} dy.$$

After multiplication and integration of the functions we shall have

$$M_{wwmm} = \rho t I_{21} \begin{bmatrix} \frac{13b}{35} & \frac{11b^2}{210} & \frac{9b}{70} & -\frac{13b^2}{420} \\ & \frac{b^3}{105} & \frac{13b^2}{420} & -\frac{3b^3}{420} \\ & & \frac{13b}{35} & -\frac{11b^2}{210} \\ symmetrical & & & \frac{b^3}{105} \end{bmatrix} \tag{22}$$

The functions Y_{wm} select one of the four types of finite strips, in dependence on the support conditions on ends.

If we present the stiffness matrix $\hat{K}_{wwmm}$ in the function Y_{wm} and its derivates, we shall have:

$$\hat{K}_{wwmn} = \int_A B_{w3m}^T D_{22} B_{w3n} dA,$$

$$D_{22} = \begin{bmatrix} D_x^{22} & D_1^{22} & 0 \\ D_1^{22} & D_y^{22} & 0 \\ 0 & 0 & D_{xy}^{22} \end{bmatrix},$$

$$B_{w3n} = L_3 A_{wn} = \begin{bmatrix} -A_{w,xx} \\ -A_{w,yy} \\ -2A_{w,xy} \end{bmatrix} \begin{bmatrix} -N_{w,xx} Y_{wn} \\ -N_w Y_{w,yyn} \\ -N_{w,x} Y_{w,yn} \end{bmatrix}, \tag{23}$$

$$\hat{K}_{wwmn} = \int_A \begin{bmatrix} N_{w,xx}^T Y_{wm} D_x^{22} N_{w,xx} Y_{wn} + \\ N_w^T Y_{w,yym} D_1^{22} N_{w,xx} Y_{wn} + \\ N_{w,xx}^T Y_{wm} D_1^{22} N_w Y_{wn} + \\ N_w^T Y_{w,yym} D_y^{22} N_w Y_{w,yyn} + \\ 4N_{w,x}^T Y_{w,ym} D_{xy}^{22} N_{w,x} Y_{w,yn} \end{bmatrix} dA,$$

where:

$$I_{21} = \int_0^a Y_{wm} Y_{wn} dy,$$

$$I_{22} = \int_0^a Y_{w,yym} Y_{wn} dy,$$

$$I_{23} = \int_0^a Y_{wm} Y_{w,yyn} dy, \tag{24}$$

$$I_{24} = \int_0^a Y_{w,yym} Y_{w,yyn} dy,$$

$$I_{25} = \int_0^a Y_{w,ym} Y_{w,yn} dy.$$

For the finite strip with simply supported ends from the orthogonality conditions (11) integrals I_{21} and I_{24} are equal to zero for $m \neq n$. Apart from that, for the function (7) the remaining integrals in (24) are also equal to zero for $m \neq n$. For this reason, all stiffness matrix blocks are equal to zero when $m \neq n$, so that the general form of the stiffness matrix has this structure,

$$\hat{K}_{ww} = \begin{bmatrix} \hat{K}_{ww11} & 0 & \dots & 0 \\ 0 & \hat{K}_{ww22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \hat{K}_{wwrr} \end{bmatrix}. \tag{25}$$

The expression (25) enables calculation of displacements for each series term seperatly. The results are summed up at the end. The general form of the stiffness matrix is:

$$\hat{K}_{wwmm} = \int_A B_{w3m}^T D_{22} B_{w3m} dA,$$

$$\hat{K}_{wwmm} = \begin{bmatrix} \hat{K}_{ww11} & \hat{K}_{ww12} & \hat{K}_{ww13} & \hat{K}_{ww14} \\ & \hat{K}_{ww22} & \hat{K}_{ww23} & \hat{K}_{ww24} \\ & & \hat{K}_{ww33} & \hat{K}_{ww34} \\ symmetrical & & & \hat{K}_{ww44} \end{bmatrix}_{mm} . \quad (26)$$

4. THE ANALYSIS OF OPTIMAL RECTANGULAR RIBBED REINFORCED BRIDGE PLATES

For our calculation we used software FSMFV. It was written by prof. dr Dragan Milašinović in FORTRAN in 1984 [1]. This program determines eigenvalues and eigenvectors of free harmonic vibration of rectangular plates with different boundary conditions. In accordance to this, the stiffness matrices $\hat{K}_{wwmm}$ and consistent mass matrices M_{wwmm} are used. For the stiffness matrices of strips with support conditions different from those of simply supported on both ends, coupling between the harmonics is neglected. Series terms are separated, and the procedure can be carried out for each term separately.

The point of our analysis was to calculate the lowest natural frequencies of different types of ribbed reinforced plates, so that we could compare them and determine which one of them is optimal. Optimal means that the plate has the lowest natural frequency for the given lenght. The purpose of this analysis is to increase the natural period of the structure so that the acceleration response of the structure is decreased during earthquake.

We analysed the reinforced plates with width of B = 10m and lenght L = 5 – 10m. The thickness of the plate is d = 0.2m and height of the ribbs is h = 0.75 – 1.5m. The amount of the ribs is n = 2 – 8. Fixed ratio of ribb's width/height is b/h = 1/3 . The width of the ribbs is b = 0.25 – 0.5m, and distance between them is b1 = 0.857 – 9.5m.

We assumed that these plates are part of bridge construction, so we analysed three boundary conditions in longitudinal direction:

- simply supported ends
- clamped ends
- one end simply supported and the other clamped,

while we assumed that the plate ends are free in transverse direction (the direction of discretization). The results are shown in Table 1.

The overview of the input parameters:

- fixed parameters:
  B=10m,
  D=0.2m,
  b/h=1/3,
- variated parameters:
  L=5, 10, 15, 20m (ΔL=5m),
  h=0.75, 1.00, 1.25, 1.5m (Δh=0.25m),
  n=2, 3, 4, 5, 6, 7, 8 .

The characterstics of plate material (reinforced concrete):

E_x = 35820000 kN/m^2 , elasticity modulus
E_y = 35820000 kN/m^2, elasticity modulus

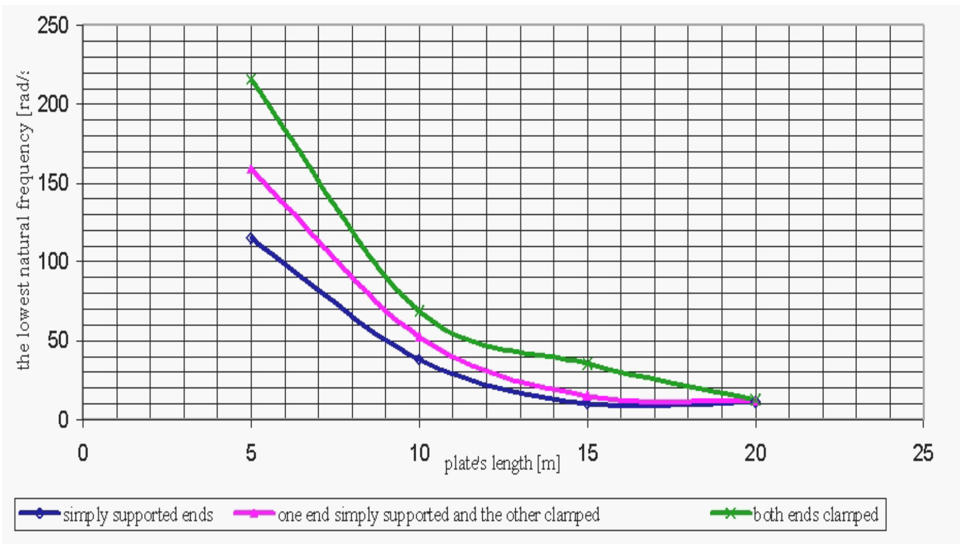
G_{xy} = 17910000 kN/m^2 , shear modulus
ν_x = ν_y =0.0, poisson's coefficients
ρ= 2.5 t/m^3 , mass density

The total amount of analysed plates, for each type of boundary conditions, is 4x4x7=112.

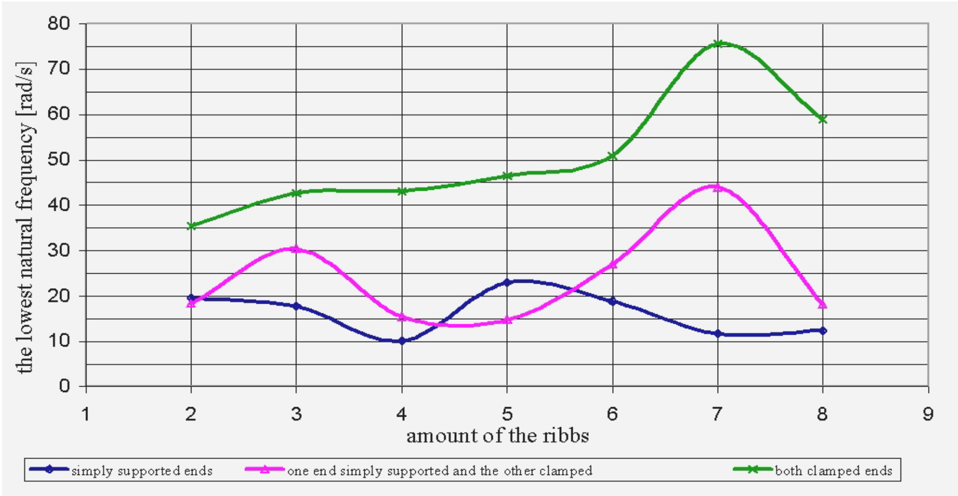
5. RESULTS

Table 1. – The lowest natural frequencies of plates for each type of boundary conditions for plate's lenght of 5, 10, 15 and 20 meters

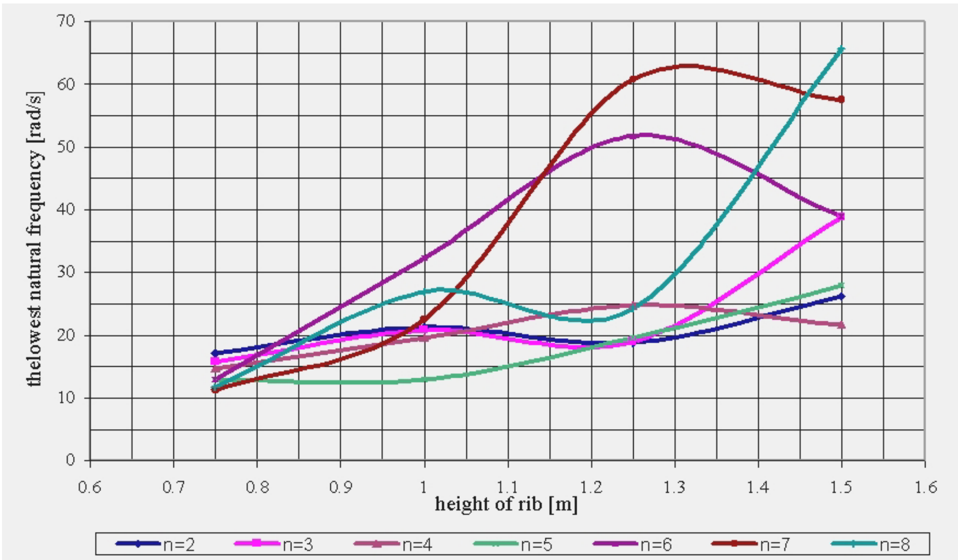
boundary conditions	L [m]	h [m]	n	ω [rad/s]	T [s]
simply supported ends	5	0.75	2	114.837	0.055
	10	0.75	2	38.255	0.164
	15	0.75	4	**10.070**	**0.624**
	20	0.75	7	11.320	0.555
one end simply supported and the other clamped	5	0.75	2	158.945	0.040
	10	0.75	2	52.331	0.120
	15	0.75	5	14.762	0.426
	20	0.75	8	**11.410**	**0.551**
both clamped ends	5	0.75	2	215.841	0.029
	10	0.75	2	68.596	0.092
	15	0.75	2	35.398	0.177
	20	0.75	8	**12.485**	**0.503**



Graph 1. Functions ω(L) for each type of boundary conditions (see Table 1.)



Graph 2. Functions ω(n) for plate's lenght L = 15m.



Graph 3. Functions ω(h) for each amount of ribbs (n = 2 – 8)
for simply supported plate L = 20m.

Acknowledgement: *The paper is a part of the research done within the project "Urban renewal and sustainable development" of Faculty of Architecture and Civil Engineering, Banjaluka*

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METODA KONAČNIH TRAKA U ANALIZI OPTIMALNIH PRAVOUGAONIH SAVIJAJUĆIH MOSTOVSKIH PLOČA

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Ako konstrukcija vibrira, moguće je dinamički problem svesti na statički primjenom D'Alembert-ovog principa dinamičke ravnoteže. Prema tom principu inercijalne sile koje djeluju na konstrukciju su jednake proizvodu mase i ubrzanja te djeluju u suprotnom smjeru u odnosu na ubrzanje. Kod slobodnih vibracija sistem osciluje u normalnom modu te je moguće transformisati jednačine ravnoteže u standarni problem svojstvenih vrijednosti. U teoriji konstrukcija su razvijeni različitii metodi za rješavanje problema svojstvenih vrijednosti, na primjer Bishop-ov metod [2]. U ovom radu metoda konačnih traka je korištena u analizi svojstvenih frekvencija i svojstvenih vektora pravougaonih savijajućih mostovskih ploča. Cilj naše analize je bio da pronađemo najniže svojstvene frekvencije raznih tipova rebrastih armirano-betonskih ploča kako bi utvrdili koje od njih su optimalne sa stanovišta dinamike konstrukcija, tj. koje od njih imaju najviše vrijednosti perioda oscilovanja.

Ključne reči: *metoda konačnih traka, slobodne vibracije, mostovske ploče.*