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ON STABILITY OF A MECHANICAL SYSTEM WITH ONE DEGREE OF FREEDOM

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Abstract. *The stability problem of the equilibrium position for mechanical system with one degree of freedom is considered in the paper. On the basis of the general theorems from [1, 2] the sufficient condition of asymptotic stability in the various assumptions with respect to viscosity and elasticity are obtained. The comparison of obtained results and the results, obtained by means of the other methodes is carried out.*

1. INTRODUCTION

Consider a mechanical system with one degree of freedom

$$\ddot{x} + k(t, x, \dot{x})\,|\,\dot{x}\,|^{\alpha}\ \dot{x} + f(x) = 0\,, \qquad (\alpha \geq 0) \tag{1.1}$$

where $k(t,x,y)$ and $f(x)$ are the functions defined and continuous for all $t\in R^+$ and $(x,y)\in R^2$, and such that

$$k(t, x, y) \geq 0(\forall t \in R^+, \forall (x, y) \in R^2),\ xf(x) > 0(\forall x \neq 0),\ f(0) = 0 \tag{1.2}$$

The equations of the form (1.1) is the subject of a number scientific papers. Here we present the most famous results. Primary the simplest equation of the form (1.1)

$$\ddot{x} + k(t)\dot{x} + hx = 0$$

have been considered, were $h>0$, $k(t)\geq 0$. Under the condition $0<k_1\leq k(t)\leq k_2$, were k_1, k_2 are constants the equilibrium position is globally asymptotically stable [3]. If the damping coefficient $k(t)$ is not bounded above, then the rest point $\dot{x} = x = 0$ is not necessarily asymptotically stable. For instance the equation $\ddot{x} + (2 + e^t)\dot{x} + x = 0$ has the solution $x=a(1+e^{-t})$ that does not tend to 0 as $t\rightarrow+\infty$ [4].

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The equation of the same form has been studied in [5]. The following conditions for the global asymptotic stability of the equilibrium point is obtained

$$\frac{1}{T^2}\int_0^T k(\tau)d\tau \le k_1 = const\ ,\ \ h > 0\ ,\ \ k(t) \ge \varepsilon > 0$$

Moreover it is pointed that this result is true for a nonlinear equation of the form

$$\ddot{x} + k(t, x, \dot{x})\dot{x} + f(x) = 0\ , \tag{1.3}$$

were $\int_0^x f(z)dz \to +\infty$ as $|x| \to +\infty$, $k(t, x, \dot{x}) \ge W(x, \dot{x}) \ge 0$ $(W(x, \dot{x}) > 0\,(\dot{x} \ne 0))$. The growth condition for the coefficient of viscosity is as follows

$$\frac{1}{T^2}\int_0^T k(t, x(t), y(t))dt \le B\ \ \forall\ (x(t),\ y(t)) : R^+ \to \{|\,x\,| \le H, |\,y\,| \le H\} \tag{1.4}$$

for some B and all $T \ge 0$.

The assumption of boundedness for all T of $1/T^2 \int_0^T k(t)\,dt$ implies that $k(t)$ grows no more than linearly in time. In paticular if $k(t) \le K + Mt$, this growth condition (1.4) is satisfied. Alternatively, it is shown that the condition $k(t) \le K + Mt^{1+\varepsilon}$ ($\varepsilon > 0$) does not assure the globally asymptotic stability.

In [6] the uniform asymptotic stability of $\dot{x} = x = 0$ is shown in the case of integrably continuous function $k = k(t)$ and when

$$\lim_{t \to +\infty, T \to +\infty} \inf \int_t^{t+T} k(\tau)\,d\tau > 0\ . \tag{1.5}$$

The investigation in detail for nonlinear equation of the form (1.1) when the damping coefficient $k(t, x, \dot{x})$ is unbounded or became "too small" is presented in [7].

It is shown that under the conditions

$$0 \le k_1(x) \le k(t, x, \dot{x}) \le k_2(t)k_3(x, \dot{x})\ ,\ \ k_2(t) > 0\ ,\ \ k_3(x, \dot{x}) \ge 0\ ,$$

$$\int_{-v}^{v} k_1(x)\,dx = m(v) > 0\ (\forall v > 0)\ ,\ \ \int_0^\infty k_2^{-1}(\tau)\,d\tau = \infty\ , \tag{1.6}$$

the equilibrium position is globally asymptotically stable, here $k_2(t)$ is the nondecreasing function.

The result is the same one when

$$0 < k_1(t) \le k(t, x, \dot{x}) \le k_2(x, \dot{x})\ ,\ \ \int_0^\infty k_1(\tau)\,d\tau = \infty\ . \tag{1.7}$$

From the other hand if $k = k(t) \ge k_1(\mathrm{t}) > 0$, $k_1(t)$ is nonincreasing and

$$\int_{t_0}^\infty k_1^{-1}(\tau)\,d\tau < +\infty$$

then the equilibrium point $x = \dot{x} = 0$ may be nonattractive for some solutions from their neighbourhood.

We note that the results obtained in [5] and [7] are independed. Indeed, the result of [7] does not follow from [5] by taking $k(t) = t\ln t$. On the other hand, one can easily choose $k(t)$, satisfying (1.4) but not (1.6).

In [8, 9] is carried out the subsequent research of the dissipation influence on the stability of the equilibrium position for a system of the form (1.3). The following conditions for the global asymptotic stability are obtained $F(x)\to+\infty$ as $|x|\to\infty$;

$$0\le a(t)\varphi(x,y)\le k(t,x,y)\le b(t)\psi(x,y);\ \varphi(x,y)>0\quad (x,y\in R),$$

$$\int_{t_0}^{\infty}\alpha=\infty\,; \tag{1.8}$$

α is nonincreasing nonnegative differentiable function on R^+, $\alpha(t)b(t)$, is bounded and there exists $v(0<v<1)$, such that for all $k,l\left(0<k<\sup_{R/\{0\}}F(x)/xf(x),\ l\ge\inf_{x,y\in R}\varphi(x,y)\right)$.

$$\overline{\lim_{t\to\infty}}\left\{\left(\int_0^t\alpha\right)^{-1}\int_0^t\left[la(\tau)\left(\int_0^t\alpha\right)-(1+k)\alpha(\tau)\right]^{-}d\tau\right\}=\mu(k)<1-v\,,$$

$$[a]^{-}=\max(0,-a)\,. \tag{1.9}$$

In particularly, it is pointed that when $a(t)\equiv1$, where $b(t)$ is the nondecreasing function on R^+ and $\int^{\infty}1/b=\infty$ where $a(t)$ is the nonincreasing function and $\int^{\infty}a(t)=\infty$ the results of this paper and the results of [7] are the same.

The equation (1.3) with the coefficient $k(t,x,\dot x)$, satisfying the inequalities

$$0\le a(t)k_1(x,\dot x)\le k(t,x,\dot x)\le b(t)k_2(x,\dot x)\,,\ k_1(x,y)>0\,,\ k_2(x,y)>0\,, \tag{1.10}$$

has been investigated in [10, 11]. It is shown that if we define $k_M=\sup(f(x)/x:0<|x|\le M)^{1/2}$, $k_0=\max\{k_2(x,y):2F(x)+y^2\}\le\min(F(M),F(-M))$, then the sufficient conditions for asymptotic stability of the equilibrium position $\dot x=x=0$ are as follows

$$\int_0^{+\infty}e^{-k_0B(t)}\int_0^t e^{k_0B(s)}dsdt=\infty\quad\left(B(t)=\int_0^t b(s)ds\right);\quad \int_{\tau}^{\tau+\alpha}a(s)ds\ge\gamma \tag{1.11}$$

for sufficiently large $\tau>0$, where: the numbers $0<\alpha<\pi/k_M$ and $\gamma>0$ are some independed constants for $k_M<+\infty$; or for all $\alpha>0$ always must exists $\gamma=\gamma(\alpha)>0$ as $k_M=+\infty$.

Papers [7, 12-14] are devoted to the investigation of the stability for the mechanical system of the following form

$$\ddot x+k(t,x)\dot x+g(t)f(x)=0\,, \tag{1.12}$$

where $g\in C^1$.

In [12] in conditions that

$$f(x)=x\,,\ 0<k_0\le k(t,x)\le k_1\,,\ g(t)>0\,,\ \dot g(t)>0\,,$$

$$\left|\dot g(t)\big/\sqrt{g^3(t)}\right|\le m=const$$

it is shown that $\lim(x(t),\ \dot x(t)\big/\sqrt{g(t)})=(0,0)$.

The more general result is obtained in [7]. If there exists a monotone function $F(t)$

such that the following expressions

$$0<\varepsilon<\frac{k(t)}{\sqrt{g(t)}}+\frac{\dot{g}(t)}{\sqrt{g^3(t)}}\le F(t) \quad \text{and} \quad \int^{\infty}\frac{\sqrt{g(t)}}{F(t)}dt=\infty \tag{1.13}$$

hold then the x-axis is an attractor. On the other hand if $k(t)\big/\sqrt{g(t)}+\dot{g}(t)>0$ and $\int^{\infty}\sqrt{g(t)}\big/F(t)dt<\infty$ then this result is not true.

In [13] is presented the further reseach of the stability properties in conditions that $g(t)>a>0$, , $\left|\dot{g}(t)\big/\sqrt{g^3(t)}\right|\le N=const$. In assumption that for any continuous function φ: $R^+\to\{|x|,|y|\le H\}$ and for all $\alpha>0$ there exist $T=T(\alpha,H,\varphi)>0$, $\xi=\xi(\alpha H,\varphi)>0$, such that

$$\left|\int_t^{t+s}k(\tau,\varphi(\tau))\right|>\alpha \text{ for all } t>T\,,\ s>\xi\,,$$

$$\frac{k(t,x)}{\sqrt{g(t)}}+\frac{\dot{g}(t)}{\sqrt{g^3(t)}}\ge\psi(t)>0\,,\quad \int_0^{\infty}\psi(t)\sqrt{g(t)}dt=\infty\,, \tag{1.14}$$

the equilibrium position of (1.5) is x-equi-asymptotically stable ($\psi(t)$ is the continuous and nonincreasing function).

In [14] the following conditions of the asymptotic stability of $\dot{x}=x=0$ in x in the case, when $k(t,x,\dot{x})\equiv 0$, $\dot{g}\ge 0$, $\lim_{t\to+\infty}g(t)=+\infty$, $2\ddot{g}(t)g(t)\le 3\dot{g}^2(t)$ are obtained. The result is obtained for the general mechanical system and is applied in the problem on the fall of the solid in the ideal liquid.

2. THE STABILITY RESEARCH IN THE CASE OF TIME DEPENDED VISCOSITY

We investigate the stability problem of the rest position $\dot{x}=x=0$ of (1.1) on the basis of limiting systems and limiting functions [1, 2].

Let the dissipation of the system is such that there exists a sequence of segments $[t_n,t_n+s_n]$, $(t_n\to+\infty,\ s_n\ge s>0)$, such that for any continuous function $(u_1(t),\ u_2(t))$: $R^+\to\{|x|\le H,|y|\le H\}$ the following relations hold

$$k[t]\le N=const\,(\forall t\in[t_n,\ t_n+s_n])\ ,\quad \liminf_{n\to\infty}\int_{t_n}^{t_n+s_n}k[\tau]d\tau>0$$

$$k[t]=k(t,\ u_1(t),\ u_2(t))\,. \tag{2.1}$$

Under the condition (2.1) it follows the precompactness of (1.1) with respect to the sequence $[t_n,t_n+s_n]$ for each $k[t]=k(t,u_1(t),u_2(t))$, where $(u_1(t),u_2(t))$:$R^+\to\{|x|\le H,|y|\le H\}$ is any continuous function [1, 2]. From here we conclude that asymptotic behaviour of bounded solution (1.1) $x=x(t)$ with respect to the sequence $[t_n,\ t_n+s_n]$ as $t_n\to+\infty$ is defined by the limiting equation of the form [1]

$$\ddot{x} + k^*(t)\,|\,\dot{x}\,| + f(x) = 0\,, \tag{2.2}$$

where k*(t) is such that

$$\int_0^t k^*(\tau)d\tau = \lim_{j\to\infty}\int_0^t k[t_{nj}+\tau]d\tau\,, \quad k[t] = k(t, x(t),\, \dot{x}(t))\,.$$

Consider the function

$$V = \dot{x}^2 + 2F(x)\,, \qquad F(x) = \int_0^x f(\tau)d\tau$$

as a Liapunov function. The derivative of V with respect to equation (1.1) has the estimate

$$\dot{V}(t,x,\dot{x}) = 2\dot{x}\ddot{x} + 2f(x)\dot{x} = 2\dot{x}(-k(t,x,\dot{x})\,|\,\dot{x}\,|^{\alpha}\ \dot{x} - f(x)) + 2f(x)\dot{x} = -2k(t,x,\dot{x})\,|\,\dot{x}\,|^{\alpha+2} \le 0$$

Hence, from here and under the condition (1.2) the equilibrium point of (1.1) $\dot{x} = x = 0$ is uniformly stable. Moreover, if $F(x)\to+\infty$ as $|x|\to+\infty$ then the motions (1.1) are uniformly bounded.

The limiting function for $W(t,\dot{x}) = k[t]\,|\,\dot{x}\,|^{\alpha+2}$ with respect to any subsequence $\{t_{nlj}\}\subset\{t_{nj}\}$ is as follows

$$\Omega(t,\dot{x}) = k^*(t)\,|\,\dot{x}\,|^{\alpha+2}\,, \text{ where } \int_0^t k^*(\tau)d\tau = \lim_{l\to\infty}\int_0^t k[t_{njl}+\tau]d\tau$$

and consequently under the condition (2.1) the function $k^*(t)>0$ on any set $E\subset[0,s]$ with $mesE \ne 0$. The set $\{V_\infty^{-1}(t,c) : c = const > 0\} \cap \{\Omega(t,\dot{x}) = 0\}$, defined by the theorems from [1] for the problem for $t \in E$ is the set $S = \{\dot{x} = 0,\ F(x) = c = const > 0\}$. By the form of the limiting equation (2.2) the solution $x = \varphi(t)$ of (2.2) on the set S for $t \in E$ must almost for all these t satisfy $f(\varphi(t)) = 0$. But this is possible when $F(\varphi(t)) = 0$.

Thus, the set $\{V_\infty^{-1}(t,c) : c = const > 0\} \cap \{\Omega(t,\dot{x}) = 0\}$ does not contain the whole solution of (2.2). Then by the theorems 3.2 and 3.3 from [1] and 1.3 from [2] we can state the following results.

Theorem 1. *Under the conditions* (1.2) *and* (2.1) *the equilibrium position of* (1.1) $\dot{x} = x = 0$ *is asymptotically stable uniformly in* $(x_0,\ \dot{x}_0)$.

Theorem 2. *Let the conditions* (1.2) *hold and instead of* (2.1) *the following conditions*

$$k[t] \le N = const\,(\forall t \in R^+)\,, \qquad \lim_{t\to+\infty,\ T\to+\infty} \inf \int_t^{t+T} k[\tau]\,d\tau > 0 \tag{2.3}$$

are true.

Then the equilibrium position $\dot{x} = x = 0$ of (1.1) is uniformly asymptotically stable.

If $F(x) \to +\infty$ as $|\,x\,| \to +\infty$, then under the conditons (1.2), (2.1), (2.3) we have accordingly global equi-asymptotic and uniform asymptotic stability of $\dot{x} = x = 0$.

By the analogy of the preceding on the basis of the theorem on instability 3.4 from [1] we can obtain the following result.

Theorem 3. *Under the conditions of the preceding theorem with respect* $k(t,x,\dot{x})$ *and the condition that* $f(x)<0$ *for* $x>0$ *or* $f(x)<0$ *for* $x<0$ *for* $x<0$ *the equilibrium position*

$\dot{x} = x = 0$ *of* (1.1) *is instable*.

The theorem 1 is true in more general case of the damping coefficient unboundedness $k(t, x, \dot{x})$. Namely, under the condition that there exists the sequence $[t_n, t_n + s]$ such that for any conditions function $(u_1(t),u_2(t))$:$R^+ \to \{|x|,|y| \le H\}$ the equalities

$$\sum_{n=1}^{\infty}\left(\int_{t_n}^{t_n+s} k[\tau]\,d\tau\right)^{-\frac{1}{\alpha+1}} = \infty\,, \quad \liminf_{n\to\infty} \int_{t_n}^{t_n+s} k[\tau]\,d\tau > 0 \tag{2.4}$$

hold. We prove the theorem 1 under the replacement of condition (2.1) by (2.4).

Let $(u_1(t),u_2(t))$:$[t_0,+\infty) \to R\{|x| \le H, |y| \le H\}$ be any continuous function satisfying the following inequality

$$\int_{t_0}^{\infty} k[\tau]\,|u_2(\tau)|^{\alpha+2} d\tau < +\infty\,, \quad k[\tau] = k(\tau, u_1(\tau), u_2(\tau))\,.$$

From the first condition of (2.4) it follows the existence of subsequence $t_{nj} \to \infty$, for which $n_j \to \infty$

$$I_{nj} = \int_{t_{nj}}^{t_{nj}+s} k[\tau]\,|u_2(\tau)|^{\alpha+1} d\tau \to 0$$

Suppose that this is not true, namely that $I_{nj} \ge \varepsilon_0 > 0$ for all $n \ge$ some n_0. By the Gjelder inequality [2] we consequently have

$$0 < \varepsilon_1 = \varepsilon^{\frac{\alpha+2}{\alpha+1}} \le \left(\int_{t_n}^{t_n+s} k^{\frac{1}{\alpha+2}}[\tau]\left(k^{\frac{1}{\alpha+2}}[\tau]\,|u_2(\tau)|\right)^{\alpha+1} d\tau\right)^{\frac{\alpha+2}{\alpha+1}} \le$$

$$\le \left(\int_{t_n}^{t_n+s} k[\tau] d\tau\right)^{\frac{1}{\alpha+1}} \left(\int_{t_n}^{t_n+s} k[\tau]\,|u_2(\tau)|^{\alpha+2} d\tau\right)$$

or for $n \ge n_0$

$$\left(\int_{t_n}^{t_n+s} k[\tau] d\tau\right)^{-\frac{1}{\alpha+1}} \le \varepsilon_1^{-1} \left(\int_{t_n}^{t_n+s} k[\tau]\,|u_2(\tau)|^{\alpha+2} d\tau\right).$$

Summarising with n_0, using the contrary assumption we obtain

$$\sum_{n=n_0}^{\infty} \left(\int_{t_n}^{t_n+s} k[\tau] d\tau\right)^{-\frac{1}{\alpha+1}} \le \varepsilon_1^{-1} \sum_{n=n_0}^{\infty} \int_{t_n}^{t_n+s} k[\tau]\,|u_2(\tau)|^{\alpha+2} d\tau \le ,$$

$$\le \varepsilon_1^{-1} \int_{t_{n0}}^{\infty} k[\tau]\,|u_2(\tau)|^{\alpha+2} d\tau < +\infty$$

that is contrary to the condition (2.4).

Thus, the subsequence $n_j \to +\infty$ exists for which

$$I_{nj} = \int_{t_{nj}}^{t_{nj}+s} k[\tau] \, | u_2(\tau) |^{\alpha+1} d\tau \to 0$$

Let $x = x(t, t_0, x_0)$ be any motion of (1.1) bounded by the domain $\{| x | \leq H, | y | \leq H\}$. The condition (1.2) implies that along this motion

$$\lim_{t \to +\infty} V(t, x(t), \ \dot{x}(t)) = V^* ;$$

$$2\int_{t_0}^{\infty} k(\tau, x(\tau), \ \dot{x}(\tau)) \, | \dot{x}(\tau) |^{\alpha+2} d\tau \leq V(t_0, x_0) - V^* < +\infty$$

hold. On the basis of the fact above the sequence $t_{nj} \to +\infty$ exists, for which

$$\int_{t_{nj}}^{t_{nj}+t} k(\tau, x(\tau), \ \dot{x}(\tau)) \, | \dot{x}(\tau) |^{\alpha} \ \dot{x}(\tau) d\tau \to 0 \tag{2.5}$$

uniformly in $t \in [0,s]$.

We construct the limiting equation for (1.1) along the considered solution $x(t, t_0, x_0)$. From (1.1) we have

$$\dot{x}(t_n + t) - \dot{x}(t_n) = -\int_{t_n}^{t_n+t} k[\tau] \dot{x}(\tau) d\tau - \int_{t_n}^{t_n+t} f(x(\tau)) d\tau \tag{2.6}$$

or

$$\dot{x}(t_n + t) - \dot{x}(t_n) = -\int_{t_n}^{t_n+t} k[\tau] \dot{x}(\tau) d\tau - \int_0^t f(x(t_n + s)) ds \, . \tag{2.7}$$

From the stability of the zero solution (0,0) it follows the boundedness $\dot{x}(t)$ and hence the family of solutions $x_n(t) = x(t_n + t)$ is uniformly bounded and continuous that is precompact. From here it follows the existence of subsequence t_{nj}, such that $x_{nj}(t) \to y^*(t)$ as $n_j \to \infty$.

Then we pass to the limit in (2.6) as $n_j \to +\infty$, using (2.5) and assuming that $\dot{x}(t_{nj}) \to z_0 = const$ as $n_j \to +\infty$. We obtain there exists a continuous function $z = z(t)$, such that $\dot{x}(t_{nj} + t) \to z(t)$ uniformly in $t \in [0,s]$ and the equality

$$z(t) = z_0 - \int_0^t f(y^*(s)) ds \tag{2.8}$$

holds. From here it follows that $z(t) \in C^1$ and $z(t) = \dot{y}^*(y)$.

Differentiating with respect to t we obtain $\ddot{y}^*(t) = -f(y^*(t))$. Whence

$$(x(t_{kj} + t), \dot{x}(t_{kj} + t)) \to (y^*(t), \dot{y}^*(t) : \ddot{y}^*(t) = -f(y^*(t))) \, .$$

Thus, we obtain some subset Q of the $\varpi^+(x(t,t_0,x_0))$ limit-set with $mes(Q) > 0$. Q is semi-invariant with respect to the limiting system of the form

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -f(x(t)) \end{cases},$$

where $0 \leq t \leq s$. At the same time the subset Q is the subset of the set

$\{V_\infty^{-1}(t,c): c = V^* = const\} \cap \{\Omega(t,y) = 0\}$, wich has the form $\{\dot{x} = 0, x: \exists x_j \to x, F(x_j) \to V^*/2\}$ by virtue of the second condition of (2.4) for $t \in E$ with *mes* $E > 0$. But this is by virtue of both the limiting system view and relation $f(x) = 0 \Leftrightarrow x = 0$ possible only if $V^* = 0$, thus, $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = 0$. So we proved the theorem.

Theorem 4. *Under the conditions* (1.2) *and* (2.4) *the equilibrium position of* (1.1) $\dot{x} = x = 0$ *is asymptotically stable*.

3. THE STABILITY RESEARCH IN THE CASE OF TIMEDEPENDED ELASTICITY

Consider the equation of the following form

$$\ddot{x} + k(t,x,\dot{x})\dot{x} + g(t)f(x) = 0 \tag{3.1}$$

Suppose that for all $t \in R^+$ and $(x,y) \in \{|x|,|y| \le H\}$ the following relations hold

$$0 < g(t) \le g_1 = const\ , \tag{3.2}$$

$$\frac{2k(t,x,y)}{g(t)} + \frac{\dot{g}(t)}{g^2(t)} \ge k_1(t) \ge 0\ , \tag{3.3}$$

and there exists a sequence of segments $[t_n, t_n+s_n]$, $t_n \to +\infty$, $s_n \ge > 0$, for which the following conditions hold

$$k(t,x,y) \le N = const\ \lim_{n\to\infty}\inf \int_{t_n}^{t_n+s} k_1(t)dt > 0\ ,\quad g(t) \ge g_0 = const > 0\ . \tag{3.4}$$

From the condition (3.2) it follows the precompactness of the equation (3.1) with respect to the sequence $[t_n, t_n+s]$, for each $k[t] = k(t,u_1(t),u_2(t))$ where $(u_1(t),u_2(t))$: $R^+ \to \{|x|,|y| \le |H|\}$ is any continuous function. Fro: here the subset Q of the limiting points $\varpi^+(x(t,t_0,x_0))$ of the bounded motion (3.1), defined by the sequences $\{t'_n \in [t_n, t_n s]\}$ is semi-invariant with respect to limiting equation

$$\ddot{x} + k^*(t)\dot{x} + g^*(t)f(x) = 0\ , \tag{3.5}$$

where

$$\int_0^t k^*(\tau)d\tau = \lim_{n_j\to\infty}\int_0^t k[t_{n_j} + \tau]d\tau\ ,$$

$$\int_0^t g^*(\tau)d\tau = \lim_{n_j\to\infty}\int_0^t g[t_{n_j} + \tau]d\tau\ . \tag{3.6}$$

From it`s definition the function $g^*(t)$ satisfies the inequality (with the exeption of the set $E \in [0,s]$ with mes $E = 0$)

$$0 < g_0 \le g^*(t) \le g_1 \tag{3.7}$$

Let $V = \dot{x}^2/g(t) + 2F(x)$. The derivative of V with respect to equation (3.1) has the estimate

$$\dot{V}(t,x,\dot{x}) = 2\dot{x}\ddot{x}/g(t) - \dot{x}^2\dot{g}(t)/g^2(t) + 2f(x)\dot{x} =$$
$$= -(2k(t,x,\dot{x})/g(t) + \dot{g}(t)/g^2(t))\dot{x}^2 \le -k_1(t)\dot{x}^2 \le 0$$

From here by taking into account (1.2) and (3.2) we have the stability of the equilibrium position $\dot{x} = x = 0$ of (3.1).

Further, as earlier, we find that the limiting function for $W(t,\dot{x}) = k_1(t)\dot{x}^2$ has a view $\Omega(t,\dot{x}) = k_1^*(t)\dot{x}^2$, where $k_1^*(t) > 0$ on the set $E \subset [0,s]$ with $\operatorname{mes} E \ne 0$. The set $\{V_\infty^{-1}(t,c) : c = const > 0\} \cap \{\Omega(t,\dot{x}) = 0\}$ for $t \in E$ is the set $S = \{\dot{x} = 0, F(x) = c/2 = const > 0\}$. Repeating the subsequent discussion as for (1.1) with respect both to the set S and the limiting equation (3.5) we have the following results.

Theorem 5. *Under the conditions* (1.2), (3.2), (3.3) *and* (3.4) *the equilibrium position* $\dot{x} = x = 0$ *of* (3.1) *is asymptotically stable uniformly in* $(x_0, \dot{x}_0)$.

If instead of conditions (3.4) we assume that for all $t \in R^+$ the following conditions

$$k(t,x,y) \le N = cnst, \lim_{t\to+\infty, T\to+\infty} \inf \int_t^{t+T} k_1(\tau)d\tau > 0,$$
$$g(t) \ge g_0 = const > 0 \tag{3.8}$$

hold, then the equilibrium position $\dot{x} = x = 0$ of (3.1) is uniformly asymptotically stable.

Theorem 6. *Under the conditions* (3.2), (3.3), (3.4), *the condition* (1.2) *with respect to* $k(t,x,\dot{x})$ *and the condition* $f(x)x < 0$ *for* $x < 0$ *or* $f(x)x < 0$ *for* $x > 0$ *the equilibrium position of* (3.1) *is instable*.

Suppose that the dissipation $k(t,x,\dot{x})$ is unbounded and the following relation hold

$$2k(t,x,y) + \frac{\dot{g}(t)}{g(t)} \ge \alpha k(t,x,y) \ge 0, \qquad \alpha = const > 0. \tag{3.9}$$

Then the following theorem is true.

Theorem 7. *Let the conditions* (1.2), (2.4), (3.2), (3.0) *and the third condition of* (3.4) *hold*.

Then the equilibrium position $\dot{x} = x = 0$ of (3.1) is asymptotically stable uniformly in $(x_0, \dot{x}_0)$.

By the conditions of the theorems 5-6 the coefficient $g(t)$ may be small but bounded above for all $t \in R^+$ and may be more than some $g_0 > 0$ on the segments $[t_n, t_n+s]$. Suppose, that $g(t)$ is a function with the same properties as above but unbounded above, that is

$$g(t) > 0, \int_0^\infty \sqrt{g(t)}d\tau = \infty.$$

Let us pass in the equation (3.1) from the variable t to τ by means of formula

$$\tau = \int_0^t \sqrt{g(s)}ds.$$

Then

$$d\tau = \sqrt{g(t)}dt, \quad \frac{dx}{dt} = \sqrt{g(t)}\frac{dt}{d\tau}, \quad \frac{d^2x}{dt^2} = g(t)\frac{d^2x}{d\tau^2} + \frac{\dot{g}(t)}{2\sqrt{g(t)}}\frac{dx}{d\tau}.$$

We have

$$g(t)\frac{d^2x}{d\tau^2}+\frac{\dot g(t)}{2\sqrt{g(t)}}\frac{dx}{d\tau}+k(t,x,\dot x)\frac{dx}{d\tau}+g(t)f(x)=0$$

or

$$x''+\left(\frac{\dot g(t(\tau))}{2\sqrt{g^3(t(\tau))}}+\frac{k\left(t(\tau),x,x'\sqrt{g(t(\tau))}\right)}{\sqrt{g(t(\tau))}}\right)x'+f(x)=0\,, \tag{3.10}$$

where $x'=\dot x\big/\sqrt{g(t)}$. The obtained equation is the equation of the type (1.1) and the reseach methods used above for the equation of such type are true. Moreover from asymptotic stability of the equilibrium position $x'=x=0$ of transformed equation (3.8) it follows the asymptotic stability of equilibrium position $x=\dot x=0$ of the primary equation (3.1) in x. Hence, on the basis of the theorem 2 we have the following result.

Theorem 8. *Let for all* $t\in R^+$ *and* $(x,\dot x)\in\{|x|,|\dot x|\le H\}$ *the following inequalities hold*

$$0\le k_0(t)\le\frac{k(t,x,\dot x)}{\sqrt{g(t)}}+\frac{\dot g(t)}{2\sqrt{g^3(t)}}\le N=const\,,$$

$$\lim_{t\to+\infty,\ T\to+\infty}\inf\int_t^{t+T}k_0(s)ds>0 \tag{3.11}$$

Then the equilibrium position $\dot x=x=0$ is uniformly asymptotically stable in $(x,\dot x\big/\sqrt{g(t)})$.

Now we study the stubility of the equilibrium position of (3.1) by the other way, proposing that $g(t)$ is a function, unbounded above. Then the Liapunov function will be positive defined with respect to x and it`s derivative with respect to (3.1) will be nonpositive. In this case the following theorems are true.

Theorem 9. *Under the conditions* (1.2), (3.3), *first and second conditions of* (3.4) *and the condition* $g(t)>0$ *for all* $t\in R^+$ *the equilibrium position* $\dot x=x=0$ *of* (3.1) *is asimptotically stable with respect to* x.

Theorem 10. *Under the conditions* (1.2), (3.3), (3.8) *the equilibrium position* $\dot x=x=0$ *is uniformly asymptotically stable with respect to* x.

As the equation (3.1) the problem on asymptotic stability of the equilibrium position of the following equation

$$\ddot x+k(t,x,\dot x)\dot x+g(t,x)f(x)=0 \tag{3.12}$$

may be considered, where $k(t,x,\dot x)$ and f are the functions eith the same properties as above and $g(t,x)\in C^1$.

Suppose that for all $t\in R^+$ and $(x,y)\in\{|x|,|y|\le H\}$ the following asumptions

$$0<g(t,x)\le g_1=const\,,\qquad \left|\frac{\partial g}{\partial x}(t,x)\right|\le k_1=const\,,$$

$$\frac{2k(t,x,\dot{x})}{g(t,x)} + \frac{1}{g^2(t,x)}\frac{\partial g}{\partial t}(t,x) \geq 2k_0 = const > 0 \tag{3.13}$$

hold, and suppose that there exists the sequence of segments $[t_n, t_n+s_n]$ ($t_n \to +\infty$, $s_n \geq s > 0$), on wich the functions $k(t,x,\dot{x})$ and $g(t,x)$ satisfy the following conditions

$$k(t,x,\dot{x}) \leq N = const\ , \qquad g(t,x) \geq g_0 = const > 0\ . \tag{3.14}$$

The corresponding limiting equations wall have the form

$$\ddot{x} + k^*(t)\dot{x} + g^*(t,x)f(x) = 0\ ,$$

where $k^*(t)$, $g^*(t,x)$ are defined by thge formulas of the form (3.6).

By (3.12) for derivative of the function $V = \dot{x}^2 / g(t,x) + 2F(t,x)$ we have the estimate

$$\dot{V}(t,x,\dot{x}) = -\left[\frac{2k(t,x,\dot{x})}{g(t,x)} + \frac{\partial g(t,x)/\partial t + \partial g(t,x)/\partial x \dot{x}}{g^2(t,x)}\right]\dot{x}^2 \leq -k_0\dot{x}^2 \leq 0\ .$$

The limiting function for $W(t,\dot{x}) = k_0\dot{x}^2$ is the same function $\Omega(t,\dot{x}) = k_0\dot{x}$. Reasoning in the same way as in the case of (3.1) we have the following results.

Theorem 11. *Under the conditions* (1.2), (3.13), (3.14) *the equilibrium positions* $\dot{x} = x = 0$ *of* (3.12) *is aszmptotically stable.*

Theorem 12. *Under the conditions* (1.2), (3.13), *the first condition of* (3.14) *and the condition* $g(t,x) \geq g_0$ *for all* $t \in R^+$ *and* $x \in \{|x| \leq H\}$ *the equilibrium position* $\dot{x} = x = 0$ *of* (3.12) *is uniformly asymptotically stable.*

4. CONCLUSION

The obtained results in contrast to the results [3-14] are deduced uniformly on the basis of the method of limiting functions and equations. The following comparative analisis takes place.

In the case of time-depended viscosity the problem is considered with more general nonlinear force of viscosity $\vec{F} = -k(t,x,\dot{x})\,|\,\dot{\vec{x}}\,|^{\alpha}\ \dot{\vec{x}}$.

The condition of asymptotic stability (1.13) and the assumptions of [5, 7-11] with respect to unboundedness from above of $k(t,x,\dot{x})$ are independed. As (1.4), (1.6), (1.8), (1.10) and (1.11) from [5, 7-9] do not suppose the boundedness of $k(t,x,\dot{x})$ on any sequence of segments $\{[t_n,\ t_n+s_n], t_n \to +\infty, s_n \geq s > 0\}$, but this is supposed in the first inequality from (1.13). But in condition (1.13) the coefficient $k(t,x,\dot{x})$ may be enough large outside of $[t_n,\ t_n+s_n]$, thus the enumerated conditions (1.4), (1.6), (1.8), (1.9), (1.10) and (1.11) may by broken. The condition (1.13) is wider in comparison with (1.7), (1.8), (1.9), (1.10), (1.11) from [7-9], becouse in contrast of pointed conditions it is possible the equality $k(t,x,\dot{x}) \equiv 0$ outside $[t_n,\ t_n+s_n]$, but from the second relation (1.7) it follows the second unequality (1.13).

In the case $k = k(t)$ the theorem 2 and the result from [6] is the same.

In the present paper in the case of time-depended elasticity more general dependences $k(t, x, \dot{x}) \equiv 0$ and $g = g(t, x)$ are admitted in comparison with [7, 13, 14]. As $k = k(t)$ or $k = k(t, x)$, $g = g(t)$ the conditions (2.4), (3.2) - (3.4) and the assumptions of [7, 13] are independed. Becouse the conditions (2.4), (3.2) - (3.4) are given only with respect to the sequence $\{[t_n, t_n + s_n]\}$ and the relations (3.4) are the conditions of uniform asymptotic stability.

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O STABILNOSTI MEHANIČKOG SISTEMA SA JEDNIM STEPENOM SLOBODE

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U radu je obrađen problem stabilnosti položaja ravnoteže mehaničkog sistema sa jednim stepenom slobode. Na osnovu opštih teorema [1,2] dobijeni su dovoljni uslovi asimptotske stabilnosti za različite pretpostavke viskoznosti i elastičnosti. Sproveden je postupak upoređenja dobijenih rezultata sa rezultatima drugih metoda.