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ON THE HOMOGENEOUS FORMALISM IN CELESTIAL MECHANICS

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V. Vujičić[*], Z. Drašković[**]

[*]Mathematical Institute, P.O. Box 368, 11001 Belgrade, Yugoslavia
[**]Military Technical Institute, Kataniceva 15, 11000 Belgrade, Yugoslavia

Abstract. *Correctness of the theory of "homogeneous formalism" in celestial mechanics is considered. It is shown that the equations (0.1a) do not include, in the classical mechanics, the "canonical forces"* Q^{α}. *Then the relation* $p_0 = -H$ *is abandoned and, instead of it, the existence of the momentum* $p_0 = \dfrac{\partial T}{\partial \dot{q}^0} = a_{00}\,\dot{q}^0 + a_{0j}\,\dot{q}^j$ *is established. Consequently, instead of the differential equations (0.1a) and (0.1b), appear the differential equations of motion (3.7a) and (3.7b).*

Keywords: *celestial mechanics, homogeneous formalism, "rheonomic coordinate" concept*

1. INTRODUCTION

A theory known as a *homogeneous formalism*, where the time is used as $n + 1$`th coordinate ($q^0 := \mathrm{t}$) and the corresponding generalized momentum as the negative Hamilton`s function H ($p_0 := -H$), is very wide-spread in the classical analytical mechanics and theoretical physics. This concept can be seen in some valuable works of celestial mechanics too, where one can find, inter alias, a system of $2n + 2$ canonical differential equations of motion of the celestial bodies in the form[1].

$$\dot{q}^{\alpha} = \frac{\partial H_h}{\partial p_{\alpha}} - Q^{\alpha} \qquad (q := \{q^0, q^1, ..., q^n\} \in M^{n+1} =: N)\,, \tag{1.1a}$$

$$\dot{p}_{\alpha} = -\frac{\partial H_h}{\partial q^{\alpha}} + P_{\alpha} \qquad (p := \{p_0, p_1, ..., p_n\} \in T^*N)\,, \tag{1.1b}$$

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with "canonical forces" Q^{α} and P^{α} (s. for example [1] or [2]); the homogeneous Hamiltonian $H_h = H + p_0$ is a first integral.

As the concept

$$q^0 = t\,, \tag{1.2}$$

$$p_0 = -H\,, \tag{1.3}$$

can be considered as "a far reaching analogy" [3], we shall look for its meaning on the basis of the axioms and principles of classical mechanics.

2. Mechanical Canonical Systems

We shall consider the motion of a celestial body as a material point (of mass m) motion with respect to the point O. The position vector is[2] $\mathbf{r} =: y^i\mathbf{e}_i = y^1\mathbf{e}_1 + y^2\mathbf{e}_2 + y^3\mathbf{e}_3$, where $\mathbf{e} =: \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are mutually orthogonal unit basis vectors, $d\mathbf{e}/dt = 0$, and $y =: \{y^1, y^2, y^3\} \in E^3$ are cartesian rectilinear coordinates. We accept Descartes' primeval notion and Newton's definition of the momentum $\mathbf{p}$ as a product of the mass m and the velocity $\mathbf{v} = \dot{y}^i\mathbf{e}_i \in \mathbf{R}_3$, i.e.

$$\mathbf{p} := m\mathbf{v} = m\dot{y}^i\mathbf{e}_i \qquad (i = 1,2,3) \tag{2.1}$$

The vector $\mathbf{p}$ projection on the coordinate axis $p_j = \mathbf{p}\cdot\mathbf{e}_j = m\delta_{ij}\dot{y}^i$ is also called the momentum. However, with respect to another, curvilinear coordinate system $x =: \{x^1, x^2, x^3\}$, the form of this covariant momentum coordinate will be somewhat complex. Namely, if the mapping $y \leftrightarrow x$ is one-to-one correspondence (i.e. if $|\partial y/\partial x| \neq 0$), we can write

$$\mathbf{p} = m\frac{d\mathbf{r}}{dt} = m\frac{\partial\mathbf{r}}{\partial x^k}\dot{x}^k = m\mathbf{g}_k\dot{x}^k \qquad (k = 1,2,3)\,, \tag{2.2}$$

where $\dot{\mathbf{x}} = \dfrac{d\mathbf{x}}{dt} = \{\dot{x}^1, \dot{x}^2, \dot{x}^3\}^T$ are the generalized velocities, and $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ are the basis vectors for the system x. By taking the scalar product of (2.2) with the vectors $\mathbf{g}_j$ we obtain the generalized momenta

$$p_j = g_{jk}(x)\dot{x}^k\,, \tag{2.3}$$

where

$$g_{jk}(x) := m\frac{\partial r}{\partial x^j}\cdot\frac{\partial r}{\partial x^k} = m\delta_{il}\frac{\partial y^i}{\partial x^j}\frac{\partial y^l}{\partial x^k} \qquad (i,j,k,l = 1,2,3)$$

is the inertial tensor. The same covariant form of momenta we meet in the body rotation description. Hence, the momenta $\mathbf{p}$ of celestial bodies are the homogeneous linear functions of the generalized velocities. At the classics like Lagrange, Jacobi, Hamilton, Helmholtz... we find that

[2] Einstein's summation convention for diagonally repeated indices will be used.

$$p_j := \frac{\partial T}{\partial \dot{x}^j} \quad \text{or} \quad p_j := \frac{\partial L}{\partial \dot{x}^j} \tag{2.4}$$

but the kinetic energy $T = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$ and the kinetic potential $L = T - V(x)$ are assigned in advance and are in full accordance with the equations (2.3). Sinse always $|g_{ij}| \neq 0$, from the equations (2.3) it follows

$$\dot{x}^i = g^{ij} p_j , \tag{2.5}$$

where g^{ij} is the contravariant inertial tensor.

If we introduce the Hamilton`s function $H = T + V$, without difficulty reducible to the form

$$H = \frac{1}{2} g^{ij} p_i p_j + V(x) , \tag{2.6}$$

the fundamental equations of motion

$$m\ddot{y}^i = -\frac{\partial V}{\partial y^i} + F_i^* \tag{2.7}$$

can be written in the canonical form

$$\dot{x}^i = \frac{\partial H}{\partial p_i} , \tag{2.8a}$$

$$\dot{p}^i = -\frac{\partial H}{\partial x^i} + P_i^* , \tag{2.8b}$$

where $P_i^* = F_j^* \cdot \frac{\partial y_i}{\partial x^i}$ are any non-potential forces.

As we can see, the equations (2.8a) do not contain any "canonical forces" Q^α appearing in the equations (1.1a). *So, the classical mechanics does not introduce the "canonical forces"* Q^α *into equations (0.1a), nor* $p_0 = -H$ *is established.*

This can be proved for the most general form of the canonical equations ([4], [5])

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha} \quad (\dot{q}^\alpha = a^{\alpha\beta}(q) p_\beta) , \tag{2.9a}$$

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} + P_\alpha^* , \tag{2.9b}$$

where $q =: \{q^0, q^1, ..., q^n\} \in M^{n+1} =: N$ are the generalized coordinates and $p =: \{p^0, p^1, ..., p^n\} \in T^*N$ the corresponding momenta. "Rheonomic coordinate" $q^0 = \tau(\kappa, t)$ is a known function of time, obtained from the existing constraints of the system, and $p_0 = a_{0\alpha}\dot{q}^\alpha = a_{00}\dot{q}^0 + a_{01}\dot{q}^1 + ... + a_{0n}\dot{q}^n \neq -H$. Hamiltonian H is here too (as in classical mechanics) equal to the mechanical energy, i.e.

$$H = T + V = \frac{1}{2} a^{\alpha\beta} p_\alpha p_\beta + V(q) = E \tag{2.10}$$

It should be noted that Hamilton`s equations (1.1) are not fundamental, primeval relations of dynamics, but arise as the consequences of Newton`s equations

$$\frac{d\mathbf{p}_\nu}{dt} = \mathbf{F}_\nu = grad_\nu V + \mathbf{F}_\nu^* \qquad (\nu = 1, ..., N) \tag{2.11}$$

or follow (by means of Legendre`s transformations) from the Lagrange`s equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = P_\alpha^* \qquad \left(= \sum_{\nu=1}^{N} \mathbf{F}_\nu^* \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \right) \tag{2.12}$$

or are, finally, obtained from the variational principle (6)

$$\int_{t_0}^{t_1} (\delta\, L + P_\alpha^* \delta\, q^\alpha) dt = \int_{t_0}^{t_1} [\delta (p_\alpha \dot{q}^\alpha - L) + P_\alpha^* \delta\, q^\alpha] dt\,, \tag{2.13}$$

where L is introduced in [4] and [5].

Therefore, none of the starting relations (2.11), (2.12) and (2.13) do not contain any other non-potential forces, except the forces P_α^*. The relations (1.1a), (2.8a) or (2.9a) are nothing else then the part of the following Legendre`s transformations

$$H = p_\alpha \dot{q}^\alpha - L(q^\beta, \dot{q}^\beta, t)\ ,\ \ p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}\ ,\ \dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}\,. \tag{2.14}$$

In essence, this is the relation between the momentum **p** and the velocity **z**, as it can be seen from the equations (2.3) or (2.5).

It has been demonstrated in the paper [5] that Q^α from the equations (1.1a) can not be the perturbational forces either. The functions Q^α can appear only as the consequences of the various transformations, but in that case too it is already known which functions are transformed and with which transformations.

2 CANONICAL TRANSFORMATIONS

It is known that Hamilton`s equations of homogeneous form

$$\dot{q}^i = \frac{\partial H}{\partial p_i}\ ,\ \dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{3.1}$$

are invariant under the canonical transformations

$$q^i = f^i(\xi^j, \eta^k)\ ,\ p_i = \varphi_i(\xi^j, \eta^k)\,. \tag{3.2}$$

Namely, with respect to the transformations (3.2) the differential equations of motion have the same form as (3.1), i.e.

$$\dot{\xi}^i = \frac{\partial \overline{H}}{\partial \eta_i}\ ,\ \dot{\eta}_i = -\frac{\partial \overline{H}}{\partial \xi_i}$$

where $\overline{H} = \overline{H}(\xi, \eta, t) = H(q(\xi, \eta), p(\xi, \eta), t)$.

However, non-homogeneous equations of motion (2.8) or (2.9) are not invariant under the transformations (3.2). Really, substituting equations (3.2) in the non-homogeneous equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \tag{3.3a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + P_i^*, \tag{3.3b}$$

and composing them with $\frac{\partial \varphi_i}{\partial \eta_k}$ and $\frac{\partial f^i}{\partial \eta_k}$ respectivelly, we obtain

$$\left[\frac{\partial f^i}{\partial \xi^j} \frac{\partial \varphi_i}{\partial \eta_k} \dot{\xi}^j + \frac{\partial f^i}{\partial \eta_j} \frac{\partial \varphi_i}{\partial \eta_k} \dot{\eta}_j \right] = \frac{\partial H}{\partial p_i} \frac{\partial \varphi_i}{\partial \eta_k} \tag{3.4a}$$

$$\left[\frac{\partial \varphi_i}{\partial \xi^j} \frac{\partial f^i}{\partial \eta_k} \dot{\xi}^j + \frac{\partial \varphi_i}{\partial \eta_j} \frac{\partial f^i}{\partial \eta_k} \dot{\eta}_j \right] = -\frac{\partial H}{\partial q^i} \frac{\partial f^i}{\partial \eta_k} + P_i^* \frac{\partial f^i}{\partial \eta_k} \tag{3.4b}$$

By subtracting equations (3.4a) and (3.4b), having in mind the properties of Lagrange - brackets

$$([\xi^j, \eta_k] :=) \qquad \left[\frac{\partial f^i}{\partial \xi^j} \frac{\partial \varphi_i}{\partial \eta_k} - \frac{\partial f^i}{\partial \eta_k} \frac{\partial \varphi_i}{\partial \xi^j} \right] = \delta_j^k$$

and putting

$$\overline{Q}^k = P_i^* \frac{\partial f^i}{\partial \eta_k}, \tag{3.5}$$

we obtain the equations

$$(\delta_j^k \, \dot{\xi}^j =) \qquad \dot{\xi}^k = \frac{\partial \overline{H}}{\partial \eta_k} - \overline{Q}^k . \tag{3.6a}$$

In the same way, composing equations (3.3a) and (3.3b) with $\frac{\partial \varphi_i}{\partial \xi^k}$ and $\frac{\partial f^i}{\partial \xi^k}$ respectivelly, we obtain the equations

$$\dot{\eta}_k = -\frac{\partial \overline{H}}{\partial \xi^k} + \overline{P}_k^*, \tag{3.6b}$$

where $\overline{P}_k^* = P_i^* \frac{\partial f^i}{\xi_k}$.

Obviously, the equations (3.6), obtained by the canonical transformations of the equations (3.3), are not of the same form as (3.3) (cf. (3.3a) and (3.6a)). Only in the case of the canonical transformations of the form

$$q^i = f^i(\xi^j) \ , \ p_i = \varphi_i(\xi^j, \eta^k)$$

the equations (3.3) are invariant, in view of the fact that the “forces” (3.5) disappear.

3. HOMOGENEOUS FORMALISM VERSUS "RHEONOMIC COORDINATE" CONCEPT

As we have pointed out at the very beginning of the paper, the fundamental point of view is now taken in the general canonical theory by introducing the $(n + 1)$'th coordinate x^0, coinciding with the independent variable t. The $(n + 1)$'th momentum p_0, conjugate to x^0, is introduced, as it is well known, by $p_0 = -H(t)$. In Hamilton's and Jacobi's mechanics, as well as at the other classics, the Hamiltonian H is usually defined by the relations (2.14) or, what is the same, by (2.6). As a natural scalar invariant, it should be invariant under any mathematical transformations. However, in the standard analytical mechanics of the system with time-dependent constraints, the kinetic energy is not a homogeneous quadratic form, but the sum of three forms T_s ($s = 0,1,2$) with the degree of homogeneity s, i.e.

$$T = T_2 + T_1 + T_0 = \frac{1}{2} a_{ij}(q,t)\,\dot{q}^i\,\dot{q}^j + b_i(q,t)\,\dot{q}^i + c(q,t)\,. \tag{4.1}$$

For these systems the Hamiltonian reduces to the form (s. for example [3])

$$H = T_2 - T_0 + V \neq E\,. \tag{4.2}$$

Obviously, the invariance of the energy is violated.

On the other side, if on the base of the system time-dependent constraints we assume $\tau(k,t)$ -as a prescribed function of the time t and a parameter k-to be an supplementary, auxiliary coordinate $q^0 = \tau(k,t)$, then the kinetic energy, instead of (3.1), represents a homogeneous quadratic form [4]

$$T = \frac{1}{2} a_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta\,, \tag{4.3}$$

where

$$a_{\alpha\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^\beta} = a_{\alpha\beta}(q^0, q^1, \ldots, q^n)$$

is the inertial tensor. It follows immediately that the momenta have the same form as in (2.3), i.e.

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial T}{\partial \dot{q}^i} = a_{i\alpha}\dot{q}^\alpha = a_{i0}\dot{q}^0 + a_{ij}\dot{q}^j\,, \tag{4.4a}$$

$$p_0 = \frac{\partial L}{\partial \dot{q}^0} = \frac{\partial T}{\partial \dot{q}^0} = a_{0\alpha}\dot{q}^\alpha = a_{00}\dot{q}^0 + a_{0j}\dot{q}^j\,. \tag{4.4b}$$

It is easy to show that now, on the base of the definition (2.14), the Hamiltonian H is equal to the mechanical energy

$$\mathrm{H} = \frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L = p_a\dot{q}^\alpha - L = 2T - (T - V) = E\,, \tag{4.5}$$

just as in the classical formulation (2.6). This is, in fact, in literature frequently encountered the "new" Hamiltonian H_h depending of $2n + 2$ variables ([4], [5])

$$H_h(q^0, q^1, \ldots, q^n; p_0, p_1, \ldots, p_n) = H(q^0, q^1, \ldots, q^n; p_1, \ldots, p_n) + p_0\,. \tag{4.6}$$

Really, if we suppose, as it is usual in the homogeneous formalism, that $q^0 = t$

$(\rightarrow \dot{q}^0 = 1)$, on the base of (1.4) and (3.1) we obtain $p_0 = T_1 + 2T_0$. Substitution of this expression and (3.2) in (3.6) gives

$$H_h = T_2 - T_0 + V + T_1 + 2T_0 = T_2 + T_1 + T_0 + V = T + V = E\,,$$

which was to be proved.

Further, from the relations (3.3) and $\dot{q}^\alpha = a^{\alpha\beta} p_\beta$ it follows

$$T = \frac{1}{2} a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta = \frac{1}{2} a_{\alpha\beta} a^{\alpha\mu} p_\mu a^{\beta\nu} p_\nu = \frac{1}{2} a^{\mu\nu} p_\mu p_\nu$$

and then

$$H = \frac{1}{2} a^{\mu\nu} p_\mu p_\nu + V\,.$$

If we write the function $V(q^0, q^1, ..., q^n)$ in the form $V = \prod(q^0, q^1, ..., q^n) + P(q^0)$, where $P = \int R_0(q^0) dq^0$ is the "rhenomic potential" [4], and the function H in the form $H = T + \Pi + \mathrm{P}(q^0) = \widetilde{H}(q^0, q^1, ..., q^n; p_0, p_1, ..., p_n) + \mathrm{P}(q^0)$, the equations (2.9) can be write down in the following way

$$\dot{q}^i = \frac{\partial \widetilde{H}}{\partial p_i}\ ,\ \dot{q}^0 = \frac{\partial \widetilde{H}}{\partial p_0}\,, \tag{4.7a}$$

$$\dot{p}_i = -\frac{\partial \widetilde{H}}{\partial q^i} + P_i^*\ ,\ \dot{p}_0 = -\frac{\partial \widetilde{H}}{\partial q^0} + P_0\,, \tag{3.7b}$$

where $P_0 = P$, and $P_0^* = \sum_{\nu=1}^N \mathbf{F}_\nu^* \cdot \dfrac{\partial \mathbf{r}_\nu}{\partial q^0}$.

Now it is clear, similarly as in the equations (2.8a) and (2.9a), that the equations (4.7a) are nothing else than the relations between the generalized velocities and the generalized momenta, and therefore they do not contain any "canonical forces". In the like manner, it becomes evident that now the momentum coordinate p_0 is not equal to the negative Hamiltonian nor $H = 0$, as it is proved in the paper [7], too.

Finally, it should be noted that the Leone`s method [2]) fits in our concept of the "rhenomic coordinate" and the corresponding momentum (4.4b).

REFERENCES

1. E. L. Stiefel and G. Scheifele, (1971) *Linear and Regular Celestial Mechanics* (Springer-Verlag, Berlin-Heidelberg-New York).
2. D. Brouwer and G. Clemence, (1961) *Methods of Celestial Mechanics* (Academic Press, New York-London,) (in Russian).
3. F. R. Gantmakher, (1966) *Lektsii po analiticheskoj mekhanike* (Fizmatgiz, Moskva).
4. V. A. Vujičić, (1987) *The modification of analytical dynamics of rheonomic systems*, Tensor (N.S.) 46, 418-431.
5. V. A. Vujičić, (1995) *Energy exchange theorems in systems with time-dependent constraints*, Theoretical and Applied MECHANICS 21, 105-121.
6. V. V. Kozlov and V.A. Vujičić, (1996) *A contribution to the theory of rhenomic systems*, Bulletin de l'Académie Serbe des Sciences et des Arts, Sciences mathématiques CXI, 21, 85-91.

7. V. A. Vujichich, (1995) *Ob odnorodnom formalizme klassicheskoj dinamiki reonomnykh sistem*, Ehlektronoe modelirovanie XVII, 4, 10-14.

O HOMOGENOM FORMALIZMU U NEBESKOJ MEHANICI

V. Vuji~i}, Z. Dra{kovi}

Komentariše se valjanost teorije "homogenog formalizma" u nebeskoj mehanici. Pokazuje se da u klasičnoj mehanici jednačine (0.1a) ne sadr`e "kanonske sile" Q^{α}. *Zatim se napušta relacija* $p_0 = -H$ *i umesto nje pokazuje se da postoji impuls* $p_0 = \frac{\partial T}{\partial \dot{q}^0} = a_{00}\,\dot{q}^0 + a_{0j}\,\dot{q}^j$. *Stoga umesto diferencijalnih jednačina (0.1a) i (0.1b) egzistiraju diferencijalne jednačine kretanja (3.7a) i (3.7b).*

Ključne reči. *nebeska mehanika, homogeni formalizam, koncept "reonommne koordinate".*