FACTA UNIVERSITATIS
Series: **Mechanics, Automatic Control and Robotics** Vol. 6, N° 1, 2007, pp. 113 - 118

THE POST-BUCKLING ANALYSIS OF PLATES. A BEM BASED MESHLESS VARIATIONAL SOLUTION

UDC 519.214

John T. Katsikadelis[1], Nick Babouskos[2]

[1]School of Civil Engineering & Academy of Athens, Office of Theoretical and Applied Mechanics
National Technical University of Athens, Zografou Campus, 157 73, Athens Greece
e-mail: jkats@central.ntua.gr
[2]School of Civil Engineering, National Technical University of Athens,
Zografou Campus, 157 73, Athens Greece
e-mail: babouskosn@yahoo.gr

Abstract. *A new BEM based variational method is presented for post-buckling analysis of thin elastic plates of arbitrary shape under general boundary conditions. The response of the plate is described by three coupled nonlinear partial differential equations in terms of displacements, which are derived on the base of the von Kármán assumption. The solution is achieved using the AEM. The numerical examples demonstrate the efficiency and validate the accuracy of the developed method.*

Key words: *Boundary element method, plates, analog equation, nonlinear, variational, optimal multiquadrics.*

1. INTRODUCTION

The post-buckling response of thin elastic plates is very important in engineering analysis. The reason is that thin plates after buckling are still capable of carrying a much increased load without failure provided that the stress is in the elastic range. Therefore, it is essential to consider the post-buckling behavior of the plate in order to benefit from this additional strength. The precice analysis of the post-buckling response proves to be quite difficult, because the governing equations are nonlinear. For this reason very few analytical or approximate solutions are available in the literature [1,2]. The BEM has been developed for large deflection analysis of plates based on the *von Kármán* plate theory [3,4], in which the Berger's assumption was used to simplify the nonlinear equations.

In this paper a new BEM solution is presented for post-buckling analysis of plates using the AEM. According to this method the original equations are converted into a linear plate (biharmonic) equation for the transverse deflection and two linear membrane equations for

Received September 11, 2007

the inplane deformation under fictitious loads and subjected to the same boundary conditions. The fictitious loads are approximated with radial basis function series of multiquadric (MQ) type. The integral representation of the solution of the substitute problems yields admissible shape functions, which are global and satisfy both essential and natural boundary conditions. Using these shape functions the solution of the original problem is represented by a Ritz expansion. The minimization of the total potential yields the Ritz coefficients and permits the evaluation of optimal values for the shape parameters of the MQs and the optimal position of the interior collocation points, minimizing thus the error. Since the arising domain integrals are converted into boundary line integrals, the method is boundary-only. First the linear buckling problem is solved and the resulting eigenmodes are employed to establish the small initial transverse load that excites possible deformation patterns. Subsequently, the post-critical response is studied by gradually increase the inplane loads. Several examples are studied which demonstrate the efficiency and validate the accuracy of the developed method and also give a better insight to this difficult engineering problem.

2. GOVERNING EQUATIONS

2.1 The Nonlinear plate problem

Consider a thin elastic plate of uniform thickness h occupying the two dimensional multiply connected domain Ω of the xy plane with boundary $\Gamma \cup_{i=0}^{K} \Gamma_i$, which may be piece-wise smooth. The plate is subjected to transverse loading $f(x,y)$. The boundary may be elastically supported with transverse and rotational restraint stiffness k_T and k_R respectively. The von Kármán assumptions for kinematic relations are adopted. That is

$$\varepsilon_x = u_{,x} + \frac{1}{2} w_{,x}^2 \qquad \varepsilon_y = v_{,y} + \frac{1}{2} w_{,y}^2 \qquad \gamma_{xy} = u_{,y} + v_{,x} + w_{,x} w_{,y} \tag{1a,bc}$$

where u, v are the membrane and w the transverse displacements. The plate undergoes bending combined with inplane deformation which is described by the following boundary value problems.
(i) Transverse deflection

$$D\nabla^4 w - (N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy}) + b_x w_{,x} + b_y w_{,y} = f \quad \text{in } \Omega \tag{2}$$

$$Vw + N_n^* w_{,n} + N_t^* w_{,t} + k_T w = V_n^* \text{ or } w = w^*, \; Mw - k_R w_{,n} = M_n^* \text{ or } w_{,n} = w_{,n}^* \text{ on } \Gamma \tag{3a,b}$$

$$k_T^{(k)} w^{(k)} - [\![Tw]\!]_k = R_k^* \quad \text{or} \quad w^{(k)} = w_k^* \quad \text{at corner point k} \tag{3c}$$

(ii) Inplane deformation

$$\nabla^2 u + \frac{1+\nu}{1-\nu}\left(u_{,x} + v_{,y}\right)_{,x} + w_{,x}\left(\frac{2}{1-\nu} w_{,xx} + w_{,yy}\right) + \frac{1+\nu}{1-\nu} w_{,x} w_{,y} + \frac{b_x}{G} = 0$$

$$\nabla^2 v + \frac{1+\nu}{1-\nu}\left(u_{,x} + v_{,y}\right)_{,y} + w_{,y}\left(\frac{2}{1-\nu} w_{,yy} + w_{,xx}\right) + \frac{1+\nu}{1-\nu} w_{,x} w_{,y} + \frac{b_y}{G} = 0$$

in Ω (4a,b)

$$N_n = N_n^* \text{ or } u_n = u_n^* \qquad N_t = N_t^* \text{ ή } u_t = u_t^* \qquad \text{on } \Gamma \tag{5a,b}$$

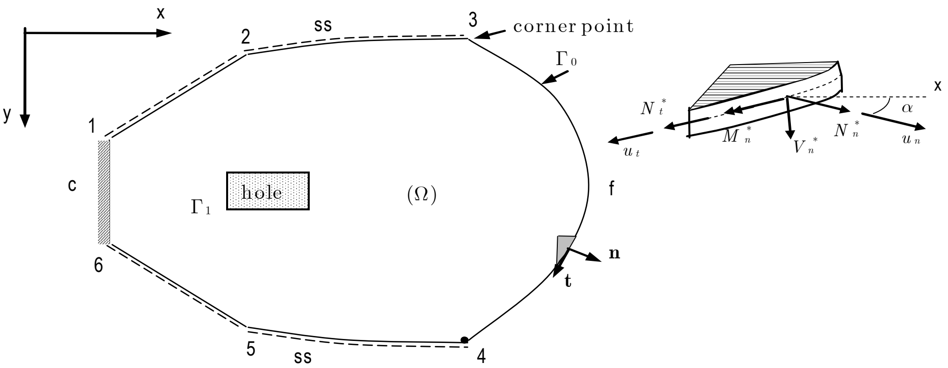


Fig. 1. Plate geometry and supports (c=clamped, ss=simply supported, f=free)

where $D = Eh^3 / 12(1-\nu^2)$ is the flexural rigidity of the plate and $G = E / 2(1+\nu)$ the shear modulus. Vw is the equivalent shear force, Mw is the normal bending moment and Tw the twisting moment on the boundary. $[\![Tw]\!]_k$ is the discontinuity jump of the twisting moment at the corner. Moreover, the quantities N_x, N_y, N_{xy} represent the inplane stress resultants. The total potential is given as

$$\begin{aligned}\Pi = {} & \frac{D}{2}\int_\Omega \left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx} w_{,yy} + 2(1-\nu) w_{,xy}^2 \right] d\Omega \\ & + \frac{C}{2}\int_\Omega \left[\left(u_{,x} + \frac{1}{2} w_{,x}^2 \right)^2 + \left(v_{,y} + \frac{1}{2} w_{,y}^2 \right)^2 + 2\nu \left(u_{,x} + \frac{1}{2} w_{,x}^2 \right)\left(v_{,y} + \frac{1}{2} w_{,y}^2 \right) + \frac{1-\nu}{2}\left(u_{,y} + v_{,x} + w_{,x} w_{,y} \right)^2 \right] d\Omega \\ & + \frac{1}{2}\int_\Gamma \left(k_T w^2 + k_R w_{,n}^2 \right) ds - \int_\Omega \left(f w + b_x u + b_y v \right) d\Omega - \int_\Gamma \left(N_n^* u_n + N_t^* u_t + V_n^* w - M_n^* w_{,n} \right) ds - \sum_k R_k^* w^{(k)}\end{aligned} \tag{6}$$

where $C = Eh / (1-\nu^2)$ is the membrane stiffness of the plate. The condition $\delta\Pi = 0$ yields the boundary value problem (2)-(5). The plate problem is linearized if the stretching of the middle surface due to bending is neglected, that is if it is set $w_{,x}^2 = w_{,y}^2 = w_{,x} w_{,y} = 0$ in Eqs (1). In this case the two problems are uncoupled. For linear buckling due to edge loading it is $f = b_x = b_y = 0$ and the eigenvalue problem is derived as

$$D\nabla^4 w - \lambda(N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy}) = 0 \qquad \text{in } \Omega \tag{7}$$

$$Vw + \lambda\left(N_n^* w_{,n} + N_t^* w_{,t} \right) + k_T w = 0 \text{ or } w = 0, \; Mw - k_R w_{,n} = 0 \text{ or } w_{,n} = 0 \text{ on } \Gamma \tag{8a,b}$$

$$k_T^{(k)} w^{(k)} - [\![Tw]\!]_k = 0 \quad \text{or} \quad w^{(k)} = 0 \quad \text{at corner point k} \tag{8c}$$

where λ is the load parameter. The forces N_x, N_{xy}, N_y are established by solving independently the boundary value problem (4a,b), (5a,b) using the BEM [5].

3.1 The AEM solution for the plate equation

The boundary value problem (2)-(5) is solved using the AEM [4]. The analog equation for the problem at hand is

$$\nabla^4 w = b(\mathbf{x}) \tag{9}$$

where $b(\mathbf{x})$ represents the fictitious load. Equation (9) under the boundary conditions (3) is solved using the BEM. Thus, the solution is obtained in integral form as

$$w(\mathbf{x}) = \int_\Omega vbd\Omega + \int_\Gamma (vVw + w_{,n}\, Mv - v_{,n}\, Mw - wVv)ds - \sum_k \left(v[\![Tw]\!] - w[\![Tv]\!]\right)_k \quad \mathbf{x} \in \Omega \tag{10}$$

which yields the following two boundary integral equations for points where the boundary is smooth

$$\frac{1}{2} w(\mathbf{x}) = \int_\Omega vbd\Omega + \int_\Gamma (vVw + w_{,n}\, Mv - v_{,n}\, Mw - wVv)ds - \sum_k \left(v[\![Tw]\!] - w[\![Tv]\!]\right)_k \quad \mathbf{x} \in \Gamma \tag{11}$$

$$\frac{1}{2} w_{,\nu}(\mathbf{x}) = \int_\Omega v_1 bd\Omega + \int_\Gamma (v_1 Vw + w_{,n}\, Mv_1 - v_{1,n}\, Mw - wVv_1)ds - \sum_k \left(v_1[\![Tw]\!] - w[\![Tv_1]\!]\right)_k \quad \mathbf{x} \in \Gamma \tag{12}$$

$v = v(\mathbf{x}, \mathbf{y}) = r^2 \ln r / 8\pi$ is the fundamental solution and v_1 its normal derivative at point $\mathbf{x} \in \Gamma$. Eqs (11) and (12) can be used to establish the not specified boundary quantities. They are solved numerically using the BEM. The boundary integrals are approximated using constant boundary elements, whereas the domain integrals are evaluated as follows. The fictitious load $b(x)$ is approximated by the series [5]

$$b(\mathbf{x}) = \sum_{k=1}^{M} \alpha_k f_k(r), \quad r = \|\mathbf{x} - \mathbf{x}_k\| \tag{13}$$

where $f_k(r)$ are radial basis functions with $\mathbf{x}_k$ being M collocation points inside Ω and a_k coefficients to be determined. Thus we can write

$$\int_\Omega vbd\Omega = \sum_{k=1}^{M} a_k \int_\Omega vf_k(r)d\Omega \tag{14}$$

In order that the method maintains its pure boundary character, the domain integral in Eq. (14) is converted to boundary line integral using the Raleigh-Green identity [6]. Similar expressions are obtained for the domain integrals including $v_1(\mathbf{x}, \mathbf{y})$. The boundary integral Eqs (11) and (12) together with the boundary conditions (3) are used to evaluate the boundary quantities. Applying them to the boundary nodal points we obtain

$$\mathbf{H}\begin{Bmatrix}\mathbf{w}\\ \mathbf{w}_{,n}\end{Bmatrix} = \mathbf{G}\begin{Bmatrix}\mathbf{V}\\ \mathbf{M}\end{Bmatrix} + \mathbf{A}\boldsymbol{a}, \quad \begin{matrix}\boldsymbol{\alpha}_1\mathbf{w} + \boldsymbol{\alpha}_2\mathbf{V} = \boldsymbol{\alpha}_3\\ \boldsymbol{\beta}_1\mathbf{w}_{,n} + \boldsymbol{\beta}_2\mathbf{M} = \boldsymbol{\beta}_3\end{matrix} \tag{15a,b}$$

where $\mathbf{H}$, $\mathbf{G}$, $\mathbf{A}$, $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \mathbf{b}_3$ are known coefficient matrices; $\mathbf{w}$, $\mathbf{w}_{,n}$ are the vectors of the boundary nodal displacements and boundary nodal slopes, respectively; $\mathbf{V}$, $\mathbf{M}$ are the vectors of boundary nodal effective shear forces and boundary nodal normal bending moments and $\mathbf{a}$ is the vector of the unknown coefficients defined in Eq. (13). Equations (15) can be combined to express the boundary quantities $\mathbf{w}, \mathbf{w}_{,\mathbf{n}}$, $\mathbf{V}$, $\mathbf{M}$ in terms of the coefficients $\mathbf{a}$, and subsequently use these expressions to eliminate the boundary quantities from the discretized counterpart of Eq. (10). Thus we obtain the following representation for the deflection

$$w(\mathbf{x}) = \sum_{k=1}^{M} a_k W_k(\mathbf{x}) + W_0(\mathbf{x}) \qquad \mathbf{x} \in \Omega \tag{16}$$

Apparently, the functions defined in Eq. (16) satisfy not only the kinematic boundary conditions but also the natural ones. The derivatives of $w(\mathbf{x})$ at points $\mathbf{x}$ inside Ω are obtained by direct differentiation of Eq. (10)

$$w_{,pqr}(\mathbf{x}) = \sum_{k=1}^{M} a_k W_{k,pqr}(\mathbf{x}) + W_{0,pqr}(\mathbf{x}), \qquad p,q,r = 0,x,y \qquad \mathbf{x} \in \Omega \tag{17}$$

Note that the above notation implies $w_{,000} = w$, $w_{,0y0} = w_{,y}$, etc.

3.2 The AEM solution for plane stress problem

The analog equations of Eqs (4) are obtained using the Laplace operator, which yields

$$\nabla^2 u = b_1(\mathbf{x}) \qquad \nabla^2 v = b_2(\mathbf{x}) \tag{18a,b}$$

The integral representation of the solution of Eq. (18a) is

$$\varepsilon u(\mathbf{x}) = \int_\Omega v^* b_1 d\Omega - \int_\Gamma (v^* q - q^* u) ds \qquad \mathbf{x} \in \Omega \cup \Gamma \tag{19}$$

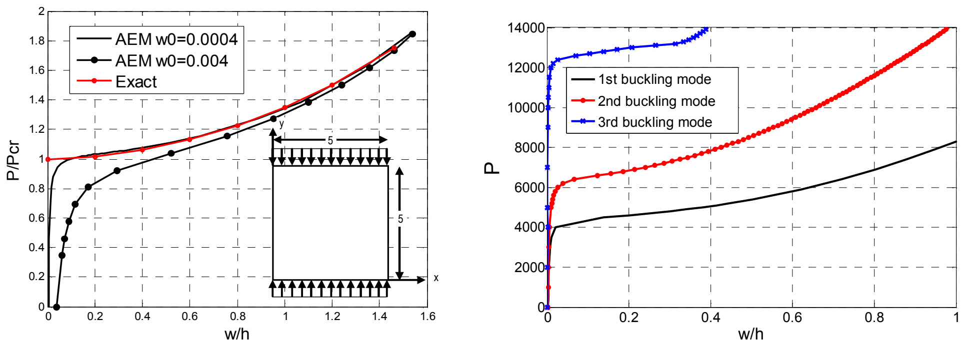
in which $q = u_{,n}$; $v^* = \ell n r / 2\pi$ is the fundamental solution to Eq. (18a) and $q^* = v^*_{,n}$ its derivative normal to the boundary with $r = \|\mathbf{y} - \mathbf{x}\|$ $\mathbf{x} \in \Omega \cup \Gamma$ and $\mathbf{y} \in \Gamma$; ε is a constant ($\varepsilon = 1$ if $\mathbf{x} \in \Omega$, $\varepsilon = 1/2$ if $\mathbf{x} \in \Gamma$ and $\varepsilon = 0$ if $\mathbf{x} \notin \Omega \cup \Gamma$). The domain integral in Eq. (19) is approximated by the series (13). Subsequently, using the BEM and the previous procedure we obtain

$$u_{,pq}(\mathbf{x}) = \sum_{k=1}^{M} a_k^{(1)} U_{k,pq}(\mathbf{x}) + U_{0,pq}(\mathbf{x}) \quad v_{,pq}(\mathbf{x}) = \sum_{k=1}^{M} a_k^{(2)} V_{k,pq}(\mathbf{x}) + V_{0,pq}(\mathbf{x}), \qquad p,q = 0,x,y \quad \mathbf{x} \in \Omega \tag{20}$$

The functions $U_k(\mathbf{x}), V_k(\mathbf{x})$, $k = 0,1,..,M$ are admissible shape functions satisfying both the kinematic and the natural boundary conditions. Once the $3M$ unknown coefficients $a_k, a_k^{(1)}, a_k^{(2)}$ are established the solution of the problem is given by Eqs (10) and (20).

3.3 The minimization of the total potential

The herein employed radial basis functions f_k are the multiquadrics (MQs), which are defined as $f_k = \sqrt{r^2 + c_k^2}$, where c_k are the shape parameters, in general different for each collocation point. It is apparent that the functional Π depends on the $3M$ coefficients $a_k, a_k^{(1)}, a_k^{(2)}$, the shape parameters c_k of MQs and the $2M$ coordinates x_k, y_k of the collocation points. Therefore, we can search for the minimum using various levels of optimization. On the base of the previously described procedure a FORTRAN code has been written for post-buckling analysis of plates. The square plate of Fig. a with $E = 3 \times 10^7$ kN/m^2, $\nu = 0.3$, $h = 0.1m$ has been analyzed. All edges are simply supported and movable in the plane of the plate. Fig. a shows the load-deflection curves at the center of the plate, when the initial deflection has the form of the first linear buckling mode as compared with the exact solution [1]. Moreover, Fig. b shows the load deflection curve at point (2.5, 1), when the initial deflection has the form of the first, second and third linear buckling mode shape. It was observed that the post-buckling response is triggered when the load reaches the value of the critical load of the corresponding mode shape ($P_{cr}^1 = 4333.3kN$, $P_{cr}^2 = 6771.70kN$, $P_{cr}^3 = 12038.6kN$).



(a) (b)

4. CONCLUSIONS

A BEM based variational method for post-buckling analysis of plates of arbitrary shape was presented. The global admissible functions are established using a BEM technique based on the concept of the analog equation. The post-buckling can start from higher critical load, if the initial deflection has the shape of the respective linear buckling mode. Thus, since the post-buckling response of thin plates is a stable state, we can conclude that the bearing capacity of the plate can be increased by specifying appropriately the shape of the initial deflection.

Acknowledgement: *This work was supported by the State Scholarships Foundation of Greece.*

REFERENCES

1. Bulson, P.S. (1970), *The Stability of Flat Plates*, Chatoo & Windus, London.
2. Elzein A. (1991), *Plate Stability by Boundary Element Method*, Springer_Verlag, Berlin.
3. Qin Q. and Huang Y. (1990), "BEM of post-buckling analysis of thin plates", Appl. Math. Modelling, Vol. 14, pp. 544-548.
4. Katsikadelis, J.T. and Neratzaki, M.S. (1994), "Nonlinear analysis of plates by the analog equation method", Comput. Mech., Vol. 14, pp. 154-164.
5. Katsikadelis, J.T. (2002), *Boundary Elements: Theory and Applications*, Elsevier Amsterdam-London
6. Katsikadelis, J.T. and Armenakas, A.E. (1989), "A new boundary equation solution to the plate problem", ASME, Journal Applied Mechanics, Vol. 56, pp. 364-374.

ANALIZA NAPREGNUTIH PLOČA METODOM GRANIČNIH ELEMENATA

John T. Katsikadelis, Nick Babouskos

Nova metoda graničnih elemanata zasnovana na varijacionoj metodi je predstavljena u primeni na analizu stanja savije netanke elastične ploče proizvoljnog oblika za slučaj opštih graničnih uslova. Odziv ploče je opisan pomoću tri spregnute parcijalne nelinearne diferencijalne jednačine izražene pomoću pomeranja, pri čemu su korišćene von Kármán-ove pretpostavke. Rešenja su dobijena korišćenjem AEM. Numeričkim primerom je prikazana efektivnost primene i kvalitet tačnosti razvijene metode.

Ključne reči: metoda graničnih elemenata, ploče, analog jednačine, nelinearni, varijacioni, optimalni, multikvadratni.