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THE FUNDAMENTAL CRITERION FOR ESTIMATION OF THE REFERENCE ELLIPSOID ACCURACY

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Abstract. *The value of the semi-major axis a, and the value of the semi-minor axis b are the geometrical characteristics of the reference ellipsoid. The other important characteristic is the normal gravity g, which represents the magnitude of the gradient of the gravity potential with the ellipsoid as an equipotential surface.*
This paper presents the fundamental criterion for estimation of accuracy of the reference ellipsoid, which stems from the Newton's second law. It has been proved, by the analysis of the motion of particle along a meridian of the reference ellipsoid, that the algebraic sum of works of gravitational attraction force and centrifugal force, excited by rotational ellipsoid around its axis, equals to zero. This is an indispensable term that the direction of gravity vector is everywhere perpendicular to the ellipsoid surface as an equipotential surface. This is what makes this criterion reliable and enables the obtaining of the quantitative evaluation of the ellipsoid's accuracy. Applying this criterion for estimation of accuracy of the ellipsoid, defined according to the Geodetic Reference System 1980 (GRS80), it can be concluded that the ellipsoid's semi-minor axis contains an error of about 599 m.

Key words: *the reference ellipsoid, semi-minor axis, gravitational attraction, GRS80*

1. INTRODUCTION

The reference ellipsoid is an ellipsoid of revolution, which would be obtained by rotational an ellipse around its minor axis. It is determined by four constants (*Moritz, 1980*), and the IUGG has chosen the following ones

a - equatorial radius, semi-major axis,
GM - geocentric gravitational constant,
J_2 - dynamic form factor and
ω - angular velocity of the Earth

The other geometric constant, semi-minor axis b, is obtained by the following formula

$$b = a\sqrt{1-e^2}\ , \tag{1}$$

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where is e first eccentricity. Therefore, the reference ellipsoid is defined as a pure geometrical figure. The corresponding reference potential (the Somigliana-Pizzetti reference potential) has been determined from the condition that reference ellipsoid should be an equipotential surface of the reference gravity potential (*Heiskanen and Moritz, 1967*). The question of the ellipsoid's accuracy is raised, and primarily of the accuracy of the value of its semi-minor axis. It can't be verified by a survey along a meridian, since it is hard to estimate for every geodesic arc, to what extent it represents the Earth as a whole (*Jeffreys, 1976*).

The reliable criterion for estimation of the reference ellipsoid accuracy should be obtained by physics. In this paper it has done by Newton's second law.

2. THE DEFINITION OF THE FUNDAMENTAL CRITERION

Let the particle of the unit mass move in the nonresistant medium along a fictional ideal smooth groove PQE (Fig. 1) along the meridian of the reference ellipsoid, with the set initial relative velocity v_0. Applying the Newton's second law, the motion of the particle is described by the equation

$$a = g^* + F_w \tag{2}$$

where g^* is the vector of gravitational attraction, and F_w is the groove reaction.

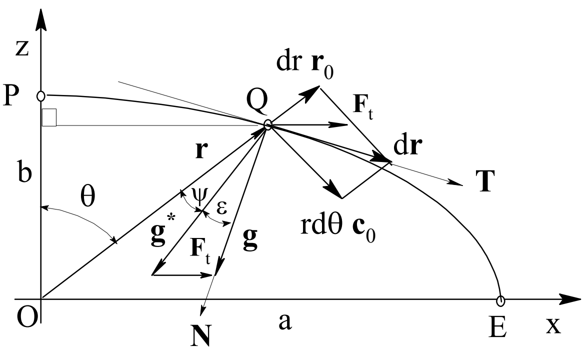

Fig. 1. Relative motion of the particle along a meridian

Absolute acceleration a is equal to vector sum of the transfer acceleration a_t, relative acceleration a_r and the Coriolis acceleration a_c, so the following equation is obtained

$$a_t + a_r + a_c = g^* + F_w \tag{3}$$

and this can be written in the form

$$a_r = g^* + F_w - a_t - a_c \tag{4}$$

or in the form

$$a_r = g^* + F_w + F_t + F_c \tag{5}$$

where F_t is the transfer inertial force (centrifugal force) and F_c is the Coriolis inertial force. Three scalar equations correspond to vectorial Eq. (5); one of them defines the relative motion of the particle in the tangential direction, whose unit vector is T. Considering that vectors F_w and F_c, and the vector sum of vectors g^* and F_t, are all perpendicular to the tangential direction, by projecting Eq. (5) onto that direction, one obtains that the relative tangential acceleration a_{rT} equals to zero, that is

$$\frac{dv_r}{dt} = 0. \tag{6}$$

where v_r is the relative velocity intensity.

Taking the dot product of Eq. (5), by the vector of the particle's elementary relative displacement $\boldsymbol{dr}$, one obtains the equation

$$d\left(\frac{v_r^2}{2}\right) = dA^{(g^*)} + dA^{(F_w)} + dA^{(F_t)} + dA^{(F_c)}, \tag{7}$$

where is, on the left-hand-side of Eq. (7), the kinetic energy of the particle's relative motion, $E_{kr} = v_r^2/2$, and on the right-hand-side are the works of the forces g^*, F_w, F_t and F_c, along the particle's elementary displacement. Since the relative velocity of a particle is of the constant intensity, then there is no change of the relative kinetic energy, and works of the constraint reaction F_w and the Coriolis force F_c are then equal to zero, since those forces are perpendicular to the direction of the particle's elementary displacement. Thus, using Eq. (7), the following equation is obtained

$$dA^{(g^*)} + dA^{(F_t)} = 0. \tag{8}$$

The work of the centrifugal force can be written in the analytical form if the force F_t and elementary relative displacement $\boldsymbol{dr}$ are expressed in terms of projection onto the radial direction $\boldsymbol{r}_0$ and the direction $\boldsymbol{c}_0$ which is perpendicular to $\boldsymbol{r}_0$ (Fig. 1), namely in the form

$$\boldsymbol{F}_t = r \cdot \omega^2 \cdot \sin^2\theta \cdot \boldsymbol{r}_0 + r \cdot \omega^2 \cdot \sin\theta \cdot \cos\theta \cdot \boldsymbol{c}_0 \tag{9}$$

and

$$\boldsymbol{dr} = dr \cdot \boldsymbol{r}_0 + r \cdot d\theta \cdot \boldsymbol{c}_0, \tag{10}$$

where $\boldsymbol{r}_0$ and $\boldsymbol{c}_0$ are the unit vectors, and ω is the angular velocity of the Earth's rotation. By the dot product of these vectors, the following equation is obtained

$$dA^{(F_t)} = \boldsymbol{F}_t \cdot \boldsymbol{dr} = r \cdot \omega^2 \cdot \sin^2\theta \cdot dr + r^2 \cdot \omega^2 \cdot \sin\theta \cdot \cos\theta \cdot \; d\theta \quad . \tag{11}$$

The work of the centrifugal force, Eq. (11), can be written as total differential, and the work of the gravitational attraction as differential of the gravitational potential dU, so that the Eq. (8) becomes

$$d\left(\frac{r^2 \cdot \omega^2 \cdot \sin^2\theta}{2}\right) + dU = 0. \tag{12}$$

The Eq. (12) consists of only actual performed works, so that it expresses the fact that the sum of centrifugal and gravitational potential is constant. From the definition of work as a dot product of the force vector and the vector of the elementary displacement, the work can be understood as a product of force intensity and the projection of elementary displacement $ds = |d\mathbf{r}|$ onto force direction (Fig. 2), so the actual work of the gravitational attraction can be written in the form

$$dA^{(g^*)} = -g^* \cdot \sin\varepsilon \cdot ds = -g^* \cdot dh_1 \tag{13}$$

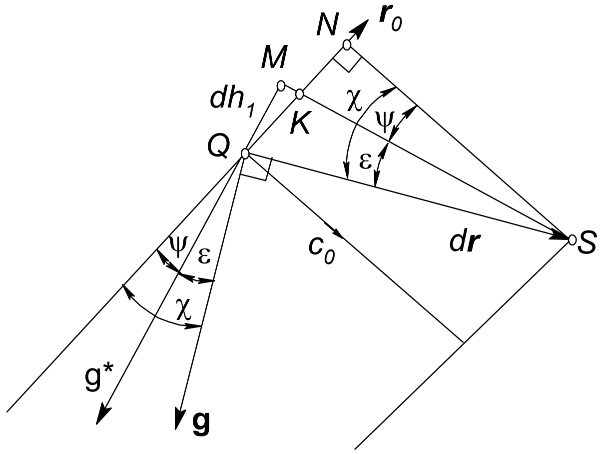

Fig. 2. Increments of meridian

The flux of the force g^* (*Raskovic, 1956*) along $\overrightarrow{QN} + \overrightarrow{NS}$ paths, is equal to the sum of actual work along $\overrightarrow{QK}$ path and works along $\overrightarrow{KN} + \overrightarrow{NS}$ paths

$$dA^{(g^*)} = -g^* \cdot \cos\psi \cdot \overline{QK} - g^* \cdot \cos\psi \cdot \overline{KN} + g^* \cdot \sin\psi \cdot \overline{NS}. \tag{14}$$

Since the work of the force g^* along $\overrightarrow{KS}$ path is equal to zero, then it can be expressed by

$$g^*(\overline{\mathrm{KN}} \cdot r_0 + \overline{\mathrm{NS}} \cdot c_0) = -g^* \cdot \cos\psi \cdot \overline{\mathrm{KN}} + g^* \cdot \sin\psi \cdot \overline{\mathrm{NS}} = 0. \tag{15}$$

The total work of the force g^* along increment $dr = \overline{QN}$ is equal to the sum of actual work along $\overline{QK}$ path and fictitious work along $\overline{KN}$ path, so that the Eq. (14) can be written in the form

$$dA^{(\mathrm{g}^*)} = -g^* \cdot \cos\psi \cdot dr + g^* \cdot \sin\psi \cdot r \cdot d\theta \tag{16}$$

Increment $\overline{QN}$ is about double as large as increment $\overline{QK}$ because the same ratio exists between angles χ and ε. Based on the Eq. (15), it can be concluded that the fictitious work occurs in two forms with the different signs but the same values.

The equation of an ellipse in the Cartesian coordinate system states

$$\left(\frac{x}{a}\right)^2+\left(\frac{z}{b}\right)^2=1\,, \tag{17}$$

and using spherical coordinates ($x = r \cdot \sin\theta$ and $z = r \cdot \cos\theta$), the equation of the ellipse is obtained in the form

$$r=\frac{a}{\sqrt{1+\left[\left(\frac{a}{b}\right)^2-1\right]\cdot\cos^2\theta}}\,. \tag{18}$$

Eq. (11) expresses flux of the force F_t along $\overrightarrow{QN}+\overrightarrow{NS}$ paths. It can be proved that is the second term, on the right-hand-side of Eq. (11), the fictitious work. The Eq. (18) can be written in the form

$$\sin^2\theta=1-\frac{b^2}{r^2}\cdot\frac{a^2}{a^2-b^2}+\frac{b^2}{a^2-b^2} \tag{19}$$

and by substituting Eq. (19) into Eq. (11), the following equation is obtained

$$dA^{(F_t)}=\frac{a^2\cdot\omega^2}{a^2-b^2}\cdot r\cdot dr-\frac{a^2\cdot b^2\cdot\omega^2}{a^2-b^2}\cdot\frac{dr}{r}+r^2\cdot\omega^2\cdot\sin\theta\cdot\cos\theta\cdot d\theta. \tag{20}$$

By differentiation of Eq. (18) one obtains the equation

$$\frac{dr}{r}=\frac{\left[\left(\frac{a}{b}\right)^2-1\right]\cdot\sin\theta\cdot\cos\theta\cdot d\theta}{1+\left[\left(\frac{a}{b}\right)^2-1\right]\cdot\cos^2\theta}, \tag{21}$$

and by substituting Eq. (21) into Eq. (20), the following equation is obtained

$$dA^{(F_t)}=\frac{a^2\cdot\omega^2}{a^2-b^2}\cdot r\cdot dr-r^2\cdot\omega^2\cdot\sin\theta\cdot\cos\theta\cdot d\theta+r^2\cdot\omega^2\cdot\sin\theta\cdot\cos\theta\cdot d\theta. \tag{22}$$

The first term in Eq. (22) is the actual work, and another two terms are fictitious works of the force F_t. The total work of this force is equal to the sum of actual work and the fictitious work. Considering that the Eq. (8) gives the relation between actual works, as well as fictitious works, of the forces F_t and g^*, and using the Eqs. (16) and (22), one obtains the following two equations

$$-r^2\cdot\omega^2\cdot\sin\theta\cdot\cos\theta\cdot d\theta+g^*\cdot\sin\psi\cdot r\cdot d\theta=0 \tag{23}$$

and

$$\frac{a^2\cdot\omega^2}{a^2-b^2}\cdot r\cdot dr+r^2\cdot\omega^2\cdot\sin\theta\cdot\cos\theta\cdot \mathrm{d}\theta-g^*\cdot\cos\psi\cdot dr=0. \tag{24}$$

The Eq. (24) represents the sum of actual works and fictitious works of centrifugal force and gravitational attraction force along increments of meridian. By integration of Eq. (24) within limits from position P to position Q (Fig. 1) where angle θ changes from zero to θ, and radius r changes from b to r, one obtains the equation

$$\frac{a^2\cdot\omega^2}{2}\cdot\frac{r^2-b^2}{a^2-b^2}+\omega^2\int_{(P)}^{(Q)} r^2\cdot\sin\theta\cdot\cos\theta\cdot d\theta-\int_{(P)}^{(Q)} g^*\cdot\cos\psi\cdot dr=0. \tag{25}$$

The first integral in Eq. (25) can be solved using Eq. (18), to obtain

$$\int_{(P)}^{(Q)} r^2\cdot\sin\theta\cdot\cos\theta\cdot d\theta=-\int_0^\theta\frac{a^2\cdot\cos\theta\cdot d(\cos\theta)}{1+\left[\left(\frac{a}{b}\right)^2-1\right]\cos^2\theta}=\frac{1}{2}\frac{a^2}{\left(\frac{a}{b}\right)^2-1}\ln\frac{\left(\frac{a}{b}\right)^2}{1+\left[\left(\frac{a}{b}\right)^2-1\right]\cos^2\theta} \tag{26}$$

and using Eq. (19), then Eq. (25) can be written in the form

$$\frac{r^2\cdot\omega^2}{2}\sin^2\theta+\frac{1}{2}\frac{a^2\cdot\omega^2}{\left(\frac{a}{b}\right)^2-1}\cdot\ln\frac{\left(\frac{a}{b}\right)^2}{1+\left[\left(\frac{a}{b}\right)^2-1\right]\cos^2\theta}-\int_{(P)}^{(Q)} g^*\cdot\cos\psi\cdot dr=0\quad . \tag{27}$$

The first term in Eq. (27) represents the centrifugal potential in position Q. It could be taken that is $\cos\psi\approx 1$ because the highest value of the angle ψ is less than 0.1^0. If the particle is moving from the point P at the pole to the point E at the equator, then in Eq. (27) the substitution should be made $\theta=\pi/2$ and $r=a$, to give

$$\frac{a^2\cdot\omega^2}{2}+\frac{1}{2}\frac{a^2\cdot\omega^2}{\left(\frac{a}{b}\right)^2-1}\ln\left(\frac{a}{b}\right)^2-\int_{(P)}^{(E)} g^*\cdot dr\approx 0. \tag{28}$$

The Eq. (28) should be satisfied if the reference ellipsoid is accurate. If it is not, then the sum of the total works of the centrifugal force and the gravitational attraction force shall not be equal to zero and shall represent the error of the ellipsoid ΔA, that is

$$\Delta A=\frac{a^2\cdot\omega^2}{2}+\frac{1}{2}\frac{a^2\cdot\omega^2}{\left(\frac{a}{b}\right)^2-1}\ln\left(\frac{a}{b}\right)^2-\int_{(P)}^{(E)} g^*\cdot dr \tag{29}$$

The Eq. (29) represents the fundamental criterion for estimation of the reference ellipsoid accuracy.

3. THE APPLICATION OF THE FUNDAMENTAL CRITERION

The fundamental criterion will be applied for the estimation of the accuracy of GRS80 ellipsoid whose semi-axes are a = 6378137 m, b = 6356752.3142 m and the gravity is determined according to the Gravity Formula 1980, which reads

$$g = 9.780327[1 + 0.0053024 \cdot \sin^2 \phi - 0.0000058 \cdot \sin^2 (2\phi)] \quad m/s^2, \tag{30}$$

where ϕ is the geodetic latitude.

The value of integral in Eq. (29) will be computed using the relationship between gravitational attraction g^* and height h. In that sense, the angle $\angle POE = \pi/2$ will be divided to n equal parts $\Delta\theta$, where one will obtain n approximately equal segments on the portion PQE of the ellipse (Fig. 3). The positions of these segments determine by the (n+1) points at their ends. For each point the magnitude of the radius vector $\boldsymbol{r}$ can be computed based on Eq. (18), and then the height of the particle's rise during the relative motion, as the difference of r and of the semi-minor axis b

$$h \approx r - b\,. \tag{31}$$

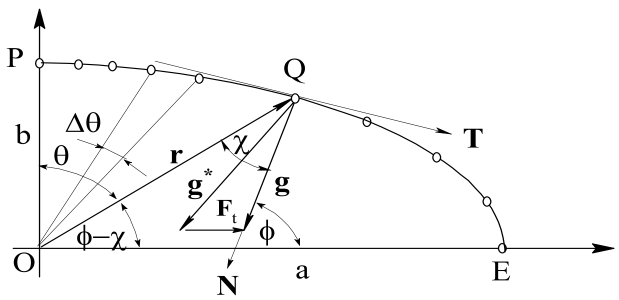

Fig. 3. Relevant variables for determination of gravitational attraction

The angle χ (Fig.2) between the vector **g** and radial direction can be computed by the equation

$$tg\,\chi = \frac{1}{r}\frac{dr}{d\theta}, \tag{32}$$

and then based on Eq. (21), the equation is obtained

$$tg\chi = \left[\left(\frac{a}{b}\right)^2 - 1\right] \frac{\sin\theta \cdot \cos\theta}{1 + \left[\left(\frac{a}{b}\right)^2 - 1\right]\cos^2\theta} \tag{33}$$

For each of the points whose position is determined by angle θ, the geodetic latitude can be obtained according to formula

$$\phi = \pi/2 - \theta + \chi \tag{34}$$

The magnitude of gravity can be computed according to Eq. (30), and of centrifugal force according to formula

$$F_t = r \cdot \sin\theta \cdot \omega^2 \tag{35}$$

where is Earth's angular velocity ω = 0.000072921 *rad / s*.

The magnitude of gravitational attraction is determined according to the cosine theorem

$$g^* = \sqrt{g^2 + F_t^2 + 2 \cdot g \cdot F_t \cdot \cos\phi} \ . \tag{36}$$

The graph of gravitational attraction g^* against height *h* has the shape shown in Figure 4.

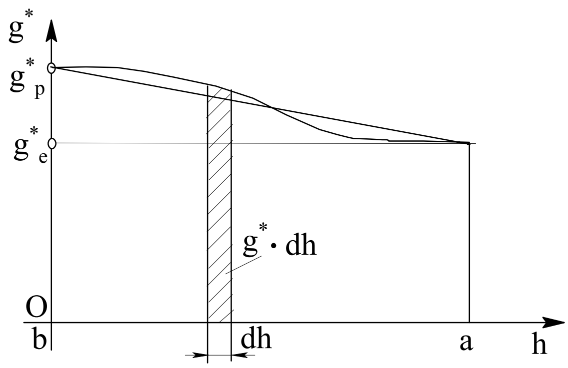

Fig. 4. Gravitational attraction versus height

The value of integral in Eq. (29) is proportional to the area under the graph, which amounts 210065 *Nm*. It is approximate equal to the area of a trapeze, namely

$$\int_{(P)}^{(E)} g^* \cdot dh \approx (a-b)\frac{g_p^* + g_e^*}{2} = 210066.35 Nm , \tag{37}$$

where g_p^* is the gravitational attraction at the pole, and g_e^* at the equator. These values are computed according to Eq. (36) and they amount to $g_p^* = 9.832186206\,N$ and $g_e^* = 9.814242566\,N$. The remaining part at the right-hand-side of Eq. (29) represents the centrifugal force work and it amounts to 215955.28 *Nm*. The algebraic sum of these works according to Eq. (29) represents the error of the ellipsoid which is 5888.66 *Nm*. The ellipsoid error wouldn't be significantly less even under the consideration of the gravitational attraction to be unchangeable with the maximum value $g_p^* = 9.832186206\,N$, in which case it would be 5697 *Nm*. The centrifugal force work weakly depends on the possible error in the value of semi-minor axis *b*, considering the Eq. (29), so it can be taken that the calculated value of this work can be considered as approximately accurate, while the value of the gravitational attraction force work is directly dependent on accuracy of the semi-axis *b*.

In order for ellipsoid to be accurate, values of these works must be equal, namely

$$(a-b)\frac{g_p^* + g_e^*}{2} \approx 215955\, Nm \ . \quad (38)$$

If one assumes that values of a, g_p^* and g_e^* are accurate, then from Eq. (38) one obtains that value of semi-minor axis b should amount to 6356153 *m*, what is smaller than the value obtained according to the model GRS80 of about 599 *m*.

4. Conclusion

The fundamental criterion for the estimation of accuracy of the reference ellipsoid is reliable criterion because it appears from Newton's second law. This is an indispensable term that the direction of gravity is everywhere perpendicular to the ellipsoid surface as an equipotential surface. Applying this criterion for estimation of accuracy of the ellipsoid defined according to the Geodetic Reference System 1980, it can be concluded that the ellipsoid's semi-minor axis contains an error of about 599 *m*.

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OSNOVNI KRITERIJUM ZA OCENU TAČNOSTI REFERENTNOG ELIPSOIDA

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Veličina veće poluose a, i veličina manje poluose b su geometrijske karakteristike referentnog elipsoida. Druga bitna karakteristika je normalna gravitacija g, koja pretstavlja intenzitet gradijenta gravitacionog potencijala na ekvipotencijalnoj površini elipsoida.

U ovom radu je pretstavljen osnovni kriterijum za odredjivanje tačnosti referentnog elipsoida koji proističe is drugog Njutnovog zakona. Analizom kretanja materijalne tačke, duž meridijana referentnog elipsoida, je dokazano da je algebarski zbir radova sile gravitacionog privlačenja i centrifugalne sile, nastale obrtanjem elipsoida oko njegove ose, jednak nuli. Ovo je neophodan uslov da pravac vektora gravitacije bude svuda upravan na površinu elipsoida kao ekvipotencijalnu površinu. Zato je ovaj kriterijum pouzdan i omogućava da se dobije kvantitativna ocena tačnosti elipsoida. Primenjujući ovaj kriterijum za ocenu tačnosti elipsoida definisanog prema the Geodetic Reference System 1980 (GRS80), može se zaključiti da manja poluosa elipsoida sadrži grešku od oko 599 m.

Ključne reči: *referentni elipsoid, manja poluosa, gravitaciono privlačenje, GRS80*